(i, j)-ξ-Open Sets in Bitopological Spaces

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 Abstract

 The aim of this paper is to introduce a new type of sets in bitopological spaces which is conditional ξ-open set in bitopological spaces called (i, j)-ξ-open set and we study its basic properties, and also we introduce some characterizations of this set.

 Keywords: ξ-open, (i, j)-ξ-open, semi-open, regular-closed
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 1 Introduction

 In 1963 Kelley J. C. [7] was first introduced the concept of bitopological spaces, where X is a nonempty set and τ₁, τ₂ are topologies on X. In 1963 Levine [8] introduced the concept of semi-open sets in topological spaces. By using this concept, several authors defined and studied stronger or weaker types of topological concept.

 In this paper, we introduce the concept of a conditional ξ-open set in a bitopological space, and we study their basic properties and relationships with other concepts of sets. Throughout this paper, (X, τ₁, τ₂) is a bitopological space, and if A ⊆ Y ⊆ X, then i-Int(A) and i-Cl(A) denote respectively the
interior and closure of \( A \) with respect to the topology \( \tau_i \) on \( X \) and \( i\text{-}Int_Y(A) \), \( i\text{-}Cl_Y(A) \) denote respectively the interior and the closure of \( A \) with respect to the induced topology on \( Y \).

2 Preliminaries

We shall give the following definitions and results.

**Definition 2.1** A subset \( A \) of a space \( (X, \tau) \) is called:

1. preopen [9], if \( A \subseteq \text{Int}(\text{Cl}(A)) \)
2. semi-open [8], if \( A \subseteq \text{Cl}(\text{Int}(A)) \)
3. \( \alpha \)-open [11], if \( A \subseteq \text{Int}(\text{Cl}(\text{Int}(A))) \)
4. regular open [5], if \( A = \text{Int}(\text{Cl}(A)) \)
5. regular semi-open [1], if \( A = \text{sInt}(\text{sCl}(A)) \)

The complement of a preopen (resp., semi-open, \( \alpha \)-open, regular open, regular semi-open) set is said to be preclosed (resp., semi-closed, \( \alpha \)-closed, regular closed, regular semi-closed). The intersection of all preclosed (resp., semi-closed, \( \alpha \)-closed) sets of \( X \) containing \( A \) is called preclosure (resp., semi-closure, \( \alpha \)-closure) of \( A \). The union of all preopen (resp., semi-open, \( \alpha \)-open) sets of \( X \) contained in \( A \) called preinterior (resp., semi-interior, \( \alpha \)-interior) of \( A \).

A subset \( A \) of a space \( X \) is called \( \delta \)-open [15], if for each \( x \in A \), there exists an open set \( G \) such that \( x \in G \subseteq \text{Int}(\text{Cl}(G)) \subseteq A \). A subset \( A \) of a space \( X \) is called \( \theta \)-semi-open [6] (resp., semi-\( \theta \)-open [2]) if for each \( x \in A \), there exists a semi-open set \( G \) such that \( x \in G \subseteq \text{Cl}(G) \subseteq A \) (resp., \( x \in G \subseteq \text{sCl}(G) \subseteq A \). A subset \( A \) of a topological space \( (X, \tau) \) is called \( \eta \)-open [13], if \( A \) is a union of \( \delta \)-closed sets. The complement of \( \eta \)-open sets is called \( \eta \)-closed.

**Definition 2.2** A topological space \( X \) is called,

1.Externally disconnected [2], if \( \text{Cl}(U) \in \tau \) for every \( U \in \tau \).
2. Locally indiscrete [4], if every open subset of \( X \) is closed.

From the above definition we obtain:

**Remark 2.3** If \( X \) is locally indiscrete space, then every semi-open subset of \( X \) is closed and hence every semi-closed subset of \( X \) is open.
Theorem 2.4 [9] A space $X$ is semi-$T_{1}$ if and only if for any point $x \in X$ the singleton set $\{x\}$ is semi-closed.

Theorem 2.5 [10] For any space $(X, \tau)$ and $(Y, \tau)$ if $A \subseteq X$, $B \subseteq Y$ then:

1. $p\text{Int}_{X \times Y}(A \times B) = p\text{Int}_{X}(A) \times p\text{Int}_{Y}(B)$
2. $s\text{Cl}_{X \times Y}(A \times B) = s\text{Cl}_{X}(A) \times s\text{Cl}_{Y}(B)$

Theorem 2.6 [10] For any topological space the following statements are true:

1. Let $(Y, \tau_{Y})$ be a subspace of a space $(X, \tau)$, if $F \in SC(X)$ and $F \subseteq Y$ then $F \in SC(Y)$.
2. Let $(Y, \tau_{Y})$ be a subspace of a space $(X, \tau)$, if $F \in SC(Y)$ and $Y \in SC(X)$ then $F \in SC(X)$
3. Let $(X, \tau)$ be a topological space, if $Y$ is an open subset of a space $X$ and $F \in SC(X)$, then $F \cap Y \in SC(X)$

Definition 2.7 [12] A space $X$ is said to be semi-regular if for any open set $U$ of $X$ and each point $x \in U$, there exists a regular open set $V$ of $X$ such that $x \in V \subseteq U$.

3 Basic Properties

In this section, we introduce and define a new type of sets in bitopological spaces and find some of its properties

Definition 3.1 A subset $A$ of a bitopological space $(X, \tau_{1}, \tau_{2})$ is said to be $(i, j)$-$\xi$-open, if $A$ is a $j$-open set and for all $x$ in $A$, there exist an $i$-semi-closed set $F$ such that $x \in F \subseteq A$. A subset $B$ of $X$ is called $(i, j)$-$\xi$-closed if $B^{c}$ is $(i, j)$-$\xi$-open.

The family of $(i, j)$-$\xi$-open (resp., $(i, j)$-$\xi$-closed) subset of $x$ is denoted by $(i, j)$-$\xi O(X)$ (resp.,$(i, j)$-$\xi C(X)$).

From the above definition we obtain:

Corollary 3.2 A subset $A$ of a bitopological space $X$ is $(i, j)$-$\xi$-open, if $A$ is $j$-open set and it is a union of $i$-semi-closed sets. This means that $A = \bigcup F_{\alpha}$, where $A$ is a $j$-open and $F_{\alpha}$ is an $i$-semi-closed set for each $\alpha$.

It is clear from the definition that every $(i, j)$-$\xi$-open set is $j$-open, but the converse is not true in general as shown in the following example.
Example 3.3 Let \( X = \{a, b, c\} \), \( \tau_1 = \{\phi, \{a\}, X\} \), \( \tau_2 = \{\phi, \{c\}, \{a, b\}, X\} \), then \((i, j)-\xi O(X)= \{\phi, \{c\}, X\}\). It is clear that \{a, b\} is \( j \)-open but not \((i, j)-\xi\)-open.

Proposition 3.4 Let \((X, \tau_1, \tau_2)\) be a bitopological space if \((X, \tau_1)\) is a semi-\(T_1\)-space , then \((i, j)-\xi O(X)= \tau_j(X)\).

Proof. Let \( A \) be any subset of a space \(X\) and \( A\) is \(j\)-open set, if \( A = \phi, \) then \( A \in (i, j)-\xi O(X)\), if \( A \neq \phi \), now let \( x \in A \), since \((X, \tau_1)\) is semi-\(T_1\)-space, then by Theorem 2.4 every singleton is \(i\)-semi-closed set , and hence \( x \in \{x\} \subseteq A \), therefore \( A \in (i, j)-\xi O(X)\), hence \( \tau_j(X)\subseteq (i, j)-\xi O(X)\) but \((i, j)-\xi O(X)\subseteq \tau_j(X)\) generally, thus \((i, j)-\xi O(X) = \tau_j(X)\).

Proposition 3.5 Let \((X, \tau_1, \tau_2)\) be a bitopological space and \( A \) be a subset the space \( X \). If \( A \in j-\delta O(X)\) and \( A \) is an \(i\)-closed set, then \( A \in (i, j)-\xi O(X)\)

Proof. If \( A = \phi\), then \( A \in (i, j)-\xi O(X)\), if \( A \neq \phi \), let \( x \in A \) since \( A \in j-\delta O(X)\) and \( j-\delta O(X) \subseteq \tau_j(X) \) in general so \( A \in \tau_j(X) \), and since \( A \) is \(i\)-closed so \( A \) is \(i\)-semi-closed and \( x \in A \subseteq \) A, and hence \( A \subseteq (i, j)-\xi O(X)\).

From Proposition 3.5 we obtain the following:

Corollary 3.6 Let \((X, \tau_1, \tau_2)\) be a bitopological space, if a subset \( A \) of \( X \) is \(i\)-regular closed and \( j\)-open then \( A \in (i, j)-\xi O(X)\)

Theorem 3.7 In a bitopological space \((X, \tau_1, \tau_2)\) if a space \((X, \tau_i)\) is locally indiscrete then \((i, j)-\xi O(X)\subseteq \tau_i\).

Proof. Let \( V \in (i, j)-\xi O(X)\), then \( V \in \tau_j(X) \) and for each \( x \in V \), there exist \(i\)-semi-closed \( F \) in \( X \) such that \( x \in F \subseteq V \), by Remark 2.3, \( F \) is \(i\)-open, it follows that \( V \in \tau_i \), and hence \((i, j)-\xi O(X)\subseteq \tau_i\).

The converse of Theorem 3.7, is not true in general, as shown in the following example:

Example 3.8 Let \( X = \{a, b, c\} \), \( \tau_1 = \{\phi, \{a\}, \{b, c\}, X\} \) and \( \tau_2=\{\phi, \{b, c\}, X\} \), then \((i, j)-\xi O(X) = \{\phi, \{b, c\}, X\} \) and it is clear that \((X, \tau_1)\) is locally indiscrete but \( \tau_1 \) is not a subset of \((i, j)-\xi O(X)\).

Theorem 3.9 Let \( X_1, X_2 \) be two bitopological space and \( X_1 \times X_2 \) be the bitopological product , let \( A_1 \in (i, j)-\xi O(X_1) \) and \( A_2 \in (i, j)-\xi O(X_2) \) then \((A_1 \times A_2) \subseteq (i, j)-\xi O(X_1 \times X_2)\)
Proof. Let \((x_1, x_2) \in A_1 \times A_2\) then \(x_1 \in A_1\) and \(x_2 \in A_2\), and since \(A_1 \in (i, j)\)-\(\xi O(X_1)\) and \(A_2 \in (i, j)\)-\(\xi O(X_2)\), there exist \(F_1 \in iSC(X_1)\) and \(F_2 \in iSC(X_2)\) such that \(x_1 \in F_1 \subseteq A_1\) and \(x_2 \in F_2 \subseteq A_2\). Therefore \((x_1, x_2) \in F_1 \times F_2 \subseteq A_1 \times A_2\), and since \(A_1 \in j\)-\(\xi O(X_1)\) and \(A_2 \in j\)-\(\xi O(X_2)\), then by Theorem 2.5 part (1) \(A_1 \times A_2 = j\)-\(\xi Int_{x_1} (A_1) \times j\)-\(\xi Int_{x_2} (A_2) = j\)-\(\xi Int_{x_1 \times x_2} (A_1 \times A_2)\), hence \(A_1 \times A_2 \in j\)-\(\xi O(X_1 \times X_2)\) and since \(F_1 \in iSC(X_1)\) and \(F_2 \in iSC(X_2)\) then by Theorem 2.5 part (2) we get \(F_1 \times F_2 = i-sCl_{x_1} (F_1) \times i-sCl_{x_2} (F_2) = i-sCl_{x_1 \times x_2} (F_1 \times F_2)\), hence \(F_1 \times F_2 \in iSC(X_1 \times X_2)\), therefore \(A_1 \times A_2 \in (i, j)\)-\(\xi O(X)\).

Theorem 3.10 For any bitopological space \((X, \tau_1, \tau_2)\), if \(A \in \tau_j(X)\) and either \(A \in i-\eta O(X)\) or \(A \in i-S\theta O(X)\), then \(A \in (i, j)-\xi O(X)\)

Proof. Let \(A \in i-\eta O(X)\) and \(A \in \tau_j(X)\), if \(A = \emptyset\), then \(A \in (i, j)-\xi O(X)\), if \(A \neq \emptyset\), since \(A \in i-\eta O(X)\), then \(A = \bigcup F_\alpha\), where \(F_\alpha \in i-\delta C(X)\) for each \(\alpha\), and since \(i-\delta C(X) \subseteq iSC(X)\), so \(F_\alpha \in iSC(X)\) for each \(\alpha\), and \(A \in \tau_j(X)\) so by Corollary 3.2 \(A \in (i, j)-\xi O(X)\).

On the other hand, suppose that \(A \in i-S\theta O(X)\) and \(A \in \tau_j(X)\), if \(A = \emptyset\), then \(A \in (i, j)-\xi O(X)\), if \(A \neq \emptyset\), since \(A \in i-S\theta O(X)\), then for each \(x \in A\), there exist \(i\)-semi-open set \(U\) such that \(x \in U \subseteq i-sCl(U) \subseteq A\), this implies that \(x \in i-sCl(U) \subseteq A\) and \(A \in \tau_j(X)\), therefore by Corollary 3.2 \(A \in (i, j)-\xi O(X)\).

Theorem 3.11 Let \(Y\) be a subspace of a bitopological space \((X, \tau_1, \tau_2)\), if \(A \in (i, j)-\xi O(X)\) and \(A \subseteq Y\), then \(A \in (i, j)-\xi O(Y)\)

Proof. Let \(A \in (i, j)-\xi O(X)\), then \(A \in \tau_j(X)\) and for each \(x \in A\), there exists \(i\)-semi-open set \(F\) in \(X\) such that \(x \in F \subseteq A\), since \(A \in \tau_j(X)\) and \(A \subseteq Y\), then by Theorem 2.6 \(A \in \tau_j(Y)\), and since \(F \in iSC(X)\) and \(F \subseteq Y\), then by Theorem 2.6 \(F \in iSC(Y)\), hence \(A \in (i, j)-\xi O(Y)\).

From the above theorem we obtain:

Corollary 3.12 Let \(X\) be a bitopological space, \(A\) and \(Y\) be two subsets of \(X\) such that \(A \subseteq Y \subseteq X\), \(Y \in RO(X, \tau_j)\), \(Y \in RO(X, \tau_i)\), then \(A \in (i, j)-\xi O(Y)\) if and only if \(A \in (i, j)-\xi O(X)\)

Proposition 3.13 Let \(Y\) be a subspace of a bitopological space \((X, \tau_1, \tau_2)\), if \(A \in (i, j)-\xi O(Y)\) and \(Y \in iSC(X)\), then for each \(x \in A\), there exists an \(i\)-semi-closed set \(F\) in \(X\) such that \(x \in F \subseteq A\).

Proof. Let \(A \in (i, j)-\xi O(Y)\), then \(A \in \tau_j(Y)\) and for each \(x \in A\) there exist an \(i\)-semi-closed set \(F\) in \(Y\) such that \(x \in F \subseteq A\), and since \(Y \in iSC(X)\) so by Theorem 2.6 \(F \in iSC(X)\), which completes the proof.

Proposition 3.14 Let \(A\) and \(Y\) be any subsets of a bitopological space \(X\), if \(A \in (i, j)-\xi O(X)\) and \(Y \in RO(X, \tau_j)\) and \(Y \in RO(X, \tau_i)\) then \(A \cap Y \in (i, j)-\xi O(X)\)
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Proof. Let $A \in (i, j)$-$\xi O(X)$, then $A \in \tau_j(X)$ and $A = \cup F_\alpha$, where $F_\alpha \in i$-$SC(X)$ for each $\alpha$, then $A \cap Y = \cup F_\alpha \cap Y = \cup(F_\alpha \cap Y)$, since $Y \in RO(X, \tau_j)$, then $Y$ is $j$-open, by Theorem 2.6 $A \cap Y \in \tau_j(X)$ and since $Y \in RO(X, \tau_i)$ then $Y \in i$-$SC(X)$ and hence $F_\alpha \cap Y \in i$-$SC(X)$, for each $\alpha$, therefore by Corollary 3.2, $A \cap Y \in (i, j)$-$\xi O(X)$.

**Proposition 3.15** Let $A$ and $Y$ be any subsets of a bitopological space $X$, if $A \in (i, j)$-$\xi O(X)$ and $Y$ is regular semi-open in $\tau_i$ and $\tau_j$, then $A \cap Y \in (i, j)$-$\xi O(Y)$

Proof. Let $A \in (i, j)$-$\xi O(X)$, then $A \in \tau_j(X)$ and $A = \cup F_\alpha$, where $F_\alpha \in i$-$SC(X)$ for each $\alpha$, then $A \cap Y = \cup F_\alpha \cap Y = \cup(F_\alpha \cap Y)$, since $Y \in RSO(X, \tau_j)$, then $Y \in j$-$SO(X)$ and by Theorem 2.6, $A \cap Y \in \tau_j(Y)$ and since $Y \in RSO(X, \tau_i)$ then $Y \in i$-$SC(X)$ and hence $F_\alpha \cap Y \in i$-$SC(X)$ for each $\alpha$, since $F_\alpha \cap Y \subseteq Y$ and $F_\alpha \cap Y \in i$-$SC(X)$ for each $\alpha$, then by Theorem 2.6, $F_\alpha \cap Y \in i$-$SC(Y)$ therefore by Corollary 3.2 $A \cap Y \in (i, j)$-$\xi O(Y)$.

**Proposition 3.16** If $Y$ is an $i$-open and $j$-open subspace of a bitopological space $X$ and $A \in (i, j)$-$\xi O(X)$, then $A \cap Y \in (i, j)$-$\xi O(Y)$

Proof. Let $A \in (i, j)$-$\xi O(X)$, then $A \in \tau_j(X)$ and $A = \cup F_\alpha$ where $F_\alpha \in i$-$SC(X)$ for each $\alpha$, then $A \cap Y = \cup F_\alpha \cap Y = \cup(F_\alpha \cap Y)$, since $Y$ is $j$-open subspace of $X$ then $Y \in j$-$SO(X)$ and hence by Theorem 2.6 $A \cap Y \in \tau_j(Y)$, and since $Y$ is an $i$-open subspace of $X$ then by Theorem 2.6 $F_\alpha \cap Y \in i$-$SC(Y)$ for each $\alpha$, then by Corollary 3.2 $A \cap Y \in (i, j)$-$\xi O(Y)$.

From the above proposition we obtain the following corollary:

**Corollary 3.17** If either $Y \in RSO(X, \tau_j)$ and $Y \in RSO(X, \tau_i)$ or $Y$ is an $i$-open and $j$-open subspace of a bitopological space $X$, and $A \in (i, j)$-$\xi O(X)$, then $A \cap Y \in (i, j)$-$\xi O(Y)$

The following result shows that any union of $(i, j)$-$\xi O(X)$ sets in bitopological space $(X, \tau_1, \tau_2)$ is $(i, j)$-$\xi O(X)$.

**Proposition 3.18** Let $\{A_\lambda : \lambda \in \Delta\}$ be family of $(i, j)$-$\xi$-open sets in bitopological space $(X, \tau_1, \tau_2)$, then $\cup \{A_\lambda : \lambda \in \Delta\}$ is an $(i, j)$-$\xi$-open set.

Proof. Let $\{A_\lambda : \lambda \in \Delta\}$ be family of $(i, j)$-$\xi$-open sets in bitopological space $(X, \tau_1, \tau_2)$. Since $A_\lambda$ is $j$-open for each $\lambda \in \Delta$ then $\cup \{A_\lambda : \lambda \in \Delta\}$ is $j$-open set in a space $X$.

Suppose that $x \in \cup A_\lambda$, this implies that there exist $\lambda_0 \in \Delta$ such that $x \in A_{\lambda_0}$ and since $A_{\lambda_0}$ is an $(i, j)$-$\xi$-open set, so there exists $i$-semi-closed set $F$ in $X$ such that $x \in F \subseteq A_{\lambda_0} \subseteq \cup A_\lambda$ for all $\lambda \in \Delta$. Therefore, $\cup \{A_\lambda : \lambda \in \Delta\}$ is an $(i, j)$-$\xi$-open set.

The following result shows that finite intersection of $(i, j)$-$\xi O(X)$ sets in bitopological space $(X, \tau_1, \tau_2)$ is $(i, j)$-$\xi O(X)$.
Proposition 3.19 Any finite intersection of \((i, j)\)-\(\xi\)-open sets in bitopological space \((X, \tau_1, \tau_2)\), is an \((i, j)\)-\(\xi\)-open set.

Proof. Let \(A_i\) be \((i, j)\)-\(\xi\)-open for \(i = 1, 2, \ldots, n\), in bitopological space \((X, \tau_1, \tau_2)\). Then \(\bigcap A_i\) is \(j\)-open in a space \(X\). Let \(x \in \bigcap A_i\), then \(x \in A_i\) for \(i = 1, 2, \ldots, n\), but \(A_i\) is \((i, j)\)-\(\xi\)-open, so there exists semi-closed \(F_i\) for each \(i = 1, 2, \ldots, n\), such that \(x \in F_i \subseteq A_i\). This implies that \(x \in \bigcap F_i \subseteq \bigcap A_i\). Therefore, \(\bigcap A_i\) is an \((i, j)\)-\(\xi\)-open set. Hence, the family \((i, j)\)-\(\xi\)-open subset of \((X, \tau_1, \tau_2)\) forms a bitopology on \(X\).

4 On \((i, j)\)-\(\xi\)- operators

Definition 4.1 A subset \(N\) of a bitopological space \((X, \tau_1, \tau_2)\) is called \((i, j)\)-\(\xi\)-neighbourhood of a subset \(A\) of \(X\) if there exists an \((i, j)\)-\(\xi\)-open set \(U\) such that \(A \subseteq U \subseteq N\). When \(A = \{x\}\), we say that \(N\) is \((i, j)\)-\(\xi\)-neighbourhood of \(x\).

Definition 4.2 A point \(x \in X\) is said to be an \((i, j)\)-\(\xi\)-interior point of \(A\) if there exists an \((i, j)\)-\(\xi\)-open set \(U\) containing \(x\) such that \(U \subseteq A\). The set of all \((i, j)\)-\(\xi\)-interior points of \(A\) is said to be \((i, j)\)-\(\xi\)-interior of \(A\) and it is denoted by \((i, j)\)-\(\xi\)-\text{Int}(\(A\))

Proposition 4.3 Let \(X\) be a bitopological space and \(A \subseteq X\), \(x \in X\), then \(x\) is \((i, j)\)-\(\xi\)-interior of \(A\) if and only if \(A\) is an \((i, j)\)-\(\xi\)-neighbourhood of \(x\).

Proposition 4.4 A subset \(G\) of a bitopological space \(X\) is \((i, j)\)-\(\xi\)-open if and only if it is an \((i, j)\)-\(\xi\)-neighbourhood of each of its points .

Proposition 4.5 Let \(A\) be any subset of a bitopological space \(X\). If a point \(x\) in the \((i, j)\)-\(\xi\)-\text{Int}(\(A\)), then there exists a \(i\)-semi-closed set \(F\) of \(X\) containing \(x\) and \(F \subseteq A\).

Proof. Suppose that \(x \in (i, j)-\xi\text{-Int}(A)\), then there exists an \((i, j)\)-\(\xi\)-open set \(U\) of \(X\) containing \(x\) such that \(x \in U \subseteq A\). Since \(U\) is an \((i, j)\)-\(\xi\)-open set, so there exists an \(i\)-semi-closed set \(F\) such that \(x \in F \subseteq U \subseteq A\). Hence, \(x \in F \subseteq A\).

Some properties of \((i, j)\)-\(\xi\)-interior operators on a set are given in the following:

Theorem 4.6 For any subsets \(A\) and \(B\) of a bitopological space \(X\), the following statements are true:

1. The \((i, j)\)-\(\xi\)-interior of \(A\) is the union of all \((i, j)\)-\(\xi\)-open sets contained in \(A\).
2. \((i, j)\)-\(\xi\)-Int\((A)\) is an \((i, j)\)-\(\xi\)-open set in \(X\) contained in \(A\).

3. \((i, j)\)-\(\xi\)-Int\((A)\) is the largest \((i, j)\)-\(\xi\)-open set in \(X\) contained in \(A\).

4. \(A\) is an \((i, j)\)-\(\xi\)-open set if and only if \(A = (i, j)\)-\(\xi\)-Int\((A)\)

5. \((i, j)\)-\(\xi\)-Int\((\phi) = \phi\).

6. \((i, j)\)-\(\xi\)-Int\((X) = X\)

7. \((i, j)\)-\(\xi\)-Int\((A) \subseteq A\).

8. If \(A \subseteq B\), the \((i, j)\)-\(\xi\)-Int\((A) \subseteq (i, j)\)-\(\xi\)-Int\((B)\).

9. \((i, j)\)-\(\xi\)-Int\((A) \cap (i, j)\)-\(\xi\)-Int\((B) = (i, j)\)-\(\xi\)-Int\((A \cap B)\).

10. \((i, j)\)-\(\xi\)-Int\((A) \cup (i, j)\)-\(\xi\)-Int\((B) \subseteq (i, j)\)-\(\xi\)-Int\((A \cup B)\).

**Proof.** Straightforward.

In general \((i, j)\)-\(\xi\)Int\((A) \cup (i, j)\)-\(\xi\)Int\((B) \neq (i, j)\)-\(\xi\)Int\((A \cup B)\) as it shown in the following example:

**Example 4.7** Let \(X = \{a, b, c\}\), \(\tau_1 = \{\phi, \{a\}, \{a, c\}, X\}\) and \(\tau_2 = \{\phi, \{b, c\}, X\}\), then \((i, j)\)-\(\xi\)O\((X) = \{\phi, \{b, c\}, X\}\) if we take \(A = \{a, b\}\) and \(B = \{b, c\}\), then \((i, j)\)-\(\xi\)Int\((A) = \phi\), and \((i, j)\)-\(\xi\)Int\((B) = \{b, c\}\), \((i, j)\)-\(\xi\)Int\((A \cup B) = (i, j)\)-\(\xi\)Int\((X) = X\).

In general \((i, j)\)-\(\xi\)Int\((A) \subseteq j\)-Int\((A)\), but \((i, j)\)-\(\xi\)Int\((A) \neq j\)-\(\xi\)Int\((A)\), which is shown in the following example:

**Example 4.8** Let \(X = \{a, b, c\}\), \(\tau_1 = \{\phi, \{a\}, \{a, c\}, X\}\) and \(\tau_2 = \{\phi, \{a\}, \{b, c\}, X\}\), then \((i, j)\)-\(\xi\)O\((X) = \{\phi, \{b, c\}, X\}\), if we take \(A = \{a\}\), then \((i, j)\)-\(\xi\)Int\((A) = \phi\), but \(j\)-\(\xi\)Int\((A) = A\). Hence \((i, j)\)-\(\xi\)Int\((A) \neq j\)-\(\xi\)Int\((A)\).

**Definition 4.9** The intersection of all \((i, j)\)-\(\xi\)-closed set containing \(F\) is called the \((i, j)\)-\(\xi\)-closure of \(F\) and we denoted it by \((i, j)\)-\(\xi\)Cl\((F)\)

**Corollary 4.10** Let \(F\) be any subset of a space \(X\). A point \(x \in X\) is in the \((i, j)\)-\(\xi\)-closed of \(F\) if and only if \(F \cap U \neq \phi\) for every \((i, j)\)-\(\xi\)-open set \(U\) containing \(x\).

**Proposition 4.11** Let \(A\) be any subset of a bitopological space \(X\). If a point \(x\) in the \((i, j)\)-\(\xi\)-closure of \(A\), then \(F \cap A \neq \phi\) for every \(i\)-semi-closed set \(F\) of \(X\) containing \(x\).
Proof. Suppose that \( x \in (i, j)\xi\text{-}cl(A) \), then by Corollary 4.10, \( A \cap U \neq \phi \) for every \((i, j)\xi\)-open set \( U \) of \( X \) containing \( x \). Since \( U \) is an \((i, j)\xi\)-open set, so there exists an \( i \)-semi-closed set \( F \) containing \( x \), such that \( F \subseteq U \). Hence, \( F \cap A \neq \phi \).

Some properties of \((i, j)\xi\)-closure operators on a set are given.

**Theorem 4.12** For any subsets \( A \) and \( B \) of a bitopological space \( X \), the following statements are true:

1. The \((i, j)\xi\)-closure of \( A \) is the intersection of all \((i, j)\xi\)-closed sets containing \( A \).
2. \((i, j)\xi\text{-}cl(A) \) is an \((i, j)\xi\)-closed set in \( X \) containing \( A \).
3. \((i, j)\xi\text{-}cl(A) \) is the smallest \((i, j)\xi\)-closed set in \( X \) containing \( A \).
4. \( A \) is an \((i, j)\xi\)-closed set if and only if \( A = (i, j)\xi\text{-}cl(A) \).
5. \((i, j)\xi\text{-}cl(\phi) = \phi \).
6. \((i, j)\xi\text{-}cl(X) = X \).
7. \( A \subseteq (i, j)\xi\text{-}cl(A) \).
8. If \( A \subseteq B \), then \((i, j)\xi\text{-}cl(A) \subseteq (i, j)\xi\text{-}cl(B) \).
9. \((i, j)\xi\text{-}cl(A) \cap (i, j)\xi\text{-}cl(B) \subseteq (i, j)\xi\text{-}cl(A \cap B) \).
10. \((i, j)\xi\text{-}cl(A) \cup (i, j)\xi\text{-}cl(B) = (i, j)\xi\text{-}Int(A \cup B) \).

**Proof.** Directly from Definition 4.9.

**Corollary 4.13** For any subset \( A \) of a bitopological space \( X \), then the following statements are true:

1. \( X \setminus ((i, j)\xi\text{Cl}(A)) = (i, j)\xi\text{Int}(X \setminus A) \)
2. \( X \setminus ((i, j)\xi\text{Int}(A)) = (i, j)\xi\text{Cl}(X \setminus A) \)
3. \((i, j)\xi\text{Int}(A) = X \setminus ((i, j)\xi\text{Cl}(X \setminus A)) \)

It is clear that \( j\text{Cl}(F) \subseteq (i, j)\xi\text{Cl}(F) \), the converse may be false as shown in the following example:

**Example 4.14** Considering a space \( X \) as defined in Example 3.3, if we take \( F = \{a, b\} \), then \( j\text{Cl}(F) = \{a, b\} \), and \((i, j)\xi\text{Cl}(F) = X \), this shows that \((i, j)\xi\text{Cl}(F) \) is not a subset of \( j\text{Cl}(F) \).
Corollary 4.15 If A is any subset of a bitopological space X, then \((i,j)\)-\(\xi\)\(\text{Int}(A) \subseteq j-\text{Int}(A) \subseteq A \subseteq j-\text{Cl}(A) \subseteq (i,j)-\xi\text{Cl}(A)\).

Definition 4.16 Let A be a subset of a bitopological space X. A point \(x \in X\) is said to be \((i,j)\)-\(\xi\)-limit point of A if for each \((i,j)\)-\(\xi\)-open set \(U\) containing \(x, U \cap (A \setminus \{x\}) \neq \emptyset\). The set of all \((i,j)\)-\(\xi\)-limit point of A is called \((i,j)\)-\(\xi\)-derived set of A and is denoted by \((i,j)-\xi D(A)\).

In general, it is clear that \((i,j)-\xi D(A) \subseteq j-D(A)\), but the converse may not be true as shown in the following example:

Example 4.17 Considering the space X as defined in Example 3.3 if we take \(A = \{a,c\}\), So \((i,j)-\xi D(A) = \{a\}\) and \(j-D(A) = \{b\}\), hence \((i,j)-\xi D(A)\) is not a subset of \(j-D(A)\).

Theorem 4.18 Let X be a bitopological space and A be a subset of X, then \(A \cup (i,j)-\xi D(A)\) is \((i,j)-\xi\)-closed.

Proof. Let \(x \notin A \cup (i,j)-\xi D(A)\). This implies that \(x \notin A\) and \(x \notin (i,j)-\xi D(A)\). Since \(x \notin (i,j)-\xi D(A)\), then there exists an \((i,j)-\xi\)-open \(U\) of \(X\) which contains no point of \(A\) other than \(x\), but \(x \notin A\), so \(U\) contains no point of \(A\), which implies that \(U \subseteq X \setminus A\). Again, \(U\) is an \((i,j)-\xi\)-open set for each of its points. But as \(U\) does not contain any point of \(A\), no point of \(U\) can be \((i,j)-\xi\)-limit point of \(A\). Therefore, no point of \(U\) can belong to \((i,j)-\xi D(A)\). This implies that \(U \subseteq X \setminus (i,j)-\xi DA\). Hence, it follows that \(x \in X \setminus A \cap (X \setminus (i,j)-\xi D(A)) = X \setminus (A \cup (i,j)-\xi D(A))\). Therefore \(A \cup (i,j)-\xi D(A)\) is an \((i,j)-\xi\)-closed. Hence \((i,j)-\xi d(A) \subseteq A \cup (i,j)-\xi D(A)\).

Corollary 4.19 If a subset A of a bitopological space X is \((i,j)-\xi\)-closed, then A contains the set of all of its \((i,j)-\xi\)-limit points.

Theorem 4.20 Let A be any subset of a bitopological space X, then the following statements are true:

1. \(((i,j)-\xi D((i,j)-\xi D(A))) \setminus A \subseteq (i,j)-\xi D(A)\)
2. \((i,j)-\xi D(A \cup (i,j)-\xi D(A)) \subseteq A \cup (i,j)-\xi D(A)\)

Proof. Obvious.

Theorem 4.21 Let X be a bitopological space and A be a subset of X, then: \((i,j)-\xi \text{Int}(A) = A \setminus ((i,j)-\xi D(X \setminus A))\)

Proof. Obvious.
**Definition 4.22** If \( A \) is a subset of a bitopological space \( X \), then \((i, j)\)-\(\xi\)-boundary of \( A \) is \((i, j)\)-\(\xi Cl(A) \cap ((i, j)\)-\(\xi Int(A)) \), and denoted by \((i, j)\)-\(\xi Bd(A)\)

**Theorem 4.23** For any subset \( A \) of a bitopological space \( X \), the following statements are true:

1. \((i, j)\)-\(\xi Bd(A) = (i, j)\)-\(\xi Bd(X \setminus A)\)
2. \( A \in (i, j)\)-\(\xi O(X) \) if and only if \((i, j)\)-\(\xi Bd(A) \subseteq X \setminus A \), that is \( A \cap (i, j)\)-\(\xi Bd(A) = \emptyset \).
3. \( A \in (i, j)\)-\(\xi C(X) \) if and only if \((i, j)\)-\(\xi Bd(A) \subseteq A \).
4. \((i, j)\)-\(\xi Bd((i, j)\)-\(\xi Bd(A)) \subseteq (i, j)\)-\(\xi Bd(A)\)
5. \((i, j)\)-\(\xi Bd((i, j)\)-\(\xi Int(A)) \subseteq (i, j)\)-\(\xi Bd(A)\)
6. \((i, j)\)-\(\xi Bd((i, j)\)-\(\xi Cl(A)) \subseteq (i, j)\)-\(\xi Bd(A)\)
7. \((i, j)\)-\(\xi Int(A) = A \setminus ((i, j)\)-\(\xi Bd(A))\)

**Proof.** Directly from Definition 4.22.

**Theorem 4.24** Let \( A \) be a subset of a bitopological space \( X \), then \((i, j)\)-\(\xi Bd(A) = \emptyset \) if and only if \( A \) is both \((i, j)\)-\(\xi\)-open and \((i, j)\)-\(\xi\)-closed set.

**Proof.** Let \( A \) be \((i, j)\)-\(\xi\)-open and \((i, j)\)-\(\xi\)-closed, then \( A = (i, j)\)-\(\xi Int(A) = (i, j)\)-\(\xi cl(A) \), hence by Definition 4.22 \( A = (i, j)\)-\(\xi Cl(A)-((i, j)\)-\(\xi Int(A)) = \emptyset \).

**References**


(i,j)-\(\xi\)-Open sets in bitopological spaces


