Gen. Math. Notes, Vol. 26, No. 1, January 2015, pp.61-73
ISSN 2219-7184; Copyright ©ICSRS Publication, 2015
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Hyers-Ulam-Rassias Stability of Orthogonal Quadratic Functional Equation in Modular Spaces

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(Received: 30-8-14 / Accepted: 28-11-14)

Abstract

In this paper, we study the Hyers-Ulam-Rassias stability of the quadratic functional equation $f(x+y)+f(x-y)=2f(x)+2f(y)$, $x \perp y$ *in which* $\perp$ *is orthogonality in the sens of Rätz in modular spaces.*

Keywords: *Hyers-Ulam-Rassias stability, Orthogonality, Orthogonally quadratic equation, Modular space.*

1 Introduction

The stability problem of functional equations has been originally raised by S. M. Ulam. In 1940, he posed the following problem: Give conditions in order for a linear mapping near an approximately additive mapping to exist (see [27]).

In 1941, this problem was solved by D. H. Hyers [7] for the first time. Subsequently, the result of Hyers was generalized by T. Aoki [2] for additive mappings and Th. M. Rassias [20] for linear mappings by considering an unbounded Cauchy difference. The paper of Th. M. Rassias [20] has provided a lot of influences in the development of the Hyers-Ulam-Rassias stability of functional equations (see [16]). During the last decades several stability problems of functional equations have been investigated by a number of mathematicians in various spaces, such as fuzzy normed spaces, orthogonal normed spaces and random normed spaces; see [3, 5, 8, 9, 15, 22, 30] and reference therein. Recently, Gh. Sadeghi [23] proved the Hyers-Ulam stability of the generalized Jensen functional equation $f(rx + sy) = rg(x) + sh(x)$ in modular space, using the fixed point method. The theory of modulars on linear spaces and the corresponding theory of modular linear spaces were founded by H. Nakano [18] and were intensively developed by his mathematical school: S. Koshi, T. Shimogaki, S. Yamamuro [10, 29] and others. Further and the most complete development of these theories are due to W. Orlicz, S. Mazur, J. Musielak, W. A. Luxemburg, Ph. Turpin [12, 14, 17, 26] and their collaborators. In the present time the theory of modulars and modular spaces is extensively applied, in particular, in the study of various W. Orlicz spaces [19] and interpolation theory [11], which in their turn have broad applications [13, 17]. The importance for applications consists in the richness of the structure of modular spaces, that-besides being Banach spaces (or $F-$spaces in more general setting)- are equipped with modular equivalent of norm or metric notions.

There are several orthogonality notions on a real normed spaces as Birkhoff-James, semi-inner product, Carlsson, Singer, Roberts, Pythagorean, isosceles and Diminnie (see, e.g., [1]). Let us recall the orthogonality space in the sense of Rätz; cf. [21].

Suppose E is a real vector space with $\dim E \geq 2$ and $\perp$ is a binary relation on E with the following properties:

(O1) totality of $\perp$ for zero: $x \perp 0$, $0 \perp x$ for all $x \in E$;

(O2) independence: if $x, y \in E - \{0\}$, $x \perp y$, then, x, y are linearly independent;

(O3) homogeneity: if $x, y \in E$, $x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;

(O4) the Thalesian property: if P is a 2-dimensional subspace of E. If $x \in P$ and $\lambda \in \mathbb{R}^+$, then there exists $y_0 \in P$ such that $x \perp y_0$ and $x + y_0 \perp \lambda x - y_0$.

The pair $(E, \perp)$ is called an orthogonality space. By an orthogonality normed space, we mean an orthogonality space having a normed structure. Some interesting examples of orthogonality spaces are:

(i) The trivial orthogonality on a vector space E defined by (O1), and for nonzero elements $x, y \in E$, $x \perp y$ if and only if x, y are linearly independent.

(ii) The ordinary orthogonality on an inner product space $(E, \langle . \rangle)$ given by $x \perp y$ if and only if $\langle x, y \rangle = 0$.

(iii) The Birkhoff-James orthogonality on a normed space $(E, \|.\|)$ defined by $x \perp y$ if and only if $\|x\| \leq \|x + \lambda y\|$ for all $\lambda \in \mathbb{R}$.

The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all $x, y \in E$.

Clearly examples (i) and (ii) are symmetric but example (iii) is not. However, it is remarkable to note, that a real normed space of dimension greater than or equal to 3 is an inner product space if and only if the Birkhoff-James orthogonality is symmetric.

Let $(E, \perp)$ be an orthogonality space and $(G, +)$ be an Abelian group. A mapping $f : E \to G$ is said to be (orthogonally) quadratic if it satisfies

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \quad x \perp y \tag{1}$$

for all $x, y \in E$. The orthogonally quadratic functional equation (1), was first investigated by Vajzović [28] when E is a Hilbert space, G is equal to $\mathbb{C}$, f is continuous and $\perp$ means the Hilbert space orthogonality. Later Drljević, Fochi and Szabó generalized this result [4, 6, 25].
J. Sikorska [24] obtained the generalized orthogonal stability of some functional equations.

In the present paper, we establish the Hyers-Ulam-Rassias Stability of Orthogonal Quadratic Functional Equation (1) in Modular spaces. Therefore, we generalized the main of theorem 5 of [24].

2 Preliminary

In this section, we give the definitions that are important in the following.

Definition 2.1. *Let X be an arbitrary vector space.*
(a) A functional $\rho : X \to [0, \infty]$ is called a modular if for arbitrary $x, y \in X$,
(i) $\rho(x) = 0$ if and only if $x = 0$,
(ii) $\rho(\alpha x) = \rho(x)$ for every scaler α with $|\alpha| = 1$,
(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if and only if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$,
(b) if (iii) is replaced by
(iii)' $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ if and only if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$,
then we say that ρ is a convex modular.

A modular ρ defines a corresponding modular space, i.e., the vector space X_ρ given by

$$X_\rho = \{x \in X : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}.$$

Let ρ be a convex modular, the modular space X_ρ can be equipped with a norm called the Luxemburg norm, defined by

$$\|x\|_\rho = \inf \left\{ \lambda > 0 : \rho(\frac{x}{\lambda}) \leq 1 \right\}.$$

A function modular is said to satisfy the Δ_2−condition if there exists $k > 0$ such that $\rho(2x) \leq k\rho(x)$ for all $x \in X_\rho$.

Definition 2.2. *Let $\{x_n\}$ and x be in X_ρ. Then*

(i) we say $\{x_n\}$ is $\rho-$convergent to x and write $x_n \xrightarrow{\rho} x$ if and only if $\rho(x_n - x) \to 0$ as $n \to \infty$,

(ii) the sequence $\{x_n\}$, with $x_n \in X_\rho$, is called $\rho-$Cauchy if $\rho(x_n - x_m) \to 0$ as $m, n \to \infty$,

(iii) a subset S of X_ρ is called $\rho-$complete if and only if any $\rho-$Cauchy sequence is $\rho-$convergent to an element of S.

The modular ρ has the Fatou property if and only if $\rho(x) \leq \lim_{n\to\infty} \inf \rho(x_n)$ whenever the sequence $\{x_n\}$ is $\rho-$convergent to x. For further details and proofs, we refer the reader to [17].

Remark 2.3. *If $x \in X_\rho$ then $\rho(ax)$ is a nondecreasing function of $a \geq 0$. Suppose that $0 < a < b$, then property (iii) of definition 2.1 with $y = 0$ shows that*

$$\rho(ax) = \rho(\frac{a}{b}bx) \leq \rho(bx).$$

Moreover, if ρ is convex modular on X and $|\alpha| \leq 1$ then, $\rho(\alpha x) \leq |\alpha|\rho(x)$ and also $\rho(x) \leq \frac{1}{2}\rho(2x) \leq \frac{k}{2}\rho(x)$ if ρ satisfy the Δ_2- condition for all $x \in X$.

Throughout this paper, $\mathbb{N}$ and $\mathbb{R}$ denote the sets of all positive integers and all real numbers, respectively. By the notation E_p we mean $E \setminus \{0\}$ provided that $p < 0$ and E otherwise. In order to avoid some definitional problems we also assume for the sake of this paper that $0^0 := 1$.

3 Orthogonal Stability of Eq (1) in Modular Spaces

In this section we assume that the convex modular ρ has the Fatou property such that satisfies the Δ_2-condition with $0 < k \leq 2$. In addition, we assume that $(E_p, \perp)$ denotes an orthogonality space, on the other hand, we give the Hyers-Ulam-Rassias stability of orthogonal quadratic functional equation in modular spaces.

Theorem 3.1. *Let $(E_p, \|.\|)$ with $\dim E_p \geq 2$ be a real normed linear space with Birkhoff-James orthogonality and X_ρ is $\rho-$complete modular space. If a function $f : E_p \to X_\rho$ satisfies*

$$\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \leq \epsilon(\|x\|^p + \|y\|^p), \tag{2}$$

for all $x, y \in E_p$ with $x \perp y$, $\epsilon \geq 0$ and $p < 2$, then there exist unique quadratic mapping $Q : E_p \to X_\rho$ such that

$$\rho(f(x) - Q(x)) \leq \begin{cases} \frac{\beta^+}{4-2^p}\|x\|^p & \text{if } 0 \leq p < 2, \\ \frac{\beta^-}{4-2^p}\|x\|^p & \text{if } p < 0, \end{cases} \tag{3}$$

for all $x \in E_p$*, where* $\beta^+ = \frac{k\alpha^+}{8}(2 + k + k.3^p)$*,* $\beta^- = \frac{k\alpha^-}{8}(2 + k + k.2^{-p})$*,* $\alpha^+ = \frac{k\epsilon}{2}(2^p + 2^{2p} + k + k.3^p)$ *and* $\alpha^- = \frac{k\epsilon}{2}(2 + k + k.2^{-p})$*.*

Proof. Fix $x \in E_p$ and choose $y_0, z_0 \in E_p$ such that $x \perp y_0$, $x \perp z_0$ and $y_0 \perp z_0$. Then as well whence $x + y_0 \perp x - y_0$ and by (2) we get

$$\rho(f(2x) + f(2y_0) - 2f(x + y_0) - 2f(x - y_0)) \leq \epsilon(\|x + y_0\|^p + \|x - y_0\|^p). \quad (4)$$

Then, from (2) and (4) we have

$$\begin{aligned}
&\rho(f(2x) + f(2y_0) - 4f(x) - 4f(y_0)) = \rho(f(2x) + f(2y_0) - 2f(x + y_0) \\
&-2f(x - y_0) + 2f(x + y_0) + 2f(x - y_0) - 4f(x) - 4f(y_0)) \\
&\leq \frac{k}{2}\rho(f(2x) + f(2y_0) - 2f(x + y_0) - 2f(x - y_0)) \\
&+\frac{k^2}{2}\rho(f(x + y_0) + f(x - y_0) - 2f(x) - 2f(y_0)) \\
&\leq \frac{k\epsilon}{2}\left\{\|x + y_0\|^p + \|x - y_0\|^p + k(\|x\|^p + \|y_0\|^p)\right\}. \qquad (5)
\end{aligned}$$

From the definition of the orthogonality, since $x \perp y_0$, we derive $\|x\| \leq \|x + y_0\|$ and $\|x\| \leq \|x - y_0\|$ (for $\lambda = 1$ and $\lambda = -1$, respectively), and analogously, from $x + y_0 \perp x - y_0$ we derive $\|x + y_0\| \leq 2\|x\|$ and $\|x + y_0\| \leq 2\|y_0\|$. From this relation and the triangle inequality we have additionally $\|y_0\| = \|y_0 + x - x\| \leq \|x + y_0\| + \|x\| \leq 3\|x\|$, $\|x - y_0\| \leq \|y_0\| + \|x\| \leq 4\|x\|$ and $\|x\| \leq \|x + y_0\| \leq 2\|y_0\|$

In case p is a non-negative real number, we have the approximation

$$\|x + y_0\|^p \leq 2^p\|x\|^p, \ \|x - y_0\|^p \leq 4^p\|x\|^p \ \text{ and } \|y_0\|^p \leq 3^p\|x\|^p$$

otherwise

$$\|y_0\|^p \leq 2^{-p}\|x\|^p, \ \|x - y_0\|^p \leq \|x\|^p \ \text{ and } \|x + y_0\|^p \leq \|x\|^p$$

Case 1: if $p < 0$ then (5) become

$$\rho(f(2x) + f(2y_0) - 4f(x) - 4f(y_0)) \leq \alpha^-\|x\|^p \qquad (6)$$

where $\alpha^- = \frac{k\epsilon}{2}(2 + k + k.2^{-p})$.
In the same way, from the conditions $x + z_0 \perp x - z_0$ and $y_0 + z_0 \perp y_0 - z_0$ we obtain

$$\rho(f(2x) + f(2z_0) - 4f(x) - 4f(z_0)) \leq \alpha^-\|x\|^p \qquad (7)$$

and

$$\rho(f(2y_0)+f(2z_0)-4f(y_0)-4f(z_0)) \le \alpha^- \|y_0\|^p \le 2^{-p}\alpha^- \|x\|^p . \qquad (8)$$

From (6), (7) and (8) we get

$$\rho(2f(2x)-8f(x)) = \rho(f(2x)+f(2y_0)-4f(x)-4f(y_0)+f(2x)+f(2z_0)-$$
$$4f(x)-4f(z_0)+4f(y_0)+4f(z_0)-f(2y_0)-f(2z_0))$$
$$\le \frac{k}{2}\rho(f(2x)+f(2y_0)-4f(x)-4f(y_0)$$
$$+\frac{k}{2}\rho(f(2x)+f(2z_0)-4f(x)-4f(z_0)+4f(y_0)+4f(z_0)-f(2y_0)-f(2z_0))$$
$$\le \frac{k\alpha^-}{2}\|x\|^p+\frac{k^2}{4}(\alpha^-\|x\|^p+2^{-p}\alpha^-\|x\|^p) \le \frac{k\alpha^-}{4}(2+k+2^{-p}.k)\|x\|^p .$$

Hence

$$\begin{aligned}\rho(f(2x)-4f(x)) &= \rho(\frac{1}{2}(2f(2x)-8f(x)) \le \frac{1}{2}\rho(2f(2x)-8f(x))\\ &\le \frac{k.\alpha^-}{8}(2+k+2^{-p}k)\|x\|^p\\ &\le \beta^-\|x\|^p , \end{aligned} \qquad (9)$$

for all $x \in E_p$, where $\beta^- = \frac{k\alpha^-}{8}(2+k+k.2^{-p})$. Thus

$$\rho(\frac{f(2x)}{4}-f(x)) = \rho(\frac{1}{4}(f(2x)-4f(x))) \le \frac{1}{4}\beta^-\|x\|^p , \qquad (10)$$

Replacing x by $2x$ in (9) we get

$$\rho(f(4x)-4f(2x)) \le \beta^-\|2x\|^p , \qquad (11)$$

for all $x \in E_p$. By (11) and (9) we have

$$\begin{aligned}\rho(\frac{f(2^2x)}{4}-4f(x)) &= \rho(\frac{f(2^2x)}{4}-f(2x)+f(2x)-4f(x))\\ &\le \frac{1}{2}\rho(\frac{f(2^2x)}{2}-2f(2x))+\frac{k}{2}\rho(f(2x)-4f(x))\\ &\le \frac{1}{4}\rho(f(2^2x)-4f(2x))+\frac{k^2}{4}\rho(f(2x)-4f(x))\\ &\le \frac{\beta^-}{4}\|2x\|^p+\frac{k^2.\beta^-}{4}\|x\|^p \le \beta^-(\frac{1}{4}\|2x\|^p+\frac{k^2}{4}\|x\|^p).\end{aligned}$$

Thus

$$\begin{aligned}\rho(\frac{f(2^2x)}{4^2}-f(x)) &= \rho(\frac{1}{4}(\frac{f(2^2x)}{4}-4f(x)))\\ &\le \beta^-(\frac{1}{4^2}\|2x\|^p+\frac{k^2}{4^2}\|x\|^p).\end{aligned} \qquad (12)$$

By mathematical induction, we can easily see that

$$\rho(\frac{f(2^n x)}{4^n} - f(x)) \leq \frac{\beta^-}{4^n} \sum_{i=1}^{n} k^{2(n-i)} \left\|2^{i-1}x\right\|^p \tag{13}$$

for all $x \in E_p$. Indeed, for $n = 1$ the relation (13) is true. Assume that the relation (13) is true for n, and we show this relation rest true for $n+1$, thus we have

$$\begin{aligned}
\rho(\frac{f(2^{n+1}x)}{4^{n+1}} - f(x)) &\leq \frac{1}{4}\rho(\frac{f(2^{n+1}x)}{4^n} - 4f(x)) \\
&= \frac{1}{4}\rho(\frac{f(2^{n+1}x)}{4^n} - f(2x) + f(2x) - 4f(x)) \\
&\leq \frac{k}{8}\left[\rho(\frac{f(2^{n+1}x)}{4^n} - f(2x)) + \rho(f(2x) - 4f(x))\right] \\
&\leq \frac{k\beta^-}{8}\left[\frac{1}{4^n}\sum_{i=1}^{n} k^{2(n-i)} \left\|2^i x\right\|^p + \|x\|^p\right] \\
&= \frac{k\beta^-}{8}\frac{1}{4^n}\sum_{i=0}^{n} k^{2(n-i)} \left\|2^i x\right\|^p \\
&\leq \frac{k\beta^-}{2}\frac{1}{4^{n+1}}\sum_{i=0}^{n} k^{2(n-i)} \left\|2^i x\right\|^p \\
&\leq \frac{\beta^-}{4^{n+1}}\sum_{i=1}^{n+1} k^{2(n+1-i)} \left\|2^{i-1} x\right\|^p,
\end{aligned}$$

hence the relation (13) is true for all $x \in E_p$ and $n \in \mathbb{N}^*$. Thus

$$\begin{aligned}
\rho(\frac{f(2^n x)}{4^n} - f(x)) &\leq \frac{\beta^-}{4^n}\sum_{i=1}^{n} k^{2(n-i)} \left\|2^{i-1}x\right\|^p \\
&\leq \beta^- \sum_{i=1}^{n} 2^{-2i} \left\|2^{i-1}x\right\|^p \\
&= \beta^- \frac{1 - 2^{n(p-2)}}{4 - 2^p} \|x\|^p
\end{aligned} \tag{14}$$

for all $x \in E_p$. Replacing x by $2^m x$ (with $m \in \mathbb{N}^*$) in (14) we obtain

$$\rho(\frac{f(2^{m+n}x)}{4^n} - f(2^m x)) \leq \frac{\beta^- 2^{mp}}{4 - 2^p}(1 - 2^{n(p-2)}) \|x\|^p \tag{15}$$

for all $x \in E_p$. Whence

$$\begin{aligned}
\rho(\frac{f(2^{m+n}x)}{4^{n+m}} - \frac{f(2^m x)}{4^m}) &= \rho(\frac{1}{4^m}(\frac{f(2^{m+n}x)}{4^{n+m}} - \frac{f(2^m x)}{4^m})) \\
&\leq \frac{\beta^- 2^{m(p-2)}}{4 - 2^p}(1 - 2^{n(p-2)}) \|x\|^p
\end{aligned} \tag{16}$$

for all $x \in E_p$. If $m, n \to \infty$ we get, the sequence $\left\{\frac{f(2^n x)}{4^n}\right\}$ is $\rho-$Cauchy sequence in the $\rho-$complete modular space X_ρ. Hence $\left\{\frac{f(2^n x)}{4^n}\right\}$ is $\rho-$convergent in X_ρ, and we well define the mapping $Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$ from E_p into X_ρ satisfying

$$\rho(f(x) - Q(x)) \leq \frac{\beta^- \|x\|^p}{4 - 2^p}, \tag{17}$$

for all $x \in E_p$, since ρ has Fatou property. For all $x, y \in E_p$ with $x \perp y$, by applying (2) and (O3) we get

$$\rho(4^{-n}(f(2^n(x+y)) + f(2^n(x-y)) - 2f(2^n x) - 2f(2^n y))) \leq \epsilon 2^{n(p-2)}(\|x\|^p + \|y\|^p). \tag{18}$$

If $n \to \infty$ then, we conclude that

$$Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y) = 0, \; x \perp y$$

for all $x, y \in E_p$ and on account of the results by F. Vajzović [28] and M. Fochi [6], Q is quadratic. To prove the uniqueness, assume $Q' : E_p \to X_\rho$ to be another quadratic mapping satisfying (17). Then, for each $x \in E_p$ and all $n \in \mathbb{N}$ one has

$$\begin{aligned}
\rho(Q(x) - Q'(x)) &= \rho(\frac{1}{n^2}(Q(nx) - Q'(nx))) \leq \frac{1}{n^2}\rho(Q(nx) - Q'(nx)) \\
&= \frac{1}{n^2}\rho(Q(nx) - f(nx) + f(nx) - Q'(nx)) \\
&\leq \frac{k}{n^2}[\rho(Q(nx) - f(nx)) + \rho(f(nx) - Q'(nx))] \\
&\leq \frac{kn^{p-2}}{4 - 2^p}\|x\|^p.
\end{aligned}$$

If $n \to \infty$ we obtain $Q = Q'$.
Case 2: if $0 \leq p < 2$ then (5) become

$$\rho(f(2x) + f(2y_0) - 4f(x) - 4f(y_0)) \leq \alpha^+ \|x\|^p \tag{19}$$

where $\alpha^+ = \frac{k\epsilon}{2}(2^p + 4^p + k + k.3^p)$, and by the case 1 we have

$$\rho(f(2x) - 4f(x)) \leq \beta^+ \|x\|^p, \tag{20}$$

for all $x \in E$, where $\beta^+ = \frac{k\alpha^+}{8}(2 + k + k.3^p)$. The rest of the proof is similar to the proof of the first case, just the constants β^+ and α^+ serve as β^- and α^-, respectively. This completes the proof of theorem. □

In the following theorem we take the integers in the set $2^{\mathbb{N}} := \{2^m : m \in \mathbb{N}\}$.

Theorem 3.2. *Let $(E, \|.\|)$ with $\dim E \geq 2$ be a real normed linear space with Birkhoff-James orthogonality and X_ρ is $\rho-$complete modular space. If a function $f : E \to X_\rho$ satisfying*

$$\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \leq \epsilon(\|x\|^p + \|y\|^p), \tag{21}$$

for all $x, y \in E$ with $x \perp y$, $\epsilon \geq 0$ and $p > 2$, then there exist unique quadratic mapping $Q : E \to X_\rho$ such that

$$\rho(f(x) - Q(x)) \leq \frac{\beta^+}{2^p - 4} \|x\|^p, \tag{22}$$

for all $x \in E$, where $\beta^+ = \frac{k\alpha^+}{8}(2 + k + k.3^p)$ and $\alpha^+ = \frac{k\epsilon}{2}(2^p + 2^{2p} + k + k.3^p)$.

Proof. Using Theorem 3.1, the case $0 \leq p < 2$ we have

$$\rho(f(2x) - 4f(x)) \leq \beta^+ \|x\|^p, \tag{23}$$

for all $x \in E$, where $\beta^+ = \frac{k\alpha^+}{8}(2 + k + k.3^p)$ and $\alpha^+ = \frac{k\epsilon}{2}(2^p + 2^{2p} + k + k.3^p)$. Replacing x by $\frac{x}{2}$ in (23) we get

$$\rho(f(x) - 4f(\frac{x}{2})) \leq \beta^+ \left\|\frac{x}{2}\right\|^p. \tag{24}$$

Replacing x by $\frac{x}{2}$ in (24) we obtain

$$\rho(f(\frac{x}{2}) - 4f(\frac{x}{2^2})) \leq \beta^+ \left\|\frac{x}{2^2}\right\|^p. \tag{25}$$

From (24) and (25) we get

$$\begin{aligned} \rho(f(x) - 4^2 f(\frac{x}{2^2})) &= \rho(f(x) - 4f(\frac{x}{2}) + 4f(\frac{x}{2}) - 4^2 f(\frac{x}{2^2})) \\ &\leq \frac{k}{2}\rho(f(x) - 4f(\frac{x}{2})) + \frac{k}{2}\rho(4f(\frac{x}{2}) - 4^2 f(\frac{x}{2^2})) \\ &\leq \frac{k^2}{4}\rho(f(x) - 4f(\frac{x}{2})) + \frac{k^3}{2}\rho(f(\frac{x}{2}) - 4f(\frac{x}{2^2})) \\ &\leq \frac{k^2}{4}\beta^+ \left\|\frac{x}{2}\right\|^p + \frac{k^4}{4}\beta^+ \left\|\frac{x}{2^2}\right\|^p \\ &= \frac{\beta^+}{4}(k^2 \left\|\frac{x}{2}\right\|^p + k^4 \left\|\frac{x}{2^2}\right\|^p) \end{aligned} \tag{26}$$

for all $x \in E$. By mathematical induction, we can easily see that

$$\rho(f(x) - 4^n f(\frac{x}{2^n})) \leq \frac{\beta^+}{4} \sum_{i=1}^{n} k^{2i} \left\|\frac{x}{2^i}\right\|^p. \tag{27}$$

Whence

$$\begin{aligned}\rho(f(x)-4^n f(\frac{x}{2^n})) &\leq \frac{\beta^+}{4}\sum_{i=1}^{n} k^{2i}\left\|\frac{x}{2^i}\right\|^p \leq \frac{\beta^+}{4}\sum_{i=1}^{n} 2^{i(2-p)}\|x\|^p \\ &= \frac{\beta^+}{2^p-4}(1-2^{n(2-p)})\|x\|^p \qquad (28)\end{aligned}$$

Same as the first case in the theorem 3.1, we find, for each $x \in E$ the sequence $\left\{4^n f(\frac{x}{2^n})\right\}$ is $\rho-$Cauchy sequence in $\rho-$complete modular space X_ρ. Hence $\left\{4^n f(\frac{x}{2^n})\right\}$ is $\rho-$convergent in X_ρ and we well define the mapping $Q(x) = \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$ from E into X_ρ satisfying

$$\rho(f(x)-Q(x)) \leq \frac{\beta^+}{2^p-4}\|x\|^p, \qquad (29)$$

for all $x \in E$, since ρ has Fatou property. For all $x, y \in E$, with $x \perp y$, we obtain

$$\rho(4^n(f(2^{-n}(x+y))+f(2^{-n}(x-y))-2f(2^{-n}x)-2f(2^{-n}y))) \leq \epsilon 2^{n(2-p)}(\|x\|^p+\|y\|^p). \qquad (30)$$

If $n \to \infty$ then, we conclude that $Q(x+y)+Q(x-y)-2Q(x)-2Q(y) = 0,\ x \perp y$ for all $x, y \in E$ and on account of the results by F. Vajzović [28] and M. Fochi [6], Q is quadratic. To prove the uniqueness, assume $Q' : E \to X_\rho$ to be another quadratic mapping satisfying (29). Then, for each $x \in E$ and for all $n \in 2^{\mathbb{N}}$ one has

$$\begin{aligned}\rho(Q(x)-Q'(x)) &= \rho(n^2(Q(\frac{x}{n})-Q'(\frac{1}{n}))) \leq k^{2m}\rho(Q(\frac{x}{2^{2m}})-Q'(\frac{x}{2^{2m}})) \\ &\leq 2^{2m}\frac{k}{2}\left[\rho(Q(\frac{x}{2^{2m}})-f(\frac{x}{2^{2m}}))+\rho(Q'(\frac{x}{2^{2m}})-f(\frac{x}{2^{2m}}))\right] \\ &\leq 2^{m(2-p)}\frac{k}{2}\|x\|^p.\end{aligned}$$

If $m \to \infty$ we obtain $Q = Q'$. This completes the proof of theorem. □

Corollary 3.3. *Let E is a real linear space with $\dim E \geq 2$ and X_ρ is $\rho-$complete modular space. If a function $f : E \to X_\rho$ satisfying*

$$\rho(f(x+y)+f(x-y)-2f(x)-2f(y)) \leq \epsilon, \qquad (31)$$

for all $x, y \in E$ with $x \perp y$ and $\epsilon \geq 0$, then there exist unique quadratic mapping $Q : E \to X_\rho$ such that

$$\rho(f(x)-Q(x)) \leq \frac{\epsilon[k(k+1)]^2}{24} \qquad (32)$$

for all $x \in E$.

Corollary 3.4. *Let* $(E_p, \|.\|)$ *with* $\dim E_p \geq 2$ *be a real normed linear space with Birkhoff-James orthogonality and* $(X, \|.\|)$ *is Banach space. If a function* $f : E_p \to X$ *satisfying*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p), \tag{33}$$

for all $x, y \in E_p$ *with* $x \perp y$, $\epsilon \geq 0$ *and* $p \in \mathbb{R} \setminus \{2\}$, *then there exist unique quadratic mapping* $Q : E_p \to X$ *such that*

$$\|f(x) - Q(x)\| \leq \begin{cases} \frac{\beta^+ sgn(p-2)}{2^p - 4}\|x\|^p & \text{if } p \in \mathbb{R}^+ \setminus \{2\}, \\ \frac{\beta^-}{4-2^p}\|x\|^p & \text{if } p < 0, \end{cases} \tag{34}$$

for all $x \in E_p$, *where* $\beta^+ = \frac{\alpha^+}{4}(4 + 2.3^p)$, $\beta^- = \frac{\alpha^-}{4}(4 + 2^{1-p})$, $\alpha^+ = \epsilon(2^p + 2^{2p} + 2 + 2.3^p)$ *and* $\alpha^- = \epsilon(4 + 2^{1-p})$.

Proof. It is well known that every normed space is a modular space with the modular $\rho(x) = \|x\|$ and $k = 2$. □

Corollary 3.5. *Let* E *is a real linear space with* $\dim E \geq 2$ *and* $(X, \|.\|)$ *is Banach space. If a function* $f : E \to X$ *satisfying*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \epsilon, \tag{35}$$

for all $x, y \in E$ *with* $x \perp y$ *and* $\epsilon \geq 0$, *then there exist unique quadratic mapping* $Q : E \to X$ *such that*

$$\|f(x) - Q(x)\| \leq \frac{3}{2}\epsilon \tag{36}$$

for all $x \in E$.

Proof. It is well known that every normed space is a modular space with the modular $\rho(x) = \|x\|$, $p = 0$ and $k = 2$. □

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