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Best One-Sided Approximation of Entire Functions in $L_{p,w}$ Spaces

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Abstract

In this paper, we estimate the degree of best one-sided approximation of an entire unbounded function by using some discrete operator in $L_{p,w}$ weighted space ($1 \le p < \infty$). We construct new theorems regarding by Jackson polynomials and Valee-Poussin operator of these functions.

Keywords: *Entire unbounded function, discrete operator and weight space.*

1 Introduction

Let X = [0, 1], we denoted by $L_\infty(X)$ the set of all bounded measurable functions with usual norm [5]:

$$\|f\|_\infty = \sup\{|f(x)| \ : x \epsilon X\} \qquad (1.1)$$

For $1 \leq p < \infty$, let $L_p = \{f$: f is bounded measurable function for which

$$\|f\|_p = \left(\int_0^1 |f(x)|^p \, dx\right)^{\frac{1}{p}} < \infty \} \quad (1.2)$$

Further for $\delta > 0$ and $(1 \leq p < \infty)$ the locally global norm of f is defined

$$\|f\|_{\delta,p} = \left(\int_0^1 (\sup\{\, |f(y)| : y \in \left[x - \frac{\delta}{2}, x + \frac{\delta}{2}\right]\})^p \, dy\right)^{\frac{1}{p}} \quad (1.3)$$

Now, let W be the set of all weight functions on X. Consider B_w the set of all functions f on X such that $|f(x)| \leq Mw(x)$ integrable, that is

$$B_w = \left\{f: X \to R; |f(x)| \leq Mw(x) and \|f\|_{p,w} = \left(\int_0^1 \left|\frac{f(x)}{w(x)}\right|^p dx\right)^{\frac{1}{p}} < \infty\right\}$$

for $\delta > 0$ and $(1 \leq p < \infty)$, the weighted locally global norm of $f \in B_w$ is defined by

$$\|f\|_{\delta,p,w} = \left(\int_0^1 (\sup\{ \left|\frac{f(y)}{w(y)}\right|^p : y \in \left[x - \frac{\delta}{2}, x + \frac{\delta}{2}\right]\}) \, dy\right)^{\frac{1}{p}} < \infty. \quad (1.4)$$

The k^{th} average modulus of $f \in L_p$ with respect to algebraic polynomials from L_p space is given by [7]:

$$\tau_k \, (f, \delta)_p = \|\omega(f, ., \delta)\|_p \quad , \ \delta > 0 \quad (1.5)$$

Where

$$\omega_k(f, x, \delta)_p = sup\left\{\Delta_h^k |f(t)| : t \in \left[x - \frac{h}{2}, x - \frac{h}{2}\right], h \leq \delta\right\} \, . \delta > 0 \quad (1.6)$$

The $k^{th} L_{p,w}$ modulus of smoothness for $f \in B_w$ is define by

$$\omega_k(f, \delta)_{p,w} = sup\left\{\left\|\Delta_h^k f\right\|_{p,w}\right\} , \quad (1.7)$$

where $\Delta_h^k f(t) = \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} f\left(x + \frac{kh}{2}\right)$

Now, we shall introduce the average modulus of smoothness for $f \in B_w$. The k^{th} average modulus of smoothness for $f \in B_w$ is define by

$$\tau_k(f, \delta)_{p,w} = \|\omega_k(f, ., \delta)\|_{p,w} \quad , \quad (1.8)$$

where

$$\omega_k(f, \delta)_{p,w} = sup\left\{\left|\Delta_h^k f(t)/w(t)\right| : t \in \left[x - \frac{kh}{2}, x - \frac{kh}{2}\right], h \leq \delta\right\}. \quad (1.9)$$

The degree of best weighted approximation to a given function $f \in$ Bw with trigonometric polynomials or algebraic polynomials on the interval X is given:

$$E_n(f)_{p,w} = \inf\{\|f - q\|_{p,w} : q \in P_n\}, \tag{1.10}$$

Where P_n denote the set of all trigonometric polynomials or algebraic polynomials of degree ≤ n.

Further, the degree of best one-sided weighted approximation of $f \in$ Bw with respect to trigonometric polynomials or algebraic polynomials of degree n on X given by [1]:

$$\tilde{E}_n(f)_{p,w} = \min_{x \in X}\{\|p - q\|_{p,w}\ , p, q \in P_n \text{ and } q(x) \le f(x) \le p(x)\} \tag{1.11}$$

Now, let us consider the Dirichlet kernel of degree n [7]

$$D_n(u) = \frac{1}{2} + \sum_{i=1}^{n} \cos(iu) \quad , u \in R\ , \quad n = 1,2,3, \ldots \tag{1.12}$$

Let $K_n(u) = \frac{1}{n+1}\,[D_0(u) + D_1(u) + \ \ldots + \ D_n(u)]$ (1.13)

be Fejer kernel of degree least than or equal n.

Let $J_n(f, x) = \frac{2}{n+1}\sum_{k=0}^{n} f(x_{k,n}) K_n(x - x_{k,n})$ (1.14)

where $x_{k,n} = \frac{2k\pi}{n+1}$, k = 0,1,…,n be the so called Jackson polynomial of function

$f \in B_w$, and $V_{2n}(t) = \frac{1}{n+1}[D_n(t) + D_{n+1}(t) + \ \ldots + \ D_{2n}(t)]$, Let

$x_j = \frac{2\pi_j}{3n+1}$, j = 0,1,2,…,3n. Then

$$V_{2n,3n}(f, x) = \frac{2}{3n+1}\sum_{j=0}^{3n} f(x_j) V_{2n}(x - x_j) \tag{1.15}$$

be the Valee-poussin operator.

The unique linear trigonometric polynomial which is interpolating a given function $f \in B_w$ at the point x_i is denoted by $I_n(t)$ which has the following representation:

$$I_n(f, x) = \frac{2}{2n+1}\sum_{i=0}^{2n} f(x_i) D_n(x - x_i). \tag{1.16}$$

Now, let $\overline{\overline{B}}_w$ be the set of all entire functions, since the derivative of polynomial exists everywhere , then we obtain every polynomial is an entire function, so we consider that $f \in \overline{\overline{B}}_w$, $J_n(f) \in \overline{\overline{B}}_w$, $V_{2n,3n}(f) \in \overline{\overline{B}}_w$ and $I_n(f) \in \overline{\overline{B}}_w$.

Further in (2002) Jiasong Deng [1] develop an analytic solution for the best one sided approximation of polynomial under L_1 norm, also (2005) Friedrich Littman [4] estimate the error of one sided approximation of function f: R→R by entire functions of exponential type $\delta > 0$ in L_p space, $(1 \leq p < \infty)$ and in (2011) S.K. Jassim and N.J. Mohamed [3] evaluate the degree of the approximation of entire function by some discrete operators in locally global quasi-norms ($L_{\delta,p}$-space).

2 Auxiliary Theorems

In this section we introduce the results that we make use of in our lemmas.

Lemma (1) [3] *If $f \in$ 2π-periodic bounded measurable function, we have for*

$$(0<p \leq 1) \quad \|f - J_n(f)\|_p \leq c(p)\tau_1(f, \tfrac{1}{n})_p. \tag{2.1}$$

Lemma (2) [3] *If $f \in$ 2π-periodic bounded measurable function, we have for*

$$(0 < p \leq 1) \quad \left\|f - V_{2n,3n}(f)\right\|_p \leq c(p,k,l)\tau_k(f, \tfrac{1}{n})_p \tag{2.2}$$

where n = 1,2,… , p,k and l are constant depends on p.

Lemma (3) [3] *If $f \in$ 2π-periodic bounded measurable function, we have then for*

$$(0< p \leq 1) \quad \|f - I_n(f)\|_p \leq c(p,k,l)\tau_k(f, \tfrac{1}{n})_p \tag{2.3}$$

where n = 1,2,… , p,k and l are constant depends on p.

Lemma (4) [6] *If f is bounded measurable function on [a, b], then we have*

$$\int_a^b f(x)dx = \frac{b-a}{n}\sum_{i=1}^n f(x_i) \tag{2.4}$$

where $x_i = a + \frac{(b-a)(2i-1)}{2n}$ and $1 \leq i \leq n$.

Lemma (5) [2] *If f is bounded measurable function, then for* $(0 < p < 1)$ and $\delta > 0$, *we have*

$$\|f\|_{\delta,p} = c(p)(1+n\delta)^{1-p}(ns)^{\frac{1}{p}}\|f\|_p . \tag{2.5}$$

3 Main Results:

In this section we introduce our main results

Theorem (1): *Let* f $\in\overline{\overline{\mathrm{B}}}_{\mathrm{w}}$, *then for* $(1 \le p < \infty)$. *Then*

$$\tau_k(f,\tfrac{1}{n})_{p,w} \le \tau_k(f,\tfrac{1}{n})_{\delta,p,w} \tag{3.1}$$

Theorem (2): *Let f* $\in\overline{\overline{\mathrm{B}}}_{\mathrm{w}}$, *then for* $(1 \le p < \infty)$. *Then*

$$\|f - J_n(f)\|_{\delta,p,w} \le c(p)\tau_k\left(f,\tfrac{1}{n}\right)_{\delta,p,w} \tag{3.2}$$

Theorem (3): *Let f* $\in\overline{\overline{\mathrm{B}}}_{\mathrm{w}}$, *for* $(1 \le p < \infty)$. *Then*

$$\left\|f - V_{2n,3n}(f)\right\|_{\delta,pw} \le c(p,k,l)\tau_k(f,\tfrac{1}{n})_{\delta,p,w} \tag{3.3}$$

Theorem (4): *Let f* $\in\overline{\overline{\mathrm{B}}}_{\mathrm{w}}$, *for* $(1 \le p < \infty)$. *Then*

$$\|f - I_n(f)\|_{\delta,p,w} \le c(p,k,l)\tau_k(f,\tfrac{1}{n})_{\delta,p,w} \tag{3.4}$$

To prove our results we need the following lemmas:

Lemma (A): *Let f* $\in\overline{\overline{\mathrm{B}}}_{\mathrm{w}}$, *then for* $(1 \le p < \infty)$ and $\delta > 0$. *Then*

$$\|f\|_{p,w} \le \|f\|_{\delta,p,w} \le c(p)(1+n\delta)^p\|f\|_{p,w}.$$

Proof:

We have $\|f\|_{p,w} = \left(\int_0^1 \left|\frac{f(x)}{w(x)}\right|^p dx\right)^{\frac{1}{p}}$

$$\le c(p)\left(\int_0^1 \left|\frac{f(x)}{w(x)}\right|^p dx\right)^{\frac{1}{p}}$$

$$= \left(\int_0^1 \sup\{\left|\frac{f(x)}{w(x)}\right|^p\}\, dx\right)^{\frac{1}{p}}$$

$$= \left(\int_0^1 (\sup\{\,|f(y)/w(y)| : y \in \left[x-\frac{\delta}{2}, x+\frac{\delta}{2}\right]\})^p\, dy\right)^{\frac{1}{p}}$$

$$= \|f\|_{\delta,p,w}.$$

We need to prove that $\|f\|_{\delta,p,w} \le c(p)(1+n\delta)^p\|f\|_{p,w}$

We have

$$\|f\|_{\delta,p,w} = \left(\int_0^1 (\sup\{\,|f(y)/w(y)| :\ y \in \left[x-\frac{\delta}{2}, x+\frac{\delta}{2}\right]\})^p \, dy\right)^{\frac{1}{p}}$$

By using (2.4) we get

$$= \left(\frac{1}{n}\sum_{i=1}^{n} \sup\left\{|f(y)/w(y)|^p, y \in [x-\frac{\delta}{2}, x-\frac{\delta}{2}]\right\} dy\right)^{\frac{1}{p}}$$

Suppose that $x_i \in [x-\frac{\delta}{2}, x+\frac{\delta}{2}]$ such that $\frac{f(x_i)}{w(x_i)} = sup\left\{\frac{f(y_i)}{w(y_i)}\right\}$ and $y_i \in [x_i - \frac{\delta}{2}, x_i + \frac{\delta}{2}]$

Thus

$$\|f\|_{\delta,p,w} \le \left(\frac{1}{n}\sum_{i=0}^{n} |\frac{f(x_i)}{w(x_i)}|^p\right)^{\frac{1}{p}} \le \left(\frac{1}{n}\sum_{i=0}^{n} \sup\{|\frac{f(x_i)}{w(x_i)}|^p\}\right)^{\frac{1}{p}}$$

$$= \left(\frac{c(p)}{n}\sum_{i=0}^{n} |\frac{f(x_i)}{w(x_i)}|^p\right)^{\frac{1}{p}} = c(p)\left(\int_0^1 \left|\frac{f(x)}{w(x)}\right|^p dx\right)^{\frac{1}{p}}$$

$$= c(p)\|f\|_{p,w} \le c(p)(1+n\delta)^p \|f\|_{p,w}$$, for $\delta > 0$ and $1 \le p < \infty$

We get $\|f\|_{\delta,p,w} \le c(p)(1+n\delta)^p\|f\|_{p,w}$.

■

Lemma (B): *Let* $f \in \overline{\overline{B}}_w$, *then for* $(1 \le p < \infty)$ and $\delta > 0$. *Then*

$\|f - J_n(f)\|_{p,w} \le c(p)\tau_1\left(f, \frac{1}{n}\right)_{p,w}$, *where c is constant depends on p.*

Proof:

From (2.1), we have for $f \in L_p \|f - J_n(f)\|_p \le c(p)\tau_1(f, \frac{1}{n})_p$

Now, for $f \in \overline{\overline{B}}_w$

$\|f - J_n(f)\|_{p,w} = \left\|\frac{f}{w} - \frac{J(f)}{w}\right\|_p$ where $w \in W$

From definition of $f \in \overline{\overline{B}}_w$ we get $\frac{f}{w}$ is integrabl and also $J_n(\frac{f}{w})$ integrable, we obtain $\frac{f}{w} - J_n(\frac{f}{w})$ is integrable.

Thus

$$\left\|\frac{f}{w}-\frac{J(f)}{w}\right\|_p \leq c(p)\tau_1\left(\frac{f}{w},\frac{1}{n}\right)_p = c(p)\tau_1(f,\frac{1}{n})_{p,w}$$

Hence $\|f-J_n(f)\|_{p,w} \leq c(p)\tau_1\left(f,\frac{1}{n}\right)_{p,w}$

■

Lemma (C): *Let* $f \in \overline{\overline{\mathrm{B}}}_{\mathrm{w}}$, then for $(1 \leq p < \infty)$ and $\delta > 0$, *we have*

$$\|f-\mathrm{I_n}(f)\|_{\mathrm{p,w}} \leq \mathrm{c(p,k,l)}\tau_{\mathrm{k}}(f,\frac{1}{\mathrm{n}})_{\mathrm{p,w}}$$

Prove of this lemma the same to proof of lemma (B).

We shall prove the theorems (3.1), (3.2), (3,3) and (3,4) to find the degree of weighted approximation of unbounded entire function by some discrete operators in weighted spaces $L_{p,w}$, $(1 \leq p < \infty)$.

4 The Proof of Theorem (1)

We have $\tau_k(f,\delta)_{p,w} = \|\omega_k(f,.,\delta)\|_{p,w}$

$$= \left\|sup\left\{\left|\Delta_h^k\frac{f(t)}{w(t)}\right| : t,t+kh \in \left[x-\frac{kh}{2},x-\frac{kh}{2}\right], h \leq \delta\right\}\right\|_{p,w}$$

$$= \left(\int_0^1 \sup\left\{\left|\sum_{i=0}^{k}(-1)^{i+k}\binom{k}{i}\frac{f(t+ih)}{w(t+ih)}\right|^p, t,t+kh \in [x-\frac{k}{2n},x-\frac{k}{2n}]\right\}dt\right)^{\frac{1}{p}}$$

$$\leq \left(\int_0^1 \sup\left\{\sup\left\{\left|\sum_{i=0}^{k}(-1)^{i+k}\binom{k}{i}\frac{f(t+ih)}{w(t+ih)}\right|^p, t,t+kh \in \left[x-\frac{k}{2n},x-\frac{k}{2n}\right]\right\}\right\}dt\right)^{\frac{1}{p}}$$

$$= \left\|sup\left\{\left|\Delta_h^k\frac{f(t)}{w(t)}\right| : t,t+kh \in \left[x-\frac{kh}{2},x-\frac{kh}{2}\right], h \leq \delta\right\}\right\|_{\delta,p,w}$$

$$= \|\omega_k(f,.,\delta)\|_{\delta,p,w} = \tau_k(f,\delta)_{\delta,p,w}$$

We obtain (3.1) hold.

■

5 The Proof of Theorem (2)

From lemma (A) we obtain

$\|f - J_n(f)\|_{\delta,p,w} \le c(p)(1+n\delta)^p \|f - J_n(f)\|_{p,w}$,

Since δ > 0, then

$$\|f - J_n(f)\|_{\delta,p,w} \le c_1(p)\|f - J_n(f\ )\|_{p,w}$$

Now, by using lemma (B) and (3.1) respectively, we obtain

$$c_1(p)\|f - J_n(f)\|_{p,w} \le c_2(p)\tau_k(f,\frac{1}{n})_{p,w} \le c_2(p)\tau_k(f,\frac{1}{n})_{\delta,p,w}$$

We obtain (3.2) hold.

■

6 The Proof of Theorem (3.3)

From lemma (A), we obtain

$$\left\|f - V_{2n,3n}(f)\right\|_{\delta,pw} \le c(p)(1+n\delta)^p \left\|f - V_{2n,3n}(f)\right\|_{p,w}$$

Since δ > 0, then

$$\left\|f - V_{2n,3n}(f)\right\|_{\delta,pw} \le c_1(p)\left\|f - V_{2n,3n}(f)\right\|_{p,w}$$

Also from theorem (2.2), we obtain

$$\left\|f - V_{2n,3n}(f)\right\|_{\delta,pw} \le c_2(p,k,l)\tau_k(f,\frac{1}{n})_{p,w}$$

So from theorem (3.1), we have

$$c_2(p,k,l)\tau_k\left(f,\frac{1}{n}\right)_{p,w} \le c(p,k,l)\tau_k\left(f,\frac{1}{n}\right)_{\delta,p,w}$$

We obtain (3.3) hold.

7 The Proof of Theorem (3.4)

From lemma (A), we obtain

$$\|f - I_n(f)\|_{\delta,p,w} \le c(p)(1+n\delta)^p \|f - I_n(f)\|_{p,w}$$

Since δ > 0, then

$$\|f - I_n(f)\|_{\delta,p,w} \le c_1(p)\|f - I_n(f)\|_{p,w}$$

From (2.3) and (3.1) respectively, we obtain

$$\|f - I_n(f)\|_{\delta,p,w} \le c(p,k,l)\tau_k\left(f,\frac{1}{n}\right)_{p,w} \le c(p,k,l)\tau_k\left(f,\frac{1}{n}\right)_{\delta,p,w}$$

We obtain (3.4) hold.

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