Gen. Math. Notes, Vol. 27, No. 2, April 2015, pp.69-84
ISSN 2219-7184; Copyright ©ICSRS Publication, 2015
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Common Fixed Theorem on Intuitionistic Fuzzy 2-Metric Spaces

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(Received: 23-2-15 / Accepted: 5-4-15)

Abstract

The aim of this paper is to prove the existence and uniqueness of common fixed point theorem for four mappings in complete intuitionistic fuzzy 2-metric spaces

Keywords: *Fuzzy metric spaces, fuzzy 2-metric spaces, intuitionistic fuzzy metric spaces, common fixed point, intuitionistic fuzzy 2-metric spaces.*

1 Introduction

The concept of fuzzy sets was introduced by L. A. Zadeh [24] in 1965, which became active field of research for many researchers. In 1975, Karmosil and Michalek [16] introduced the concept of a fuzzy metric space based on fuzzy sets, this notion was further modified by George and Veermani [11] with the help of t-norms. Many authors made use of the definition of a fuzzy metric space in proving fixed point theorems. In 1976, Jungck [14] established common fixed point theorems for commuting maps generalizing the Banach's fixed point theorem. Sessa [23] defined a generalization of commutativity, which is called weak commutativity. Further Jungck [15] introduced more generalized commutativity, so called compatibility. Mishra et. al. [21] introduced the concept of compatibility in fuzzy metric spaces. Atanassov [1-8] introduced the notion of intuitionistic fuzzy sets and developed its theory. Park [22] using the idea of intuitionistic fuzzy sets to define the notion of intuionistic fuzzy metric spaces with the help of continuous t-norm and continuous t co-norm as a generalization of fuzzy metric space. Muralisankar and Kalpana [20] proved a common fixed point theorem in an intuitionistic fuzzy metric space for pointwise R-weakly commuting mappings using contractive condition of integral type and established a situation in which a collection of maps has a fixed point which is a point of discontinuity. Gahler [10] introduced and studied the concept of 2-metric spaces in a series of his papers. Iseki et. al. [13] investigated, for the first time, contraction type mappings in 2-metric spaces. In 2002 Sharma [18] introduced the concept of fuzzy 2- metric spaces. Mursaleen et. al. [19] introduced the concept of intuitionistic fuzzy 2-metric space. In this paper, we prove the existence and uniqueness of common fixed point theorem for four mappings in complete intuitionistic fuzzy 2-metric spaces

2 Preliminaries

Definition 2.1 (17) *A binary operation* $* : [0,1] \times [0,1] \longrightarrow [0,1]$ *is called continuous* t*-norm if* $*$ *is satisfying the following conditions:*

(TN1) $*$ is commutative and associative;

(TN2) $*$ is continuous;

(TN3) $a * 1 = a$ for all $a \in [0,1]$;

(TN4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0,1]$.

Examples of t-norms are $a * b = ab$ and $a * b = min\{a, b\}$

Definition 2.2 (16) *A binary operation* $\diamondsuit : [0,1] \times [0,1] \longrightarrow [0,1]$ *is called continuous* t*-conorm if* $\diamondsuit$ *is satisfying the following conditions:*

(TCN1) $\diamondsuit$ is commutative and associative;

(TCN2) $\diamondsuit$ is continuous;

(TCN3) $a \diamondsuit 0 = a$ for all $a \in [0,1]$;

(TCN4) $a \diamondsuit b \leq c \diamondsuit d$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0,1]$.

Definition 2.3 (16) *A fuzzy metric space (shortly,* FM*-space) is a triple* $(X, M, *)$*, where* X *is a nonempty set,* $*$ *is a continuous* t*-norm and* M *is a fuzzy set on* $X^2 \times [0, \infty)$ *satisfying the following conditions : for all* $x, y, z \in X$ *and* $s, t > 0$*,*

(FM1) $M(x, y, 0) = 0$

(FM2) $M(x, y, t) = 1$, for all $t > 0$ if and only if $x = y$,

(FM3) $M(x, y, t) = M(y, x, t)$,

(FM4) $M(x, y, t + s) \geq M(x, z, t) * M(z, y, s)$,

(FM5) $M(x, y, .) : [0, 1) \longrightarrow [0, 1]$ is left continuous.

Note that $M(x, y, t)$ can be thought of as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$ and $M(x, y, t) = 0$ with ∞.

Definition 2.4 (9) *The 5-tuple $(X, M, N, *, \diamondsuit)$ is said to be an intuitionistic fuzzy metric space (shortly, IFM-space) if X is an arbitrary set, $*$ is a continuous t-norm, $\diamondsuit$ is a continuous t-conorm, and M, N are fuzzy sets on $X^2 \times [0, \infty)$ satisfying the following conditions:*

(IFM1) $M(x, y, t) + N(x, y, t) \leq 1$;

(IFM2) $M(x, y, 0) = 0$;

(IFM3) $M(x, y, t) = 1$, for all $t > 0$ if and only if $x = y$;

(IFM4) $M(x, y, t) = M(y, x, t)$;

(IFM5) $M(x, y, t + s) \geq M(x, z, t) * M(z, y, s)$ for all $x, y, z \in X$ and $s, t > 0$;

(IFM6) $M(x, y, .) : [0, \infty) \longrightarrow [0, 1]$ is left continuous.

(IFM7) $\lim_{t \longrightarrow \infty} M(x, y, t) = 1$ for all $x, y \in X$;

(IFM8) $N(x, y, 0) = 1$;

(IFM9) $N(x, y, t) = 0$, for all $t > 0$ if and only if $x = y$;

(IFM10) $N(x, y, t) = N(y, x, t)$;

(IFM11) $N(x, z, t + s) \leq N(x, y, t) \diamondsuit N(y, z, s)$ for all $x, y, z \in X$ and $s, t > 0$;

(IFM12) $N(x, y, .) : [0, \infty) \longrightarrow [0, 1]$ is right continuous.

(IFM13) $\lim_{t \longrightarrow \infty} N(x, y, t) = 0$ for all $x, y \in X$;

Then (M, N) is called an intuitionistic fuzzy metric on X.

The function $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t respectively.

Remark 2.5 *Every fuzzy metric $(X, M, *)$ is an intuitionistic fuzzy metric space of the form $(X, M, 1 - M, *, \diamondsuit)$ such that t-norm $*$ and t-conorm $\diamondsuit$ are associated [12] i.e., $x \diamondsuit y = 1 - ((1 - x) * (1 - y))$ for any $x, y \in X$.*

Remark 2.6 *In intuitionistic fuzzy metric space X, $M(x, y, .)$ is non-decreasing and $N(x, y, .)$ is non-increasing for any $x, y \in X$.*

Definition 2.7 (10) *A 2-metric space is a set X with a real-valued function d on X^3 satisfying the following conditions:*

(2M1) For distinct elements $x, y \in X$, there exists $z \in X$ such that $d(x, y, z) \neq 0$.

(2M2) $d(x, y, z) = 0$ if at least two of x, y and z are equal.

(2M3) $d(x, y, z) = d(x, z, y) = d(y, z, x)$ for all $x, y, z \in X$.

(2M4) $d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z) \quad \forall\ x, y, z, w \in X$.

The function d is called a 2-metric for the space X and the pair (X, d) denotes a 2-metric space. It has shown by Gähler [10] that a 2-metric d is non-negative and although d is a continuous function of any one of its three arguments, it need not be continuous in two arguments. A 2-metric d which is continuous in all of its arguments is said to be continuous.

Geometrically a 2-metric $d(x, y, z)$ represents the area of a triangle with vertices x, y and z.

Example 2.8 *Let $X = \Re^3$ and let $d(x, y, z)$ is the area of the triangle spanned by x, y and z which may be given explicitly by the formula, $d(x, y, z) = [x_1(y_2 z_3 - y_3 z_2) - x_2(y_1 z_3 - y_3 z_1) + x_3(y_1 z_2 - y_2 z_1)]$, where $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$, $z = (z_1, z_2, z_3)$. Then (X, d) is a 2-metric space.*

Definition 2.9 (18) *The 3-tuple $(X, M, N, *)$ is said to be a fuzzy 2-metric space (shortly, F2M-space) if X is an arbitrary set, $*$ is a continuous t-norm, and M is fuzzy sets on $X^3 \times [0, \infty)$ satisfying the following conditions: for all $x, y, z, u \in X$ and $r, s, t > 0$.*

(IFM2) $M(x, y, z, 0) = 0$,

(IFM3) $M(x, y, z, t) = 1$, if and only if at least two of the three points are equal,

(IFM4) $M(x, y, z, t) = M(x, z, y, t) = M(y, z, x, t)$.
(Symmetry about first three variables)

(IFM5) $M(x,y,z,r+s+t) \geq M(x,y,u,r) * M(x,u,z,s) * M(u,y,z,t)$.
(This corresponds to tetrahedron inequality in 2-metric space, the function value $M(x,y,z,t)$ may be interpreted as the probability that the area of triangle is less than t.)

(IFM6) $M(x,y,z,.) : [0,\infty) \longrightarrow [0,1]$ is left continuous.

Definition 2.10 (19) *The 5-tuple $(X, M, N, *, \diamondsuit)$ is said to be an intuitionistic fuzzy 2-metric space (shortly, IF2M-space) if X is an arbitrary set, $*$ is a continuous t-norm, $\diamondsuit$ is a continuous t-conorm, and M, N are fuzzy sets on $X^3 \times [0,\infty)$ satisfying the following conditions:*

for all $x, y, z, w \in X$ and $r, s, t > 0$.

(IF2M1) $M(x,y,z,t) + N(x,y,z,t) \leq 1$,

(IF2M2) given distinct elements x, y, z of X there exists an element z of X such that $M(x,y,z,0) = 0$,

(IF2M3) $M(x,y,z,t) = 1$, if at least two of x, y, z of X are equal,

(IF2M4) $M(x,y,z,t) = M(x,z,y,t) = M(y,z,x,t)$,

(IF2M5) $M(x,y,z,r+s+t) \geq M(x,y,w,r) * M(x,w,z,s) * M(w,y,z,t)$;

(IF2M6) $M(x,y,z,.) : [0,\infty) \longrightarrow [0,1]$ is left continuous,

(IF2M7) $N(x,y,z,0) = 1$,

(IF2M8) $N(x,y,z,t) = 0$, if at least two of x, y, z of X are equal,

(IF2M9) $N(x,y,z,t) = N(x,z,y,t) = N(y,z,x,t)$,

(IF2M10) $N(x,y,z,r+s+t) \leq N(x,y,w,r) \diamondsuit N(x,w,z,s) \diamondsuit N(w,y,z,t)$;

(IF2M11) $N(x,y,z,.) : [0,\infty) \longrightarrow [0,1]$ is left continuous,

In this case (M, N) is called an intuitionistic fuzzy 2-metric on X. The function $M(x,y,z,t)$ and $N(y,x,z,t)$ denote the degree of nearness and the degree of non-nearness between x, y and z with respect to t, respectively.

Example 2.11 *Let (X, d) be a 2-metric space. Denote $a * b = ab$ and $a \diamondsuit b = min\{1, a+b\}$ for all $a, b \in [0,1]$ and M_d and N_d be fuzzy sets on $X^3 \times [0,\infty)$ defined by*

$$M_d(x,y,z,t) = \frac{ht^n}{ht^n + md(x,y,z)}, N_d(x,y,z,t) = \frac{d(x,y,z)}{kt^n + md(x,y,z)}$$

for all $h, k, m, n \in R^+$. Then $(X, M_d, N_d, *, \diamondsuit)$ is IF2M-space.

Definition 2.12 *Let $(X, M, N, *, \diamondsuit)$ be an IF2M-space.*

(a) A sequence $\{x_n\}$ in IF2M-space X is said to be convergent to a point $x \in X$ (denoted by $\lim_{n \longrightarrow \infty} x_n = x$ or $x_n \longrightarrow x$) if for any $\lambda \in (0, 1)$ and $t > 0$, there exists $n_0 \in N$ such that for all $n \geq n_0$ and $a \in X$, $M(x_n, x, a, t) > 1 - \lambda$ and $N(x_n, x, a, t) < \lambda$. That is $\lim_{n \longrightarrow \infty} M(x_n, x, a, t) = 1$ and $\lim_{n \longrightarrow \infty} N(x_n, x, a, t) = 0$, for $a \in X$ and $t > 0$.

(b) A sequence $\{x_n\}$ in IF2M-space X is called a Cauchy sequence, if for any $\lambda \in (0, 1)$ and $t > 0$, there exists $n_0 \in N$ such that for all $m, n \geq n_0$ and $a \in X$, $M(x_m, x_n, a, t) > 1 - \lambda$ and $N(x_m, x_n, a, t) < \lambda$. That is $\lim_{m,n \to \infty} M(x_m, x_n, a, t) = 1$ and $\lim_{m,n \to \infty} N(x_m, x_n, a, t) = 0$, for $a \in X$ and $t > 0$.

(c) The IF2M-space X is said to be complete if and only if every Cauchy sequence is convergent.

Definition 2.13 *Self mappings A and B of an IF2M-space $(X, M, N, *, \diamondsuit)$ is said be be compatible, if $\lim_{n \longrightarrow \infty} M(ABx_n, BAx_n, a, t) = 1$ and $\lim_{n \longrightarrow \infty} N(ABx_n, BAx_n, a, t) = 0$ for all $a \in X$ and $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \longrightarrow \infty} Ax_n = \lim_{n \longrightarrow \infty} Bx_n = z$ for some $z \in X$*

3 Main Results

Lemma 3.1 *Let $(X, M, N, *, \diamondsuit)$ be an IF2M-space. Then $M(x, y, z, t)$ is non-decreasing and $N(x, y, z, t)$ is non-increasing for all $x, y, z \in X$.*

Proof: Let $s, t > 0$ be any points such that $t > s$. $t = s + \frac{t-s}{2} + \frac{t-s}{2}$. Hence we have

$$
\begin{aligned}
N(x, y, z, t) &= N(x, y, z, s + \frac{t-s}{2} + \frac{t-s}{2}) \\
&\leq N(x, y, z, s) \diamondsuit N(x, z, z, \frac{t-s}{2}) \diamondsuit N(z, y, z, \frac{t-s}{2}) \\
&= N(x, y, z, s)
\end{aligned}
$$

Thus $N(x, y, z, t) < N(x, y, z, s)$. Similarly, $M(x, y, z, t) > M(x, y, z, s)$. Therefore, $M(x, y, z, t)$ is non-decreasing and $N(x, y, z, t)$ is non-increasing.

From Lemma 3.1, let $(X, M, N, *, \diamondsuit)$ be an IF2M-space with the following conditions:

$$
\lim_{t \to \infty} M(x, y, z, t) = 1, \quad \lim_{t \to \infty} N(x, y, z, t) = 0
$$

Lemma 3.2 *Let $(X, M, N, *, \diamondsuit)$ be an IF2M-space. If there exists $q \in (0,1)$ such that $M(x, y, z, qt+0) \geq M(x, y, z, t)$ and $N(x, y, z, qt+0) \leq N(x, y, z, t)$ for all $x, y, z \in X$ with $z \neq x$, $z \neq y$ and $t > 0$. Then $x = y$.*

Proof: Since

$$M(x, y, z, t) \geq M(x, y, z, qt+0) \geq M(x, y, z, t), \quad and$$

$$N(x, y, z, t) \leq N(x, y, z, qt+0) \leq N(x, y, z, t)$$

for all $t > 0$, $M(x, y, z, .)$ and $N(x, y, z, .)$ are constant. Since $lim_{t\to\infty} M(x, y, z, t) = 1$, $lim_{t\to\infty} N(x, y, z, t) = 0$. Then $M(x, y, z, t) = 1$ and $N(x, y, z, t) = 0$. Consequently, for all $t > 0$. Hence $x = y$ because $z \neq x$, $z \neq y$.

Lemma 3.3 *Let $(X, M, N, *, \diamondsuit)$ be an IF2M-space and let $lim_{t\to\infty} x_n = x, lim_{t\to\infty} y_n = y$. Then the following are satisfied for all $a \in X$ and $t \geq 0$*

(1) $\underline{lim}_{n\to\infty} \; inf M(x_n, y_n, a, t) \geq M(x, y, a, t)$ and $\underline{lim}_{n\to\infty} \; sup N(x_n, y_n, a, t) \leq N(x, y, a, t)$

(2) $M(x, y, a, t+0) \geq \underline{lim}_{n\to\infty} \; sup M(x_n, y_n, a, t)$ and $N(x, y, a, t+0) \leq \underline{lim}_{n\to\infty} \; inf N(x_n, y_n, a, t)$

Proof: (1) For all $a \in X$ and $t \geq 0$ we have

$$\begin{aligned} M(x_n, y_n, a, t) &\geq M(x_n, y_n, x, t_1) * M(x_n, x, a, t_2) * M(x, y_n, a, t), t_1 + t_2 = 0 \\ &\geq M(x_n, y_n, x, t_1) * M(x_n, x, a, t_2) * M(x, y_n, y, t_3) \\ &\quad * M(x, y, a, t_4) * M(y, y_n, a, t), t_3 + t_4 = 0 \end{aligned}$$

which implies $\underline{lim}_{n\to\infty} \; inf M(x_n, y_n, a, t) \geq 1*1*1*M(x, y, a, t)*1 = M(x, y, a, t)$ Also,

$$\begin{aligned} N(x_n, y_n, a, t) &\leq N(x_n, y_n, x, t_1) \diamondsuit N(x_n, x, a, t_2) \diamondsuit N(x, y_n, a, t), t_1 + t_2 = 0 \\ &\leq N(x_n, y_n, x, t_1) \diamondsuit N(x_n, x, a, t_2) \diamondsuit N(x, y_n, y, t_3) \\ &\quad \diamondsuit N(x, y, a, t_4) \diamondsuit N(y, y_n, a, t), t_3 + t_4 = 0 \end{aligned}$$

which implies $\underline{lim}_{n\to\infty} \; sup N(x_n, y_n, a, t) \leq 0 \diamondsuit 0 \diamondsuit 0 \diamondsuit N(x, y, a, t) \diamondsuit 0 = N(x, y, a, t)$

(2) Let $\epsilon > 0$ be given. For all $a \in x$ and $t > 0$ we have

$$\begin{aligned} M(x, y, a, t+2\epsilon) &\geq M(x, y, x_n, \frac{\epsilon}{2}) * M(x, x_n, a, \frac{\epsilon}{2}) * M(x_n, y, a, t+\epsilon) \\ &\geq M(x, y, x_n, \frac{\epsilon}{2}) * M(x, x_n, a, \frac{\epsilon}{2}) * M(x_n, y, y_n, \frac{\epsilon}{2}) \\ &\quad * M(x_n, y_n, a, t) * M(y_n, y, a, \frac{\epsilon}{2}). \end{aligned}$$

Consequently,

$$M(x,y,a,t+2\epsilon) \geq \underset{n\to\infty}{lim} \quad supM(x_n,y_n,a,t).$$

Letting $\epsilon \to 0$, we have

$$M(x,y,a,t+0) \geq \underset{n\to\infty}{lim} \quad supM(x_n,y_n,a,t).$$

Also, we have

$$\begin{aligned} N(x,y,a,t+2\epsilon) &\leq N(x,y,x_n,\frac{\epsilon}{2})\diamondsuit N(x,x_n,a,\frac{\epsilon}{2})\diamondsuit N(x_n,y,a,t+\epsilon) \\ &\leq N(x,y,x_n,\frac{\epsilon}{2})\diamondsuit N(x,x_n,a,\frac{\epsilon}{2})\diamondsuit N(x_n,y,y_n,\frac{\epsilon}{2}) \\ &\quad \diamondsuit N(x_n,y_n,a,t)\diamondsuit N(y_n,y,a,\frac{\epsilon}{2}). \end{aligned}$$

Consequently,

$$N(x,y,a,t+2\epsilon) \leq \underset{n\to\infty}{lim} \quad infN(x_n,y_n,a,t).$$

Letting $\epsilon \to 0$, we have

$$N(x,y,a,t+0) \leq \underset{n\to\infty}{lim} \quad infN(x_n,y_n,a,t).$$

Lemma 3.4 *Let $(X, M, N, *, \diamondsuit)$ be an IF2M-space and let A and B be continuous self mappings of X and $[A, B]$ are compatible. Let x_n be a sequence in X such that $Ax_n \to z$ and $Bx_n \to z$. Then $ABx_n \to Bz$.*

Proof: Since A, B are continuous maps, $ABx_n \to Az$, $BAx_n \to Bz$ and so, $M(ABx_n, Az, a, \frac{t}{3}) \to 1$ and $M(BAx_n, Bz, a, \frac{t}{3}) \to 1$ for all $a \in X$ and $t > 0$.
Since the pair $[A, B]$ is compatible, $M(BAx_n, ABx_n, a, \frac{t}{3}) \to 1$ for all or all $a \in X$ and $t > 0$. Thus

$$\begin{aligned} M(ABx_n,Bz,a,t) &\geq M(ABx_n,Bz,BAx_n,\frac{t}{3}) * M(ABx_n,BAx_n,a,\frac{t}{3}) \\ &\quad *M(BAx_n,Bz,a,\frac{t}{3}) \\ &\geq M(BAx_n,Bz,ABx_n,\frac{t}{3}) * M(BAx_n,ABx_n,a,\frac{t}{3}) \\ &\quad *M(BAx_n,Bz,a,\frac{t}{3}) \\ &\to 1 \end{aligned}$$

Also we have

$$\begin{aligned} N(ABx_n, Bz, a, t) &\leq N(ABx_n, Bz, BAx_n, \frac{t}{3}) \diamondsuit N(ABx_n, BAx_n, a, \frac{t}{3}) \\ &\quad \diamondsuit N(BAx_n, Bz, a, \frac{t}{3}) \\ &\leq N(BAx_n, Bz, ABx_n, \frac{t}{3}) \diamondsuit N(BAx_n, ABx_n, a, \frac{t}{3}) \\ &\quad \diamondsuit N(BAx_n, Bz, a, \frac{t}{3}) \\ &\to 0 \end{aligned}$$

for all $a \in X$ and $t > 0$.
Hence $ABx_n \to Bz$.

Theorem 3.5 *Let $(X, M, N, *, \diamondsuit)$ be a complete IF2M-space with continuous t-norm * and continuous t-conorm $\diamondsuit$. Let S and T be continuous self mappings of X. Then S and T have a unique common fixed point in X if and only if there exists two self mappings A, B of X satisfying*

(1) $AX \subset TX$, $BX \subset SX$,

(2) the pair $\{A, S\}$ and $\{B, T\}$ are compatible,

(3) there exists $q \in (0, 1)$ such that for every $x, y, a \in X$ and $t > 0$
$M(Ax_n, By, a, qt) \geq min\{M(Sx, Ty, a, t), M(Ax, Sx, a, t), M(By, Ty, a, t), M(Ax, Ty, a, t)\}$.
$N(Ax_n, By, a, qt) \leq max\{N(Sx, Ty, a, t), N(Ax, Sx, a, t), N(By, Ty, a, t), N(Ax, Ty, a, t)\}$. Then A, B, S and T have a unique common fixed point in X.

Proof: Suppose that S and T have a (unique) common fixed point say $z \in X$. Define $A : X \to X$ be $Ax = z$ for all $x \in X$, and $B : X \to X$ be $Bx = z$ for all $x \in X$.
Then one can see that (1)-(3) are satisfied.

Conversely, assume that there exist two self mappings A, B of X satisfying condition (1)-(3). From condition (1) we can construct two sequences x_n and y_n of X such that $y_{2n-1} = Tx_{2n-1} = Ax_{2n-2}$ and $y_{2n} = Sx_{2n} = Bx_{2n-1}$ for $n = 1, 2, 3, ...$ Putting $x = x_{2n}$ and $x = x_{2n+1}$ in condition (3), we have that for all $a \in X$ and $t > 0$

$$\begin{aligned} M(yx_{2n+1}, yx_{2n+2}, a, qt) &= M(Ax_{2n}, Bx_{2n+1}, a, qt) \\ &\geq min\{M(Sx_{2n}, Tx_{2n+1}, a, t), M(Ax_{2n}, Sx_{2n}, a, t) \end{aligned}$$

$$
\begin{aligned}
& M(Bx_{2n+1}, Tx_{2n+1}, a, t), M(Ax_{2n}, Tx_{2n+1}, a, t)\} \\
\geq \;& min\{M(yx_{2n}, yx_{2n+1}, a, qt), M(yx_{2n+1}, yx_{2n+1}, a, qt)\}
\end{aligned}
$$

and

$$
\begin{aligned}
N(yx_{2n+1}, yx_{2n+2}, a, qt) &= N(Ax_{2n}, Bx_{2n+1}, a, qt) \\
&\leq max\{N(Sx_{2n}, Tx_{2n+1}, a, t), N(Ax_{2n}, Sx_{2n}, a, t) \\
&\quad N(Bx_{2n+1}, Tx_{2n+1}, a, t), N(Ax_{2n}, Tx_{2n+1}, a, t)\} \\
&\leq max\{N(yx_{2n}, yx_{2n+1}, a, qt), N(yx_{2n+1}, yx_{2n+1}, a, qt)\}
\end{aligned}
$$

which implies $M(yx_{2n+1}, yx_{2n+2}, a, qt) \geq M(yx_{2n+1}, yx_{2n+1}, a, qt)$ and $N(yx_{2n+1}, yx_{2n+2}, a, qt) \leq N(yx_{2n+1}, yx_{2n+1}, a, qt)$,
by Lemma 3.1, Also, letting $x = x_{2n+2}$ and $y = x_{2n+1}$ in condition (3), we have that
$M(y_{2n+2}, y_{2n+3}, a, qt) \geq M(y_{2n+1}, y_{2n+2}, a, t)$ and
$N(y_{2n+2}, y_{2n+3}, a, qt) \leq N(y_{2n+1}, y_{2n+2}, a, t)$, for all $a \in X$ and $t > 0$.

In general we obtain that for all $a \in X$ and $t > 0$ and $n = 1, 2, ...$

$M(y_n, y_{n+1}, a, qt) \geq M(y_{n-1}, y_n, a, t)$ and
$N(y_n, y_{n+1}, a, qt) \leq N(y_{n-1}, y_n, a, t)$. Thus, for all $a \in X$ and $t > 0$ and $n = 1, 2, ...$

$$M(y_n, y_{n+1}, a, t) \geq M(y_0, y_1, a, \frac{t}{q^n}) \tag{3.1}$$

and

$$N(y_n, y_{n+1}, a, t) \leq N(y_0, y_1, a, \frac{t}{q^n}) \tag{3.2}$$

We now show that $\{y_n\}$ is a Cauchy sequence in X.
Let $m > n$. Then for all $a \in X$ and $t > 0$ we have

$$
\begin{aligned}
M(y_m, y_n, a, t) &\geq M(y_m, y_n, y_{n+1}, \frac{t}{3}) * M(y_{n+1}, y_n, a, \frac{t}{3}) * \\
&\quad M(y_m, y_{n+1}, a, \frac{t}{3}) \\
&\geq M(y_m, y_n, y_{n+1}, \frac{t}{3}) * M(y_{n+1}, y_n, a, \frac{t}{3}) * \\
&\quad M(y_m, y_{n+1}, y_{n+2}, \frac{t}{3^2}) * M(y_{n+2}, y_{n+1}, a, \frac{t}{3^2}) \\
&\quad M(y_m, y_{n+2}, a, \frac{t}{3^2})
\end{aligned}
$$

$$\begin{aligned} & . \\ & . \\ & . \\ & M(y_m, y_{m-n}, a, \frac{t}{3^{m-n}}) \end{aligned}$$

and

$$\begin{aligned} N(y_m, y_n, a, t) \;\leq\; & N(y_m, y_n, y_{n+1}, \frac{t}{3}) \Diamond N(y_{n+1}, y_n, a, \frac{t}{3}) \Diamond \\ & N(y_m, y_{n+1}, a, \frac{t}{3}) \\ \leq\; & N(y_m, y_n, y_{n+1}, \frac{t}{3}) \Diamond N(y_{n+1}, y_n, a, \frac{t}{3}) \Diamond \\ & N(y_m, y_{n+1}, y_{n+2}, \frac{t}{3^2}) \Diamond N(y_{n+2}, y_{n+1}, a, \frac{t}{3^2}) \\ & N(y_m, y_{n+2}, a, \frac{t}{3^2}) \\ & . \\ & . \\ & . \\ & N(y_m, y_{m-n}, a, \frac{t}{3^{m-n}}) \end{aligned}$$

letting $m, n \to \infty$ we have
$\underset{n\to\infty}{lim}\ M(y_m, y_n, a, t) = 1$, $\underset{n\to\infty}{lim}\ N(y_m, y_n, a, t) = 0$. Thus $\{y_n\}$ is a Cauchy sequence in X.
It follows from completeness of X that there exists $z \in X$ such that $\underset{n\to\infty}{lim}\ y_n = z$. Hence $\underset{n\to\infty}{lim}\ y_{2n-1} = \underset{n\to\infty}{lim}\ Tx_{2n-1} = \underset{n\to\infty}{lim}\ Ax_{2n-2} = z$ and $\underset{n\to\infty}{lim}\ y_{2n} = \underset{n\to\infty}{lim}\ Sx_{2n} = \underset{n\to\infty}{lim}\ Bx_{2n-1} = z$. From Lemma 3.4, $ASx_{2n+1} = Sz$ and $BTx_{2n+1} = Tz$ (3.3)
Mean while, for all $a \in X$ with $a \neq Sz$ and $a \neq Tz$ and $t > 0$.

$$\begin{aligned} M(ASx_{2n+1}, BTx_{2n+1}, a, qt) \;\geq\; & min\{M(SSx_{2n+1}, TTx_{2n+1}, a, t), \\ & M(ASx_{2n+1}, SSx_{2n+1}, a, t), \\ & M(BTx_{2n+1}, TTx_{2n+1}, a, t), \\ & M(ASx_{2n+1}, TTx_{2n+1}, a, t)\} \end{aligned}$$

and

$$\begin{aligned} N(ASx_{2n+1}, BTx_{2n+1}, a, qt) \;\leq\; & max\{N(SSx_{2n+1}, TTx_{2n+1}, a, t), \\ & N(ASx_{2n+1}, SSx_{2n+1}, a, t), \\ & N(BTx_{2n+1}, TTx_{2n+1}, a, t), \\ & N(ASx_{2n+1}, TTx_{2n+1}, a, t)\}. \end{aligned}$$

Taking limit as $n \to \infty$ and using (3.3), we have for all $a \in X$ with $a \neq Sz$ and $a \neq Tz$ and $t > 0$.

$$\begin{aligned} M(Sz,Tz,a,qt+0) &\geq min\{M(Sz,Tz,a,t), M(Sz,Sz,a,t),\\ &\quad M(Tz,Tz,a,t), M(Sz,Tz,a,t)\}\\ &\quad M(Sz,Tz,a,t) \end{aligned}$$

and

$$\begin{aligned} N(Sz,Tz,a,qt+0) &\leq max\{N(Sz,Tz,a,t), N(Sz,Sz,a,t),\\ &\quad N(Tz,Tz,a,t), N(Sz,Tz,a,t)\}\\ &\quad N(Sz,Tz,a,t) \end{aligned}$$

By Lemma 3.2, we have S z = T z (3.4)
From condition (3), we get for all $a \in X$ with $a \neq Az$, $a \neq Tz$ and $t > 0$

$$\begin{aligned} M(Az,BTx_{2n+1},a,qt) &\geq min\{M(Sz,TTx_{2n+1},a,t), M(Az,Sz,a,t),\\ &\quad M(BTx_{2n+1},TTx_{2n+1},a,t), M(Az,TTx_{2n+1},a,t)\} \end{aligned}$$

and

$$\begin{aligned} N(Az,BTx_{2n+1},a,qt) &\leq max\{N(Sz,TTx_{2n+1},a,t), N(Az,Sz,a,t),\\ &\quad N(BTx_{2n+1},TTx_{2n+1},a,t), N(Az,TTx_{2n+1},a,t)\} \end{aligned}$$

Taking limit as $n \to \infty$ and using condition (3), and Lemma 3.3, we have for all $a \in X$

$$\begin{aligned} M(Az,Tz,a,qt+0) &\geq min\{M(Sz,Tz,a,t), M(Az,Sz,a,t),\\ &\quad M(Tz,Tz,a,t), M(Az,Tz,a,t)\}\\ &\quad M(Az,Tz,a,t) \end{aligned}$$

and

$$\begin{aligned} N(Az,Tz,a,qt+0) &\leq max\{N(Sz,Tz,a,t), N(Az,Sz,a,t),\\ &\quad N(Tz,Tz,a,t), N(Az,Tz,a,t)\}\\ &\quad N(Az,Tz,a,t) \end{aligned}$$

By Lemma 3.2, we have, $Az = Tz$ (3.5)
And for all $a \in X$ with $a \neq Az$ and $a \neq Bz$, and $t > 0$.

$$\begin{aligned} M(Az,Bz,a,qt) &\geq min\{M(Sz,Tz,a,t), M(Az,Sz,a,t),\\ &\quad M(Bz,Tz,a,t), M(Az,Tz,a,t)\}\\ &\geq min\{M(Tz,Tz,a,t), M(Tz,Tz,a,t),\\ &\quad M(Bz,Az,a,t), M(Tz,Tz,a,t)\}\\ &\quad M(Az,Bz,a,t) \end{aligned}$$

and

$$\begin{aligned} N(Az, Bz, a, qt) &\leq min\{N(Sz, Tz, a, t), N(Az, Sz, a, t), \\ &\quad N(Bz, Tz, a, t), N(Az, Tz, a, t)\} \\ &\leq max\{N(Tz, Tz, a, t), N(Tz, Tz, a, t), \\ &\quad N(Bz, Az, a, t), N(Tz, Tz, a, t)\} \\ &\quad N(Az, Bz, a, t) \end{aligned}$$

By Lemma 3.2, $Az = Bz$ (3.6)
It follows that $Az = Bz = Sz = Tz$. For all $a \in X$ with $a \neq Bz$ and $a \neq z$, and $t > 0$.

$$\begin{aligned} M(Ax_{2n}, Bz, a, qt) &\geq min\{M(Sx_{2n}, Tz, a, t), M(Ax_{2n}, Sx_{2n}, a, t), \\ &\quad M(Bz, Tz, a, t), M(Ax_{2n}, Tz, a, t)\} \end{aligned}$$

and

$$\begin{aligned} N(Ax_{2n}, Bz, a, qt) &\leq max\{N(Sx_{2n}, Tz, a, t), N(Ax_{2n}, Sx_{2n}, a, t), \\ &\quad N(Bz, Tz, a, t), N(Ax_{2n}, Tz, a, t)\} \end{aligned}$$

Taking limit as $n \to \infty$ and using (3.3) and Lemma 3.3, we have for all $a \in X$ with $a \neq Bz$, $a \neq z$ and $t > 0$.

$$\begin{aligned} M(z, Bz, a, qt + 0) &\geq min\{M(z, Tz, a, t), M(z, z, a, t), \\ &\quad M(Bz, Bz, a, t), M(z, Tz, a, t)\} \\ &\geq M(z, Tz, a, t) \geq M(z, Bz, a, t) \end{aligned}$$

and

$$\begin{aligned} N(z, Bz, a, qt + 0) &\leq max\{N(z, Tz, a, t), N(z, z, a, t), \\ &\quad N(Bz, Bz, a, t), N(z, Tz, a, t)\} \\ &\leq N(z, Tz, a, t) \leq N(z, Bz, a, t), \end{aligned}$$

and so we have, $M(z, Bz, a, qt) \geq M(z, Bz, a, t)$ and $N(z, Bz, a, qt) \leq N(z, Bz, a, t)$, and hence $Bz = z$. Thus, $z = Az = Bz = Sz = Tz$, and so z is a common fixed point of A, B, C and T.
For uniqueness, let w be another common fixed point of A, B, S, T. Then, for all $a \in X$ with $a \neq z$, $a \neq w$ and $t > 0$.

$$M(z, w, a, qt) = M(Az, Bw, a, qt)$$

$$
\begin{aligned}
&\geq min\{M(Sz,Tw,a,t), M(Az,Sz,a,t),\\
&\quad M(Bw,Tw,a,t), M(Az,Tw,a,t)\}\\
&\geq min\{M(z,w,a,t), M(z,z,a,t),\\
&\quad M(w,w,a,t), M(z,w,a,t)\}\\
&\geq M(z,w,a,t).
\end{aligned}
$$

and

$$
\begin{aligned}
N(z,w,a,qt) &= N(Az,Bw,a,qt)\\
&\leq max\{N(Sz,Tw,a,t), N(Az,Sz,a,t),\\
&\quad N(Bw,Tw,a,t), N(Az,Tw,a,t)\}\\
&\leq max\{N(z,w,a,t), N(z,z,a,t),\\
&\quad N(w,w,a,t), N(z,w,a,t)\}\\
&\leq N(z,w,a,t).
\end{aligned}
$$

which implies that $M(z,w,a,qt) \geq M(z,w,a,t)$ and $N(z,w,a,qt) \geq N(z,w,a,t)$, hence $z = w$. This complete the proof of.

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