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Common Fixed Point Theorem on Compatible Mappings of Type (P)

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Abstract

The purpose of this paper is to prove a common fixed point theorem in a metric space which generalizes the result of Bijendra Singh and M.S. Chauhan using the weaker conditions such as compatible mappings of type (P) and associated sequence in place of compatibility and completeness of the metric space.

Keywords: *Fixed point, self maps, compatible mappings, compatible mappings of type (P), associated sequence.*

1 Introduction

G. Jungck [1] introduced the concept of compatible maps which is weaker than weakly commuting maps. Afterwards Jungck and Rhoades [4] defined weaker class of maps known as weakly compatible maps.

2 Definitions and Preliminaries

2.1 Compatible Mappings

Two self maps S and T of a metric space (X,d) are said to be compatible mappings if $\lim_{n\to\infty} d(STx_n,TSx_n)=0$, whenever $<x_n>$ is a sequence in X such that $\lim_{n\to\infty} Sx_n=\lim_{n\to\infty} Tx_n= t$ for some $t\in X$.

2.2 Compatible Mappings of Type (A)

Two self maps S and T of a metric space (X, d) are said to be compatible mappings of type(A) if $\lim_{n\to\infty} d(STx_n ,TTx_n) = 0$ and $\lim_{n\to\infty} d(TSx_n,SSx_n) = 0$ whenever $<x_n>$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t\in X$.

2.3 Compatible Mappings of Type (B)

Two self maps S and T of a Metric Space(X,d) are said to be compatible mappings of type(B) if

$$\lim_{n\to\infty} d(STx_n ,TTx_n) \le \frac{1}{2}[\lim_{n\to\infty} d(STx_n ,St) + \lim_{n\to\infty} d(St, SSx_n)]$$ and

$$\lim_{n\to\infty} d(TSx_n, SSx_n) \le \frac{1}{2}[\lim_{n\to\infty} d(TSx_n,Tt) + \lim_{n\to\infty} d(Tt, TTx_n)]$$ whenever

$<x_n>$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t\in X$.

2.4 Compatible Mappings of Type (P)

Two self maps S and T of a Metric Space (X,d) are said to be compatible mappings of type (P) if $\lim_{n\to\infty} d(SSx_n,TTx_n) = 0$, when ever $<x_n>$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t\in X$.

It is clear that every compatible pair is weakly compatible but its converse need not be true.

Bijendra Singh and M.S. Chauhan [5] proved the following theorem.

2.5 Theorem: *Let A, B, S and T be self mappings from a complete metric space (X,d) into itself satisfying the following conditions*

A(X) $\subseteq$ T(X) and B(X) $\subseteq$ S(X) …….(2.5.1)

one of A, B, S or T is continuous ……(2.5.2)

$$[d(Ax,By)]^2 \le k_1 [d(Ax,Sx)d(By,Ty)+d(By,Sx)d(Ax,Ty)] + k_2 [d(Ax,Sx)d(Ax,Ty)+d(By,Ty)d(By,Sx)] \quad \text{…… (2.5.3)}$$

where $0 \le k_1 + 2k_2 < 1, k_1, k_2 \ge 0$

The pairs (A, S) and (B,T) are compatible on X ..…. (2.5.4)

Further, if X is a complete metric space then A, B, S and T have a unique common fixed point in X.

Now, we generalize the theorem using compatible mappings of type (P) and associated sequence.

2.6 Associated Sequence

Suppose A, B, S and T are self maps of a metric space (X, d) satisfying the condition (2.5.1). Then for an arbitrary x0$\in$X such that Ax0 = Tx_1 and for this point x_1, there exist a point x_2 in X such that Bx1= Sx_2 and so on. Proceeding in the similar manner, we can define a sequence $<y_n>$ in X such that y2n=Ax_{2n}= Tx_{2n+1} and y2n_{+1}=Bx_{2n+2} = Sx_{2n+1} for n ≥ 0. We shall call this sequence as an "Associated sequence of x_0 "relative to the four self maps A, B, S and T.

Now we prove a lemma which plays an important role in our main Theorem.

2.7 Lemma: *Let A, B, S and T be self mappings from a complete metric space (X,d) into itself satisfying the conditions (2.5.1) and (2.5.3). Then the associated sequence $\{y_n\}$ relative to four self maps is a Cauchy sequence in X.*

Proof: From the conditions (2.5.1), (2.5.3) and from the definition of associated sequence we have

$$\begin{aligned}[d(y_{2n+1},y_{2n})]^2 &= [d(Ax_{2n},Bx_{2n-1})]^2 \\ &\le k_1 [d(Ax_{2n},Sx_{2n})\ d(Bx_{2n-1},Tx_{2n-1})+d(Bx_{2n-1},Sx_{2n})\ d(Ax_{2n},Tx_{2n-1}] \\ &+ k_2 [d(Ax_{2n},Sx_{2n})\ d(Ax_{2n},Tx_{2n-1})+d(Bx_{2n-1},Tx_{2n-1})\ d(Bx_{2n-1}Sx_{2n})]\end{aligned}$$

$$= k_1\left[d(y_{2n+1},y_{2n})\ d(y_{2n},y_{2n-1})+0\right]$$
$$+k_2\left[d(y_{2n+1},y_{2n})\ d(y_{2n+1},y_{2n-1})+0\right]$$

This implies

$$d(y_{2n+1},y_{2n})\le k_1\ d(y_{2n},y_{2n-1})+k_2\left[d(y_{2n+1},y_{2n})+d(y_{2n},y_{2n-1})\right]$$
$$d(y_{2n+1},y_{2n})\le h\ d(y_{2n},y_{2n-1})$$

where $h=\dfrac{k_1+k_2}{1-k_2}<1$

For every integer p>0, we get

$$d(y_n,y_{n+p})\le d(y_n,y_{n+1})+d(y_{n+1},y_{n+2})+\ldots\ldots\ldots\ldots+d(y_{n+p-1},y_{n+p})$$
$$\le h^n d(y_0,y_1)+h^{n+1}d(y_0,y_1)+\ldots\ldots\ldots\ldots+h^{n+p-1}d(y_0,y_1)$$
$$\le\left(h^n+h^{n+1}+\ldots\ldots\ldots\ldots+h^{n+p-1}\right)d(y_0,y_1)$$
$$\le h^n\left(1+h+h^2+\ldots\ldots\ldots\ldots+h^{p-1}\right)d(y_0,y_1)$$

Since h<1, $h^n\to 0$ as $n\to\infty$, so that $d\left(y_n,y_{n+p}\right)\to 0$. This shows that the sequence $\{y_n\}$ is a Cauchy sequence in X and since X is a complete metric space, it converges to a limit, say $z\in X$.

The converse of the Lemma is not true, that is A,B,S and T are self maps of a metric space (X,d) satisfying (2.5.1) and (2.5.3), even if for $x_0\in X$ and for associated sequence of x_0 converges, the metric space (X,d) need not be complete. The following example establishes this.

Example: Let X = [0, 1/2) with d(x,y) = $|x-y|$

Define self maps A, B, S and T of X by

Sx = Tx = $\dfrac{1}{2}-x$ if $x\in[0, 1/2)$ and

$$Ax=Bx=\begin{cases}1/4 & \text{if } x\in[0,1/4]\\ 1/3 & \text{if } x\in(1/4,1/2)\end{cases}$$

Clearly $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$ and the associated sequence $Ax_0, Bx_1Ax_2, Bx_3, .., Ax_{2n}, Bx_{2n+1}$……. converges to the point 1/4 ,but X is not a complete metric space.

Now, we generalize the above Theorem 2.5 in the following form.

3 Main Result

3.1 Theorem: *Let A, B, S and T are self maps of a metric space (X,d) satisfying the conditions*

$A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$ ....(3.1.1)

$$[d(Ax,By)]^2 \le k_1[d(Ax,Sx)d(By,Ty)+d(By,Sx)d(Ax,Ty)] + k_2[d(Ax,Sx)d(Ax,Ty)+d(By,Ty)d(By,Sx)] \quad \text{........(3.1.2)}$$

for all x, y in X where $0 \le k_1 + 2k_2 < 1, k_1, k_2 \ge 0$

One of A, B, S or T is continuous ....(3.1.3)

The pairs (A,S) and (B,T) are compatible mappings of type-P. ....(3.1.4)

The sequence Ax_0, Bx_1Ax_2, Bx_3,..,Ax_{2n}, Bx_{2n+1}…., converges to $z \in X$. ......(3.1.5)

Then A, B, S and T have a unique common fixed point in X.

Proof: From the condition (3.1.5), we have

$$Ax_{2n} \to z \quad and \quad Bx_{2n+1} \to z \quad as\ n \to \infty .$$

Suppose A is continuous. Then $AAx_{2n} \to Az$, $ASx_{2n} \to Az$ as $n \to \infty$

Since (A,S) is compatible of type-P, $\lim_{n\to\infty} d(AAx_{2n}, SSx_{2n}) = 0$. This gives $\lim_{n\to\infty} SSx_{2n} = Az$.

$$\lim_{n\to\infty} SSx_{2n} = \lim_{n\to\infty} AAx_{2n} = Az.$$

Put $x = Sx_{2n}$ $y = x_{2n+1}$

$$[d(ASx_{2n}, Bx_{2n+1})]^2 \le k_1 [d(ASx_{2n}, SSx_{2n})\ d(Bx_{2n+1}, Tx_{2n+1}) + d(Bx_{2n+1}, SSx_{2n})\ d(ASx_{2n}, Tx_{2n+1}]$$
$$+ k_2 [d(ASx_{2n}, SSx_{2n})\ d(ASx_{2n}, Tx_{2n+1}) + d(Bx_{2n+1}, Tx_{2n+1})\ d(Bx_{2n+1}, SSx_{2n}]$$

Letting $n \to \infty, Bx_{2n+1}, Tx_{2n+1} \to z$, then we get

$$[d(Az, z)]^2 \le k_1 [d(Az, Az)\ d(z, z) + d(z, Az)\ d(Az, z)]$$
$$+ k_2 [d(Az, Az)\ d(Az, z) + d(z, z)\ d(z, Az)]$$
$$[d(Az, z)]^2 \le k_1 [d(Az, z)]^2$$
$$[d(Az, z)]^2 (1 - k_1) \le 0$$
$$d(Az, z) \le 0$$
$$d(Az, z) = 0$$
$$Az = z$$

Since $A(X) \subseteq T(X)$ implies there exists $u \in X$ such that z=Az=Tu

To prove $Bu = z$

Put $x = x_{2n}, y = u$

$$[d(Ax_{2n}, Bu)]^2 \le k_1 [d(Ax_{2n}, Sx_{2n}) d(Bu, Tu) + d(Bu, Sx_{2n}) d(Ax_{2n}, Tu)]$$
$$+ k_2 [d(Ax_{2n}, Sx_{2n}) d(Ax_{2n}, Tu) + d(Bu, Tu) d(Bu, Sx_{2n})]$$
$$[d(z, Bu)]^2 \le k_1 [d(z, z) d(Bu, z) + d(Bu, z) d(z, z)]$$
$$+ k_2 [d(z, z) d(z, z) + d(Bu, z) d(Bu, z)]$$
$$[d(Bu, z)]^2 \le k_2 [d(Bu, z)]^2$$
$$(1 - k_2)[d(Bu, z)]^2 \le 0$$
$$[d(Bu, z)] = 0$$
$$Bu = z$$

Therefore $Bu = Tu = z$

Since (B, T) is compatible of type-P and z=Bu=Tu, we get

$d(BBu, TTu) = 0$.*This gives* $d(Bz, Tz) = 0$ *or* $Bz = Tz$.
Since $B(X) \subseteq S(X)$, *there exists* $v \in V$ *such that* $z = Bz = Sv$.

To prove $Av = z$ *put* $x = v$ *and* $y = z$

$$[d(Av, Bz)]^2$$
$$\leq k_1[d(Av, Sv)\ d(Bz, Tz) + d(Bz, Sv)\ d(Av, Tz)]$$
$$+ k_2[d(Av, Sv)\ d(Av, Tz) + d(Bz, Tz)\ d(Bz, Sv)]$$

$$[d(Av, z)]^2 \leq k_1[d(Av, z)d(z, z) + d(z, z)d(Av, z)]$$
$$+ k_2[d(Av, z)d(Av, z) + d(z, z)d(z, z)]$$

$$[d(Av, z)]^2 \leq k_2[d(Av, z)]^2$$

$$(1 - k_2)[d(Az, z)]^2 \leq 0$$

$$[d(Av, z)] = 0$$

$$Av = z$$

Therefore $z = Av = Sv$

Since the pair (A.S) is compatible type-P and z=Av=Sv, we get d(AAv.SSv)=0

which implies AAv=SSv or Az=Sz.

Since Az=Bz=Sz=Tz=z, we get z is a common fixed point of A, B, S and T. The uniqueness of the fixed point can be easily proved.

3.2 Remark: From the example given above, clearly the pairs (A, S) and (B, T) are not commutative and it can be easily verified that the mappings are not compatible, compatible of type (A), and also not compatible of type (B) but they are compatible of type (P). Also the rational inequality (3.1.2) holds for the values of $0 \leq k_1 + 2k_2 < 1$, *where* $k_1, k_2 \geq 0$. We note that X is not a complete metric space and it is easy to prove that the associated sequence $Ax_0, Bx_1 Ax_2, Bx_3, .., Ax_{2n}, Bx_{2n+1}$……. converges to the point 1/4 which is a common fixed point of A, B, S and T. Infact 1/4 is the unique common fixed point of A, B, S and T.

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