Shooting Method via Taylor Series for Solving Two Point Boundary Value Problem on an Infinite Interval

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(Received: 14-7-12/ Accepted: 14-3-14)

Abstract

A modified method of finding a guess for the starting point in the shooting method was applied to third order two point boundary value problems in an infinite domain. The Boundary Value Problems (BVPS) in the infinite domain were reduced to Initial Value Problems (IVPS) to the original differential equations. The systems of equations obtained were solved iteratively using Taylor series method of order nine. The initial guess obtained for each problem and the usage of Taylor series allow for quick convergence. The results obtained were compared with those obtained in the literature and they were found to be better.

Keywords: Starting point, Shooting method, BVPs, IVPs, Taylor series method.

1 Introduction

Many attempts have been made to solving third order boundary value problems on infinite intervals which do arise in engineering and Applied sciences. Numerical methods such as shooting via Runge Kutta method, finite difference method have been so far helpful. The main hurdle in the solution of such problems is the absense of second derivative \(f''(0)\) for a third order differential
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equation. Once this derivative have been evaluated an analytical solution of the boundary value problem may be found. Wang[9] derived a transformation for a third order BVPS of this type where adomian decomposition method was used, Faiz Ahmad and Wafaa Alhasan Albarakati[4] used Pade approximation to solve Blasius equation which is in the same category equation. Odejide and Aregbesola[11] used the method of weighted residual via partition method to solve this type of problem. Various other methods have also been applied such as differential transform method(DTM) by Xiao-hong SU and Lian-eun Zheng[12].

In this paper shooting method was used with a modified way of finding the initial starting point which were updated using the powerful Newton iteration scheme. The initial value problems so obtained were approximated by the Taylor series method. Though the method involves usage of higher derivatives but results obtained with the method is of very high accuracy.

2 Method of Solution

2.1 Initial Guess

Suppose we have a boundary value problem in a semi infinite domain governed by the differential equations

\[ f'''(\eta) + B_1 f(\eta)f''(\eta) + B_2 (1 - f'(\eta)^2) = 0 \] (1)

\[ f(0) = 0, f'(0) = 0, f'(\infty) = 1 \] (2)

and

\[ f'''(\eta) + f(\eta)f''(\eta) + B_3 f'(\eta) + B_4 f'^2 = 0 \] (3)

\[ f(0) = 0, f'(0) = 1, f'(\infty) = 0 \] (4)

where \( f' \) is the derivative of \( f \) with respect to \( \eta \), \( B_1, B_2, B_3 \) and \( B_4 \) are constants. Considering equation(1) with the boundary condition (2), we derive a function

\[ f_0 = \eta + a + be^{-p\eta} \] (5)

that satisfies the boundary conditions.

Imposing the boundary conditions (2) on equation (5) and solving for \( a \) and \( b \) simultaneously we have

\[ f_0 = \eta + \frac{1}{p}(e^{-p\eta} - 1) \] (6)

Equation (6) in equation (1) gives the residual \( F(\eta) \).

Multiplying the residual by \( e^{-p\eta} \) and integrating over the whole semi-infinite domain, we have
\[ \int_0^\infty e^{-p\eta}F(\eta)d\eta = 0 \] (7)

We note that equation (7) is the Laplace Transform of the function \( F(\eta) \).

Evaluating the integral equation (7), we have
\[ p = \frac{1}{6} \sqrt{6B_1 + 48B_2} \]

But from equation (6) \( f''(0) = p \) which can then serve as the initial guess for the shooting angle in the shooting technique for the third order differential equation.

Following the same procedure as discussed for equation (1) and (2) for equation (3) and (4), we have
\[ f = \frac{1}{p}(1 - e^{-pn}) \] (8)

with
\[ p = \frac{1}{3} \sqrt{3 - 9B_3 - 6B_4} \]

and \( f''(0) = -p \)

### 2.2 Shooting Method

Applying the shooting method discussed in [6] and suppose that the solution of equation (1) depends on the parameters \( \eta \) and \( \alpha \) then
\[
\frac{\partial}{\partial \alpha} f''' = \frac{\partial}{\partial \eta} (-B_1ff'' - B_2(1 - f^2)) \frac{\partial f}{\partial \alpha} + \frac{\partial}{\partial \eta} (-B_1ff'' - B_2(1 - f^2)) \frac{\partial f'}{\partial \alpha} + \frac{\partial}{\partial \eta^2} (-B_1ff'' - B_2(1 - f^2)) \frac{\partial f''}{\partial \alpha}
\]
giving
\[ \frac{\partial}{\partial \alpha} f''' = -B_1f'' \frac{\partial f}{\partial \alpha} + 2B_2f' \frac{\partial f'}{\partial \alpha} - B_1f \frac{\partial f''}{\partial \alpha} \] (9)

\[ \frac{\partial f}{\partial \alpha}(0) = 0, \quad \frac{\partial f'}{\partial \alpha}(0) = 0, \quad \frac{\partial f''}{\partial \alpha}(0) = 1 \] (10)

Equation (1) can also be expressed in the form
\[ f''' = -B_1ff'' - B_2(1 - f^2) \] (11)
\[ f(0) = 0, \quad f'(0) = 0, \quad f''(0) = \alpha_m \] (12)

where \( \alpha_m \) is a sequence of initial guess that can be updated by Newton iterative formular as given in [6]. For this problem it will be
\[ \alpha_{m+1} = \alpha_m - \frac{f'(L, \alpha_m) - 1}{\frac{\partial f'}{\partial \alpha}(L, \alpha_m)} \] (13),
where \( L \) is the truncated value of infinity in the boundary condition, equations (9), (10), (11), (12) are reduced to systems of initial value problems which are solved alongside with equation (13) using Taylor series method of order nine repeatedly until

\[
\lim_{n \to \infty} f'(L, \alpha_n) \approx f'(L) = 1
\]

Now considering equations (3) and (4), using the same procedure for shooting as in equations (1) and (2) we have the following initial value problems

\[
f''' + f''f' + B_3f' + B_4f'^2 = 0 \tag{14}
\]

\[
f(0) = 0, \ f'(0) = 0, \ f''(0) = \alpha_m \tag{15}
\]

\[
\frac{\partial f'''}{\partial \alpha} = -f''\frac{\partial f}{\partial \alpha} - (B_3 + 2B_4f') \frac{\partial f'}{\partial \alpha} - f \frac{\partial f''}{\partial \alpha} \tag{16}
\]

\[
\frac{\partial f}{\partial \alpha}(0) = 0, \ \frac{\partial f'}{\partial \alpha}(0) = 0, \ \frac{\partial f''}{\partial \alpha}(0) = 1 \tag{17}
\]

and the Newton iterative formular to determine the next guess for this problem becomes

\[
\alpha_{m+1} = \alpha_m - \frac{f'(L, \alpha_m) - 0}{\frac{\partial f'}{\partial \alpha}(L, \alpha_m)} \tag{18}
\]

Equations (14) to (17) are solved using Taylor series method of order nine with equation (18) repeatedly untill

\[
\lim_{n \to \infty} f'(L, \alpha_n) \approx f'(L) = 0
\]

### 2.3 Taylor Series Approach

Considering equation (1),

\[
f''' = -B_1ff'' - B_2(1 - f'^2)
\]

\[
f'''' = -2B_1(f'f'' + f''f') + 2B_2(f'f''') + 2B_2(f'f'') \tag{19}
\]

other higher derivatives like \( f^v, f^vi, f^vii \) and \( f^viii \) are obtained as well for order nine. \( f(0), f'(0) \) are known from the given boundary conditions, also \( f''(0) \) have been calculated. This implies that

\[
f''(0) = -B_1f(0)f''(0) - B_2(1 - f'(0)^2) \tag{20}
\]

\[
f''''(0) = -2B_1(f(0)f''(0) + f'(0)f''(0)) + 2B_2(f'(0)f''(0)) \tag{21}
\]

where other higher derivatives \( f^v(0), f^vi(0), f^vii(0) \) and \( f^viii(0) \) are obtained in a similar manner. The general Taylor scheme for equation (1) are

\[
f_{n+1} = f_n + hf_n + \frac{h^2}{2!}f''_n + \frac{h^3}{3!}f'''_n + \ldots + \frac{h^8}{8!}
\]
\[ \begin{align*}
f'_{n+1} &= f'_{n} + hf''_{n} + \frac{h^2}{2!}f'''_{n} + \frac{h^3}{3!}f''''_{n} + \ldots + \frac{h^7}{7!}f^{viii}_{n} \\
f''_{n+1} &= f''_{n} + hf'''_{n} + \frac{h^2}{2!}f''''_{n} + \frac{h^3}{3!}f''''''_{n} + \ldots + \frac{h^6}{6!}f^{viii}_{n} \\
f'''_{n+1} &= -B_1 f'_{n+1} - B_2 (1 - f'^2_{n+1}) \\
f^{iv}_{n+1} &= -B_1 (f'_n f'''_{n+1} + f''_{n} + 1f''_{n+1}) + 2B_2 (f'_n f''_{n+1})
\end{align*} \]

The scheme for \( f^v_{n+1}, f^vi_{n+1}, f^vii_{n+1}, f^viii_{n+1} \) are also obtained from their corresponding derivatives. The same Taylor series procedure are applied to equations (3), (9) and (16) to obtain every necessary derivatives.

### 3 Numerical Examples

**Example 1:** Consider equation (3), if \( B_3 = B_4 = -1 \) this becomes the Hydromagnetic fluid problem obtained by Mostafa Mahmoud et al[2]

\[ f''' + ff'' - f' - f'^2 = 0 \quad (21) \]

\[ f(0) = 0, f'(0) = 1, f'(\infty) = 0 \]

\[ f''_{initial guess}(0) = -1.4142135623731 \]

For some problems values very close to the starting guess point and the origin give better convergence. This means that at times the starting value only provide information on where to guess, for this problem if we choose \( f''_{initial guess}(0) = -1.414 \) and using the procedure of shooting via Taylor discussed in Section 2 with \( L = 10 \), Solving the resulting initial value problems with \( h = 0.0025 \) that is 4000 iterations eight times we obtained

\[ f''_{Taylor iterate}(0) = -1.41421356006213 \]

which compares favourably with the exact solution

\[ f''_{exact}(0) = -1.41421356237310 \]

The iterative result is nearly equal to the exact solution to nineth place of decimal. Figure 1 shows how the residual of the original equation tends to zero.
Example 2: Consider equation (3), if $B_3 = 0, B_4 = -2$ this becomes the heat and mass problem obtained by E. Magyari and B. Keller [3]

\[ f''' + f''f - 2f'^2 = 0 \]  

\[ f(0) = 0, f'(0) = 1, f'(\infty) = 0 \]

Using the procedure of shooting via Taylor discussed in section two with $L = 10$, $f''_{\text{initial guess}}(0) = -1.290$. For some problems and this problem in particular, values very close to the starting guess point and the origin give better convergence. This means that at times the starting value only provide information on where to guess. For this problem if we choose $f''_{\text{initial guess}}(0) = -1.286, -1.28, -1.27$ the iteration will converge.

Solving the resulting initial value problems with $h = 0.0025$ that is 4000 iterations eight times we obtained

\[ f''_{\text{Taylor iterate}}(0) = -1.28181638009430 \]

which compares favourably with the solution

\[ f''_{\text{exact}}(0) = -1.281808 \]

obtained by E. Magyari and B. Keller [3]. Figure 2 shows how the residual of the original equation tends to zero.
Example 3: In equation (1), if $B_1 = B_2 = 1$ this becomes the popular Fakner Skan equation govern by

$$f''' + f''f + (1 - f'^2) = 0$$  \hspace{1cm} (21)$$

$$f(0) = 0, \ f'(0) = 0, \ f'(\infty) = 1$$

Using the procedure discussed in section two with $L = 10$, $f''_{\text{initial guess}}(0) = 1.224744871391$.

Solving the resulting initial value problems with $h = 0.001$ that is 6000 iterations eight times we obtained

$$f''_{\text{Taylor iterate}}(0) = 1.23258765688780$$

which compares favourably with the solution obtained through Runge Kutta method [11] which is

$$f''(0) = 1.232588$$

Figure 3 shows how the residual of the original equation tends to zero.
Example 4: In equation (1), if $B_1 = 1.0, B_2 = 0$ we have equation govern by

$$f''' + f f'' = 0$$

Using the procedure discussed in section two with $L = 10$, $f''_{\text{initial guess}}(0) = 0.4082482905$. Solving the resulting initial value problems with $h = 0.005$ that is 2000 iterations eight times we obtained $f''_{\text{Taylor iterate}}(0) = 0.469599988361082$ which compares favourably with the solution obtained by Asaithambi[1] which is $f''(0) = 0.469600$

Figure 4 shows how the residual of the original equation tends to zero.

Example 5: In equation (1), if $B_1 = 0.5, B_2 = 0$ we have equation govern by

$$f''' + 0.5 f f'' = 0$$

Using the procedure discussed in section two with $L = 10$, $f''_{\text{initial guess}}(0) = 0.4082482905$. Solving the resulting initial value problems with $h = 0.005$ that is 2000 iterations eight times we obtained $f''_{\text{Taylor iterate}}(0) = 0.332057337203765$ which compares favourably with the solution obtained by Howarth[8] which is $f''(0) = 0.33206$

Figure 5 shows how the residual of the original equation tends to zero.
4 Conclusion

Taylor series method of order nine was presented for third order non linear two point boundary value problems in an infinite domain via shooting method. The Taylor iterative technique was found very suitable and efficient. This is evident from the residual plot of each problem considered. Results obtained were compared with the exact solution when they are available and to the results in the literature where they are not, to determine the efficiency of the method discussed. The technique is found to be accurate and easy to use.

References


