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Co-PI Index of Some Chemical Graphs

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Abstract

The Co-PI index of a graph G, denoted by $Co-PI(G)$, is defined as $Co-PI(G) = \sum_{uv=e\in E(G)} \left|n_u^G(e) - n_v^G(e)\right|$. In this paper, the upper bounds for the Co-PI indices of the unilateral hexagonal chain, unilateral polyomino chain, V-phenylenic nanotubes and nanotori are obtained.

Keywords: *Co-PI Index, Chemical graphs.*

1 Introduction

All the graphs considered in this paper are connected and simple. A vertex $x \in V(G)$ is said to be *equidistant* from the edge $e = uv$ of G if $d_G(u,x) = d_G(v,x)$, where $d_G(u,x)$ denotes the distance between u and x in G. The degree of the vertex u in G is denoted by $d_G(u)$. For an edge $uv = e \in E(G)$, the number of vertices of G whose distance to the vertex u is smaller than the distance to the vertex v in G is denoted by $n_u^G(e)$; analogously, $n_v^G(e)$ is the number of vertices of G whose distance to the vertex v in G is smaller than the distance to the vertex u; the vertices equidistant from both the ends of the edge $e = uv$ are not counted.

The *vertex PI index* of G, denoted by $PI(G)$, is defined as $PI(G) =$

$\sum_{e=uv \in E(G)} \left(n_u^G(e) + n_v^G(e)\right)$. The *Co-PI index* of G, denoted by *Co-PI(G),* is defined as *Co-PI(G)=* $\sum_{e=uv \in E(G)} \left|n_u^G(e) - n_v^G(e)\right|$. The PI index of the graph G is a topological index related to equidistant vertices. Another topological index of G related to distance of G is the Wiener index of G, first introduced by Wiener, see [26]. Khadikar, Karmarkar and Agrawal [16] first introduced edge Padmakar-Ivan index of graphs and they investigated the chemical applications of the Padmakar-Ivan index. The mathematical properties of the PI_v and its applications in chemistry and nanoscience are well studied by Ashrafi and Loghman [1, 3], Ashrafi and Rezaei [2], Deng, Chen and Zhang [6], Khadikar [14], Khalifeh, Yousefi-Azari and Ashrafi [15], Klavzar [17] and Yousefi-Azari, Manoochehrian and Ashrafi [25]. The vertex PI indices of the tensor and strong products of graphs are studied in [22, 24]. The Co-PI indices of join, composition, corona product, generalized hierarchical product of two connected graphs are obtained in [12]. In [18, 19, 20], the PI indices of bridge graphs and chain graphs are discussed. In this paper, the upper bounds for the Co-PI indices of unilateral polyomino chain, unilateral hexagonal chain, V-phenylenic nanotubes and nanotori are obtained.

2 Co-PI Index of Hexagonal Chain

Hexagonal chain is one class of hexagonal system consisting of hexagonal. In hexagonal chain, each of two hexagonal has one common edge or no common vertex. Two hexagonal are adjacent if they have commen edge. No three or more hexagonal share one vertex. Each hexagonal has two adjacent hexagonal except hexagonal in terminus, and each hexagonal chain has two hexagonal in terminus.

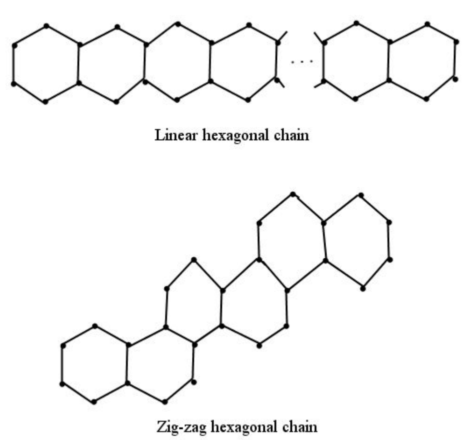


Figure 1. The structure of L_n^6 and Z_n^6

One can easy to check that the hexagonal chain with n hexagonal has $4n+1$ vertices and $5n+1$ edges. Let L_n^6 andZ_n^6 be the liner hexagonal chain and zig−zag hexagonal chain, respectively, see Figure 1.

The following cut method was presented in Gutman and klavazar [9]. Choose an edge e of the hexagonal system and draw a straight line through the center of e, orthogonal on e. This line will intersect the perimeter in two end points P_1 and P_2. The straight line segment C whose end points are P_1 and P_2 is the elementary cut,intersecting the edge e. A fragment S in hexagonal chain is just maximal linear chain which includes the hexagonal in start and end vertices. Let $l'(S)$ be the length of fragment which denotes the number of hexagonal contained. LetH_n^6 be a hexagonal chain with n hexagonal consisting of fragment sequence $S_1, S_2, \ldots, S_m,\ m \geq 1$. Denote $l'(S_i) = l'_i,\ i = 1, 2, \ldots, m$. Then we check that $l'_1 + l'_2 + \cdots + l'_m = n + m - 1$ since each two adjacent fragment have one common hexagonal. For the k^{th} fragment of hexagonal chain, the cut of this fragment is the cut which intersects with$l'_k + 1$ parallel edges of hexagonal in this fragment. A fragment is called horizontal fragment if its cut parallels the horizontal direction; otherwise, it is called inclined fragment. Unilateral hexagonal chain is a special class of hexagonal chain such that the cut for each inclined fragment at the same angle with a horizontal direction. Clearly, the linear hexagonal chain L_n^6 is a unilateral hexagonal chain with one fragment, and zig-zag is a unilateral hexagonal chain with $n-1$ fragments, see Figure 2.

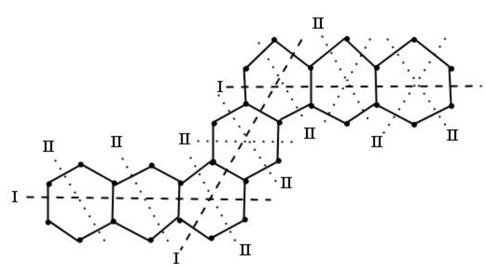


Figure 2. Type I and type II cuts of unilateral hexagonal chain

Theorem 2.1. *Let H_n^6be a unilateral hexagonal chain consisting of m fragment $S_1, S_2, \ldots, s_m,\ m \geq 1$, and let $l'(S_i) = l'_i,\ i = 1, 2, \ldots, m$ be the length of each fragment. Then*

$Co-PI(H_n^6) \leq 16l'_1(1'_1-1) - 16(n-1)(l'_1-1) + 16m(m-2)(l'_k-2) - 16(l'_k-2)(m(m-1)-2) + 16(m-2)(l'_k(l'_k-1)-2) + 16(l'_m(l'_m+1)-2) - 16m(l'_m-1) + 4(m+1)(n+2m-3-l'_1-l'_m) - 4(m(m-1)-2) + 2\sum_{k=1}^{m-1}\Big(4\sum_{i=1}^{k} l'_i -$

$4\sum_{i=k+1}^{m} l'_i\Big) + 4\sum_{k=2}^{m-1}\sum_{j=2}^{l'_k-1}\Big(4\sum_{i=1}^{k-1} l'_i - 4\sum_{i=k}^{m} l'_i\Big) + 4\sum_{j=2}^{l'_m}\Big(4\sum_{i=1}^{m-1} l'_i - 4l'_m\Big) + \sum_{k=2}^{m-1}(l'_k +$
$1)\Big(4\sum_{i=1}^{k-1} l'_i - 4\sum_{i=k+1}^{m} l'_i\Big) - 8\sum_{k=2}^{m-1} kl'_k.$

Proof: The cuts in H_n^6 are divided into two types: type I and type II, see Figure 2. An edge is called type I if it intersects with type I cut. Also, an edge is called type II if it intersects with type II cut.

Case 1: If edge e is type I in j^{th} square of k^{th} fragment, then we have the following subcases.

- If $k = 1$, then we have $n_1(e) = 2l'_1+1$ and $n_2(e) = 4\sum_{i=1}^{m}(l'_i-m+1)+2l'_1+1$.

- If $k = m$, then we obtain $n_1(e) = 4\sum_{i=1}^{m-1}(l'_k - m + 1) + 2l'_1 + 1$ and $n_2(e) = 2l'_m + 1$.

- If $2 \leq k \leq m-1$, then we get $n_1(e) = 4\sum_{i=1}^{k-1}(l'_i - k + 1) + 2l'_k + 1$ and $n_2(e) = 4\sum_{i=1}^{m}(l'_i - m + 1) + 2l'_k + 1$.

Case 2: If edge e is type II in j^{th} square of k^{th} fragment, then we have the following subcases.

- If $k = 1$,then we have $n_1(e) = 4j - 1$ and $n_2(e) = 4\sum_{i=2}^{m} l'_i + 4l'_1 - 4j + 3$.

- If $k = m$, then we obtain $n_1(e) = 4\sum_{i=1}^{m-1}(l'_i - k + 1) + 4j - 1$ and $n_2(e) = 4l'_m - 4j + 3$.

- If $2 \leq k \leq m-1$, then we deduce $n_1(e) = 4\sum_{i=1}^{k-1}(l'_i - k + 1) + 4j - 1$ and $n_2(e) = 4\sum_{i=k+1}^{m} l'_i + 4(l'_k - j) + 3$. In particular, if $j = l'_k$, we have $n_1(e) = 4\sum_{i=1}^{k}(l'_i - k + 1) - 1$ and $n_2(e) = 4\sum_{i=k+1}^{m}(l'_i - m + k) + 3$.

Thus, the following is obtained by combining the above cases and the definition of Co-PI index.

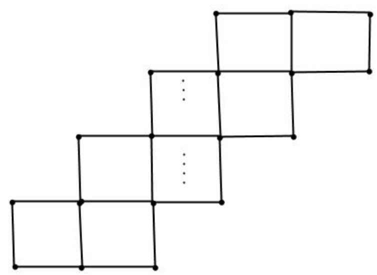
$$
\begin{aligned}
Co-PI(H_n^6) &= 4\sum_{j=1}^{l'_1-1}\left|(4j-1)-((4\sum_{i=2}^{m} l'_i-m+1)+4l'_1-4j+3)\right| \\
&+ 2\sum_{k=1}^{m-1}\left|(4(\sum_{i=1}^{k} l'_i-k+1)-1)-(4(\sum_{i=k+1}^{m} l'_i-m+k)+3)\right| \\
&+ 4\sum_{k=2}^{m-1}\sum_{j=2}^{l'_k-1}\left|(4(\sum_{i=1}^{k-1} l'_i-k+1)+4j-1)-(4(\sum_{i=k+1}^{m} l'_i-m+k)+4l'_k-4j+3)\right| \\
&+ 4\sum_{j=2}^{l'_m}\left|(4(\sum_{i=1}^{m-1} l'_i-m+1)+4j-1)-(4l'_m-4j+3)\right| \\
&+ \sum_{k=2}^{m-1}(l'_k+1)\left|(4(\sum_{i=1}^{k-1} l'_i-k+1)+2l'_k+1)-(4(\sum_{i=k+1}^{m} l'_i-m+k)+2l'_k+1)\right| \quad (1)
\end{aligned}
$$

Deduce (1) we obtain the required result.

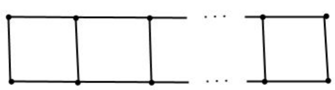
3 $Co-PI$ Index of Polyomino Chain

Polyomino is a finite 2-connected planar graph and each interior face is surrounded by a square with length 4. Polyomino chain is one class of polyomino such that the connection of centers for adjacent squares constitutes a path $c_1c_2\ldots c_n$, where c_i is the center of i^{th} square. Polyomino chain H_n^4 is called a linear chain if the subgraph induced by all 3- degree vertices is a graph with n-2 squares. Furthermore, polyomino chain H_n^4 is called a zig-zag chain if the subgraph induced by all vertices with degree > 2 is path with $n-1$ edges. In what follows, we use L_n^4 and Z_n^4 to denote linear polyomino chain and zig-zag polyomino chain, respectively. For the structure of L_n^4 and Z_n^4, see Figure 3.

Use the similar technology raised in Gutman and Klavzar [9] and we define elementary cut as follows. Choose an edge e of the polyomino system and draw a straight line through the center of e, orthogonal on e. This line will intersect the perimeter in two end points, P_1 and P_2. The straight line segment C whose end points are P_1 and P_2 is the elementary cut, intersecting the edge e. A fragment S in polyomino chain is just maximal linear chain which includes the squares in start and end vertices. Let $l(S)$ be the length of fragment which denotes the number of squares it contained. Let H_n^4 be a polyomino chain with n squares consisting of frgment sequence $S_1, S_2, \ldots S_m (m \geq 1)$. Denote $l(S_i) = l_i (i = 1, \ldots, m)$. It is not difficult to verify that $l_1 + l_2 + \ldots + l_m = n + m - 1$ and $|V(H_n^4)| = 2n + 2, |E(H_n^4)| = 3n + 1$. For the k^{th} fragment of polyomino chain, the cut of this fragment is the cut which intersects with $l_k + 1$ parallel



Zig-zag polyomino chain Z_n^4



Linear polyomino chain L_n^4

Figure 3.

edges of squares in this fragment. A fragment is called horizontal fragment if its cut parallels the horizontal direction and called vertical fragment if its cut parallels the vertical direction. Unilateral polyomino chain is a special kind of polyomino chain such that, for each vertical fragment, two horizontal fragments (if exists) are adjacent and it appears in the left and right sides, respectively. Now we obtain the $co-PI$ index of unilateral polyomino chain.

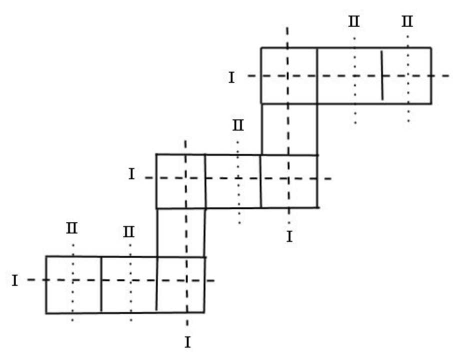


Figure 2. Type I and type II cut of unilateral polyomino chain

Theorem 3.1. *Let H_n^4 be a unilateral polyomino chain consisting of m fragment $S_1, S_2, \ldots, S_m (m \geq 1)$, and let $l(S_i) = l_i (i = 1, \ldots, m)$ be the length of each fragment. Then*

$Co-PI(H_n^4) \leq 3l_1(l_1-1) - 4(l_1-1)(n+m-1) + 2(1-2m)(l_m-1) + 2(l_m(l_m+1)-2) + 2(m-2)(l_k-2) + 3(m-2)(l_k(l_k-1)-2) - 3(l_k-2)(m(m-$

$1)-2)+2\sum_{k=2}^{m-1}\sum_{j=2}^{l_k-1}\Big(2\sum_{i=1}^{k-1}l_i-\sum_{i=k}^{m}l_i)+2\sum_{j=2}^{l_m}(2\sum_{i=1}^{m-1}l_i-l_m)+\sum_{k=1}^{m}(l_k+1)\Big(2\sum_{i=1}^{k-1}l_i-$
$2\sum_{i=k+1}^{m}l_i\Big)+2(m+1)\sum_{k=1}^{m}(l_k+1)-4\sum_{k=1}^{m}k(l_k+1).$

Proof: The cuts in H_n^4 are divided into two types type I and type II, see Figure 4. An edge is called type I(resp. type II) if it intersects with type I(resp. type II)cut. Now, we consider the following two cases.

Case 1: If edge e is $I-$type in j^{th} square of k^{th} fragment, we observe that there is l_k+1 such edges in k^{th} fragment.

- If $k=1$, then we have $n_1(e)=l_1+1$ and $n_2(e)=2\sum_{i=2}^{m}l_i-2m+l_1+3$.
- If $k=m$,then we obtain $n_1(e)=2\sum_{i=1}^{m-1}l_i-2m+3$ and $n_2(e)=l_m+1$.
- If $2\leq k\leq m-1$, then we get $n_1(e)=2\sum_{i=1}^{k-1}l_i-2k+l_k+3$, $n_2(e)=2\sum_{i=k+1}^{m}l_i-2(m-k)+l_k+1$.

Case 2: If edge e is type II in j^{th} square of k^{th} fragment, then we observe the following subcases.

- If $k=1$, then we have $n_1(e)=2j$ and $n_2(e)=2\sum_{i=2}^{m}l_i-2m+2(l_1-j)+4$.
- If $k=m$, then we have $n_1(e)=2\sum_{i=1}^{m-1}l_i-2m+2j+2$ and $n_2(e)=2l_m-2j+2$.
- If $2\leq k\leq m-1$, then we deduce $n_1(e)=2\sum_{i=1}^{k-1}l_i-2k+2j+2$ and $n_2(e)=2\sum_{i=k+1}^{m}l_i-2(m-k)+2(l_k-j)+2$.

Hence, the following is obtained by combining the above case and the definition of the Co-PI index.

$$
\begin{aligned}
& Co-PI(H_n^4) = 2\sum_{j=1}^{l_1-1}\left|j-(2\sum_{i=2}^{m} l_i - 2m + 2(l_1-1)+4)\right| \\
+ \quad & 2\sum_{j=2}^{l_m}\left|(2\sum_{i=1}^{m-1} l_i - 2m + 2j + 2) - (l_m - j + 1)\right| \\
& +2\sum_{k=2}^{m-1}\sum_{j=2}^{l_k-1}\left|(2\sum_{i=1}^{k-1} l_i - 2k + 2j + 2) - (\sum_{j=k+1}^{m} l_i - (m-k) + (l_k - j) + 1)\right| \\
+ \quad & \sum_{k=1}^{m}(l_k+1)\left|(2\sum_{i=1}^{k-1} l_i - 2k + l_k + 3) - (2\sum_{i=k+1}^{m} l_i - 2(m-k) + l_k + 1)\right| \\
\leq \quad & 2\sum_{j=1}^{l_1-1}(3j - 2n + 2m - 2) + 2\sum_{j=2}^{l_m}\Big(2\sum_{i=1}^{m-1} l_i - l_m - 2m + 2j + 1\Big) \\
& +2\sum_{k=2}^{m-1}\sum_{j=2}^{l_k-1}\Big(2\sum_{i=1}^{k-1} l_i - \sum_{j=k}^{m} l_i - 3k + 3j + 1\Big) \\
+ \quad & \sum_{k=1}^{m}(l_k+1)\Big(2\sum_{i=1}^{k-1} l_i - 2\sum_{i=k+1}^{m} l_i - 4k + 2m + 2\Big). \qquad (2)
\end{aligned}
$$

Deduce (2), we obtain the required result.

4 $Co-PI$ Index of V-Phenylenic Nanotubes

The molecular structures $V-$phenylenic nanotubes is denoted by $VPHX[m,n]$, see Figure 5. One can see that the number of vertices of $VPHX[m,n]$ is $6mn$. We calculate the $Co-PI$ index of $VPHX[m,n]$, we assume that E_1, E_2 and E_3 are the set of all vertical, oblique and horizontal edges, respectively.

Theorem 4.1. *The $Co-PI$ index of $VPHX[m,n]$ is*

$$Co-PI(VPHX[m,n]) \leq \alpha - \tfrac{3mn(n-1)(8m+6n-5)}{2}, \text{ where}$$

$$
\alpha = \begin{cases} 2(|m-n|-1)((2\beta+1)S_\beta - 6mn) \\ +(2\beta+1)(6\beta-3)(|m-n|)\frac{|m-n|-1}{2}, & \text{if } m \neq n, \\ (4n+1)S_n + 24n^2 - 6mn - 6n - 3, & \text{if } m = n. \end{cases}
$$

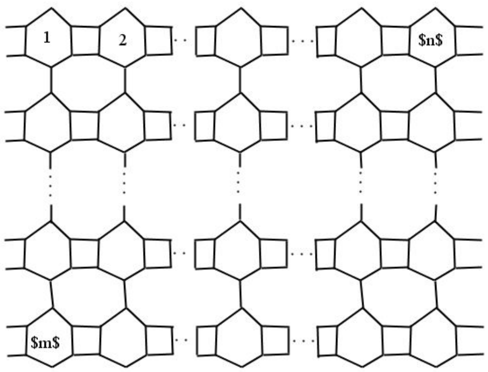


Figure 5. The structure of V-phenylenic nanotube

Proof: Let $G = VPHX[m, n]$. Then

$$\begin{aligned} co-PI(G) &= \sum_{e\in E(G)} |n(u)-n(v)| \\ &= \Big[\sum_{e\in E_1} + \sum_{e\in E_2} + \sum_{e\in E_3}\Big] |n(u)-n(v)|, \end{aligned} \tag{3}$$

where E_1,E_2 and E_3 are the set of all vertical, oblique and horizontal edges, respectively.

First we obtain $\sum_{e\in E_1} |n(u)-n(v)|$.

$$\begin{aligned} \sum_{e\in E_1} |n(u)-n(v)| &= 2\sum_{i=1}^{n-1} |6mi-(6mn-6mi)(2m)| \\ &\leq 2\sum_{i=1}^{n-1} (6mi-12m^2n+12m^2i) \\ &= 2(-12m^2n)(n-1)+2(6m+12m^2)\tbinom{n}{2} \\ &= (6m+12m^2)n(n-1)-2(n-1)(12m^2n). \end{aligned} \tag{4}$$

Next we obtain $\sum_{e\in E_2} |n(u)-n(v)|$.

$$\begin{aligned} \sum_{e\in E_2} |n(u)-n(v)| &= \sum_{i=1}^{n-1} |3mi-(6mn-3mi)(2n)| \\ &\leq \sum_{i=1}^{n-1} [3mi-12mn^2+6mni] \\ &= (3m+6mn)\tbinom{n}{2}-12mn^2(n-1). \end{aligned} \tag{5}$$

Finally, we calculate $\sum_{e\in E_3} |n(u)-n(v)|$ by considering following two cases.

Case 1: If $m \neq n$, then

$$\begin{aligned}\sum_{e\in E_3} |n(u)-n(v)| &= 2\sum_{i=1}^{|m-n|-1} |2\beta(S_\beta+(6\beta-3)i)-(6mn-S_\beta-(6\beta-3)i)| \\ &\leq 2\sum_{i=1}^{|m-n|-1} ((2\beta S_\beta-6mn+S_\beta)+(2\beta+1)(6\beta-3)i)) \\ &= 2(|m-n|-1)(2\beta+1)s_\beta-6mn) \\ &\quad +(2\beta+1)(6\beta-3)(|m-n|)\frac{|m-n|-1}{2}, \qquad (6)\end{aligned}$$

where $S_i = 3+9+\ldots+(6i-3)$ and $\beta = min\{m,n\}$.

Case 2: If $m = n$, then

$$\begin{aligned}\sum_{e\in E_3} |n(u)-n(v)| &= |4n(S_n+6n-3)-(6mn-S_n-(6n-3))| \\ &\leq (4nS_n+24n^2-12n-6mn+S_n+6n-3) \\ &= (4n+1)S_n+24n^2-6mn-6n-3). \qquad (7)\end{aligned}$$

Using (4),(5),(6) and (7) in (3), we obtain the required result.

5 $Co-PI$ Index of V-Phenylenic Nanotori

The molecular structures $V-$phenylenic nanotorus is denoted by $VPHY[m,n]$, see Figure 3. One can see that the number of vertices of $VPHY[m,n]$ is $6mn$. We calculate the $Co-PI$ index of $VPHY[m,n]$, we assume that E_1, E_2 and E_3 are the set of all vertical, oblique and horizontal edges, respectively.

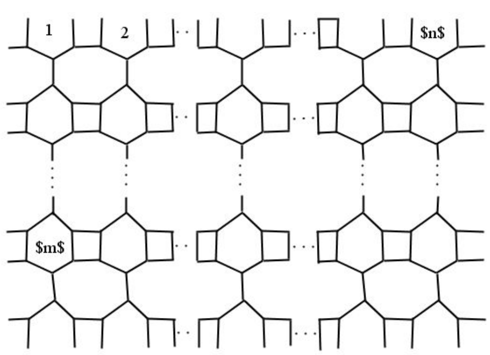


Figure 6. The structure of V-phenylenic nanotorus

Theorem 5.1. *The* $Co-PI$ *index of* $VPHY[m,n]$ *is*

$Co-PI(VPHY[m,n]) \leq \alpha_2 - 3mn(n-1)(4m+2n-3)$, where

$$\alpha_2 = \begin{cases} 4(|m-n|-1)((2\beta+1)S_\beta - 6mn) \\ +2(2\beta+1)(6\beta-3)\,|m-n|\,(|m-n|-1), & \text{if } m \neq n, \\ (4n+1)S_n + 24n^2 - 6mn - 6n - 3, & \text{if } \; m = n. \end{cases}$$

Proof: Let $G = VPHY[m,n]$. Let E_1, E_2 and E_3 be the set of all verticle, oblique and horizontal edges, respectively. Then by the definition of Co-PI

$$\begin{aligned} co-PI(G) &= \sum_{e\in E(G)} |n(u)-n(v)| \\ &= \left(\sum_{e\in E_1} + \sum_{e\in E_2} + \sum_{e\in E_3}\right) |n(u)-n(v)|\,. \end{aligned} \tag{8}$$

First we obtain $\sum\limits_{e\in E_1} |n(u)-n(v)|$.

$$\begin{aligned} \sum_{e\in E_1} |n(u)-n(v)| &= 2\sum_{i=1}^{n=1} |6mi - (6mn-6mi)2m| \\ &= 2\sum_{i=1}^{n-1} \left|6mi - 12m^2n + 12m^2i\right| \\ &= 2\sum_{i=1}^{n-1} \left|(6m+12m^2)i - 12m^2n\right| \\ &\leq 2\sum_{i=1}^{n-1} (6m+12m^2)i - 24m^2n(n-1) \\ &= 2(6m+12m^2)\tbinom{n}{2} - 24m^2n(n-1) \\ &= (6m+12m^2)n(n-1) - 24m^2n(n-1). \end{aligned} \tag{9}$$

Next we obtain $\sum\limits_{e\in E_2} |n(u)-n(v)|$.

$$\begin{aligned} \sum_{e\in E_2} |n(u)-n(v)| &= 2\sum_{i=1}^{n-1} |3mi - (6mn-3mi)(2n)| \\ &= 2\sum_{i=1}^{n-1} \left|3mi - 12mn^2 + 6mni\right| \\ &\leq 2\sum_{i=1}^{n-1} (3m+6mn)i - 24mn^2(n-1) \\ &= (3m+6mn)n(n-1) - 24mn^2(n-1). \end{aligned} \tag{10}$$

we calculate $\sum_{e\in E_2} |n(u)-n(v)|$ according to the relationship between m and n.

Case 1: If $m \neq n$, then we have

$$\begin{aligned}
\sum_{e\in E_3} |n(u)-n(v)| &= 4\sum_{i=1}^{|m-n|-1} |2\beta(S_\beta+(6\beta-3)i)-(6mn-S_\beta-(6\beta-3)i| \\
&\leq 4(|m-n|-1)[2\beta S_\beta - 6mn + S_\beta] \\
&\quad +4\sum_{i=1}^{|m-n|-1} [(2\beta(6\beta-3)+(6\beta-3))i] \\
&= 4(|m-n|-1)((2\beta+1)S_\beta-6mn) \\
&\quad +4(2\beta+1)(6\beta-3)\frac{|m-n|}{2}(|m-n|-1) \\
&= 4(|m-n|-1)((2\beta+1)S_\beta-6mn) \\
&\quad +2(2\beta+1)(6\beta-3)\,|m-n|\,(|m-n|-1). \qquad (11)
\end{aligned}$$

Case 2: If $m = n$, then we have

$$\begin{aligned}
\sum_{e\in E_3} |n(u)-n(v)| &= |4n(S_n+6n-3)-(6mn-S_n-(6n-3))| \\
&\leq (4nS_n+24n^2-12n-6mn+S_n+6n-3) \\
&= (4n+1)S_n+24n^2-6mn-6n-3. \qquad (12)
\end{aligned}$$

Using (9),(10),(11) and (12) in (8), we obtain the required result.

References

[1] A.R. Ashrafi and A. Loghman, PI index of zig-zag polyhex nanotubes, *MATCH Commun. Math. Comput. Chem.*, 55(2006), 447-452.

[2] A.R. Ashrafi and F. Rezaei, PI index of polyhex nanotori, *MATCH Commun. Math. Comput. Chem.*, 57(2007), 243-250.

[3] A.R. Ashrafi and A. Loghman, PI index of armchair polyhex nanotubes, *Ars Combin.*, 80(2006), 193-199.

[4] L. Barriere, F. Comellas, C. Dalfo and M.A. Fiol, The hierarchical product of graphs, *Discrete Appl. Math.*, 157(2009), 36-48.

[5] L. Barriere, C. Dalfo, M.A. Fiol and M. Mitjana, The generalized hierarchical product of graphs, *Discrete Math.*, 309(2009), 3871-3881.

[6] H. Deng, S. Chen and J. Zhang, The PI index of phenylenes, *J. Math. Chem.*, 41(2007), 63-69.

[7] J. Devillers and A.T. Balaban (Eds.), *Topological Indices and Related Descriptors in QSAR and QSPR*, Gordon and Breach, Amsterdam, The Netherlands, (1999).

[8] M. Eliasi and A. Iranmanesh, The hyper-Wiener index of the generalized hierarchical product of graphs, *Discrete Appl. Math.*, 159(2011), 866-871.

[9] I. Gutman and S. Klavzar, An algorithm for the calculation of the Szeged index of benzenoid hydrocarbons, *Journal of Chemical Information and Computer Sciences*, 35(1995), 1011-1014.

[10] M. Hoji, Z. Luo and E. Vumar, Wiener and vertex PI indices of Kronecker products of graphs, *Discrete Appl. Math.*, 158(2010), 1848-1855.

[11] A. Illić and N. Milosavljević, The Weighted vertex PI index, *Math. Comput. Model.*, 57(2013), 623-631.

[12] V. Kaladevi and G. Sharmiladevi, On Co-Pi index of graphs, *Ars Combin.*, (in Press).

[13] M.H. Khalifeh, H.Y. Azari and A.R. Ashrafi, Vertex and edge PI indices of cartesian product graphs, *Discrete Appl. Math.*, 156(2008), 1780-1789.

[14] P.V. Khadikar, On a novel structural descriptor PI, *Nat. Acad. Sci. Lett.*, 23(2000), 113-118.

[15] M.H. Khalifeh, H.Y. Azari and A.R. Ashrafi, The hyper-Wiener index of graph operations, *Comput. Math. Appl.*, 56(2008), 1402-1407.

[16] P.V. Khadikar, S. Karmarkar and V.K. Agrawal, A novel PI index and its application to QSPR/QSAR studies, *J. Chem. Inf. Comput. Sci.*, 41(2001), 934-949.

[17] S. Klavžar, On the PI index: PI-partitions and Cartesian product graphs, *MATCH Commun. Math. Comput. Chem.*, 57(2007), 573-586.

[18] T. Mansour and M. Schork, The PI index of bridge and chain graphs, *MATCH Commun. Math. Comput. Chem.*, 61(2009), 723-734.

[19] T. Mansour and M. Schork, The vertex PI index and Szeged index of bridge graphs, *Discrete Appl. Math.*, 157(2009), 1600-1606.

[20] T. Mansour and M. Schork, The PI index of polyomino chains of 4k-cycles, *Acta Appl. Math.*, 109(2010), 671-681.

[21] K. Pattabiraman, Weighted PI index of corona product of graphs, *Accepted in Discrete Math. Algorithm Appl.*, 1450055(6) (2014), 9 pages.

[22] K. Pattabiraman and P. Paulraja, On some topological indices of the tensor products of graphs, *Discrete Appl. Math.*, 160(2012), 267-279.

[23] K. Pattabiraman and P. Paulraja, Vertex and edge Padmakar-Ivan indices of generalized hierarchical product of graphs, *Discrete Appl. Math.*, 160(2012), 13761384.

[24] K. Pattabiraman and P. Paulraja, Wiener and vertex PI indices of the strong product of graphs, *Discuss. Math. Graph Theory*, 32(2012), 749-769.

[25] H.Y. Azari, B. Manoochehrian and A.R. Ashrafi, The PI index of product graphs, *Appl. Math. Lett.*, 21(2008), 624-627.

[26] H. Wiener, Structural determination of the paraffin boiling points, *J. Am. Chem. Soc.*, 69(1947), 17-20.