Co-PI Index of Some Chemical Graphs

G. Sharmiladevi\textsuperscript{1} and V. Kaladevi\textsuperscript{2}

\textsuperscript{1}Research Scholar, Research and Development Centre
Bharathiar University, Department of Mathematics
Kongu Arts and Science College, Erode, India
E-mail: sharmilashamritha@gmail.com

\textsuperscript{2}Department of Mathematics
Bishop Heber College, Trichy, India
E-mail: kaladevi1956@gmail.com

(Received: 3-4-16 / Accepted: 13-5-16)

Abstract
The Co-PI index of a graph $G$, denoted by $\text{Co-PI}(G)$, is defined as $\text{Co-PI}(G) = \sum_{uv \in E(G)} |n_u^G(e) - n_v^G(e)|$. In this paper, the upper bounds for the Co-PI indices of the unilateral hexagonal chain, unilateral polyomino chain, $V$-phenylenic nanotubes and nanotori are obtained.

Keywords: Co-PI Index, Chemical graphs.

1 Introduction
All the graphs considered in this paper are connected and simple. A vertex $x \in V(G)$ is said to be equidistant from the edge $e = uv$ of $G$ if $d_G(u, x) = d_G(v, x)$, where $d_G(u, x)$ denotes the distance between $u$ and $x$ in $G$. The degree of the vertex $u$ in $G$ is denoted by $d_G(u)$. For an edge $uv = e \in E(G)$, the number of vertices of $G$ whose distance to the vertex $u$ is smaller than the distance to the vertex $v$ in $G$ is denoted by $n^G_u(e)$; analogously, $n^G_v(e)$ is the number of vertices of $G$ whose distance to the vertex $v$ in $G$ is smaller than the distance to the vertex $u$; the vertices equidistant from both the ends of the edge $e = uv$ are not counted.

The vertex PI index of $G$, denoted by $PI(G)$, is defined as $PI(G) = \sum_{uv \in E(G)} |n^G_u(e) - n^G_v(e)|$. In this paper, the upper bounds for the Co-PI indices of the unilateral hexagonal chain, unilateral polyomino chain, $V$-phenylenic nanotubes and nanotori are obtained.
The \textit{Co-PI index} of $G$, denoted by $\text{Co-PI}(G)$, is defined as

$$\sum_{e = uv \in E(G)} \left( n_u^G(e) + n_v^G(e) \right).$$

The \textit{PI index} of the graph $G$ is a topological index related to equidistant vertices. Another topological index of $G$ related to distance of $G$ is the Wiener index of $G$, first introduced by Wiener, see [26]. Khadikar, Karmarkar and Agrawal [16] first introduced edge Padmakar-Ivan index of graphs and they investigated the chemical applications of the Padmakar-Ivan index. The mathematical properties of the $PI_v$ and its applications in chemistry and nanoscience are well studied by Ashrafi and Loghman [1, 3], Ashrafi and Rezaei [2], Deng, Chen and Zhang [6], Khadikar [14], Khalifeh, Yousefi-Azari and Ashrafi [15], Klavzar [17] and Yousefi-Azari, Manoochehrian and Ashrafi [25]. The vertex PI indices of the tensor and strong products of graphs are studied in [22, 24]. The Co-PI indices of join, composition, corona product, generalized hierarchical product of two connected graphs are obtained in [12]. In [18, 19, 20], the PI indices of bridge graphs and chain graphs are discussed. In this paper, the upper bounds for the Co-PI indices of unilateral polyomino chain, unilateral hexagonal chain, $V$-phenylenic nanotubes and nanotori are obtained.

\section{Co-PI Index of Hexagonal Chain}

Hexagonal chain is one class of hexagonal system consisting of hexagonal. In hexagonal chain, each of two hexagonal has one common edge or no common vertex. Two hexagonal are adjacent if they have common edge. No three or more hexagonal share one vertex. Each hexagonal has two adjacent hexagonal except hexagonal in terminus, and each hexagonal chain has two hexagonal in terminus.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{hexagonal-chain.png}
\caption{The structure of $SL_n$ and $SZ_n$}
\end{figure}
One can easily check that the hexagonal chain with \( n \) hexagonal has
\( 4n + 1 \) vertices and \( 5n + 1 \) edges. Let \( L^6_n \) and \( Z^6_n \) be the linear hexagonal chain and zig-zag hexagonal chain, respectively, see Figure 1.

The following cut method was presented in Gutman and klavazar [9]. Choose an edge \( e \) of the hexagonal system and draw a straight line through the center of \( e \), orthogonal on \( e \). This line will intersect the perimeter in two end points \( P_1 \) and \( P_2 \). The straight line segment \( C \) whose end points are \( P_1 \) and \( P_2 \) is the elementary cut, intersecting the edge \( e \). A fragment \( S \) in hexagonal chain is just maximal linear chain which includes the hexagonal in start and end vertices. Let \( l'(S) \) be the length of fragment which denotes the number of hexagonal contained. Let \( H^6_n \) be a hexagonal chain with \( n \) hexagonal consisting of fragment sequence \( S_1, S_2, \ldots, S_m, m \geq 1 \). Denote \( l'(S_i) = l'_i \), \( i = 1, 2, \ldots, m \). Then we check that \( l'_1 + l'_2 + \cdots + l'_m = n + m - 1 \) since each two adjacent fragment have one common hexagonal. For the \( k^{th} \) fragment of hexagonal chain, the cut of this fragment is the cut which intersects with \( l'_k + 1 \) parallel edges of hexagonal in this fragment. A fragment is called horizontal fragment if its cut parallels the horizontal direction; otherwise, it is called inclined fragment. Unilateral hexagonal chain is a special class of hexagonal chain such that the cut for each inclined fragment at the same angle with a horizontal direction. Clearly, the linear hexagonal chain \( L^6_n \) is a unilateral hexagonal chain with one fragment, and zig-zag is a unilateral hexagonal chain with \( n - 1 \) fragments, see Figure 2.

\[ C - PI(H^6_n) \leq 16l'_1(l'_1 - 1) - 16(n - 1)(l'_1 - 1) + 16m(m - 2)(l'_k - 2) - 16(l'_k - 2)(m(m - 1) - 2) + 16(l'_k - l'_k - 1 - 2) + 16(l'_m(l'_m + 1) - 2) - 16m(l'_m - 1) + 4(m + 1)(n + 2m - 3 - l'_1 - l'_m) - 4(m(m - 1) - 2) + 2 \sum_{k=1}^{m-1} \left( 4 \sum_{i=1}^{k} l'_i - \right. \]
\[
4 \sum_{i=k+1}^{m} l'_i \bigg) + 4 \sum_{k=2}^{m-1} \sum_{j=2}^{k-1} \left( 4 \sum_{i=1}^{k-1} (l'_i - 4 \sum_{i=k}^{m} l'_i) + 4 \sum_{j=2}^{l_m} \left( 4 \sum_{i=1}^{m-1} l'_i - 4l'_m \right) + \sum_{k=2}^{m-1} (l'_k + 1) \right) + 1 \left( 4 \sum_{i=1}^{k-1} l'_i - 4 \sum_{i=k+1}^{m} l'_i \right) - 8 \sum_{k=2}^{m-1} k l'_k.
\]

**Proof:** The cuts in \( H_n^6 \) are divided into two types: type I and type II, see Figure 2. An edge is called type I if it intersects with type I cut. Also, an edge is called type II if it intersects with type II cut.

**Case 1:** If edge \( e \) is type I in \( j^{th} \) square of \( k^{th} \) fragment, then we have the following subcases.

- If \( k = 1 \), then we have \( n_1(e) = 2l'_1 + 1 \) and \( n_2(e) = 4 \sum_{i=1}^{m} (l'_i - m + 1) + 2l'_1 + 1 \).

- If \( k = m \), then we obtain \( n_1(e) = 4 \sum_{i=1}^{m-1} (l'_k - m + 1) + 2l'_1 + 1 \) and \( n_2(e) = 2l'_m + 1 \).

- If \( 2 \leq k \leq m - 1 \), then we get \( n_1(e) = 4 \sum_{i=1}^{k-1} (l'_i - k + 1) + 2l'_1 + 1 \) and \( n_2(e) = 4 \sum_{i=1}^{m} (l'_i - m + 1) + 2l'_k + 1 \).

**Case 2:** If edge \( e \) is type II in \( j^{th} \) square of \( k^{th} \) fragment, then we have the following subcases.

- If \( k = 1 \), then we have \( n_1(e) = 4j - 1 \) and \( n_2(e) = 4 \sum_{i=2}^{m} l'_i + 4l'_1 - 4j + 3 \).

- If \( k = m \), then we obtain \( n_1(e) = 4 \sum_{i=1}^{m-1} (l'_i - k + 1) + 4j - 1 \) and \( n_2(e) = 4l'_m - 4j + 3 \).

- If \( 2 \leq k \leq m - 1 \), then we deduce \( n_1(e) = 4 \sum_{i=1}^{k-1} (l'_i - k + 1) + 4j - 1 \) and \( n_2(e) = 4 \sum_{i=k+1}^{m} l'_i + 4(l'_k - j) + 3 \). In particular, if \( j = l'_k \), we have \( n_1(e) = 4 \sum_{i=k+1}^{k} (l'_i - k + 1) - 1 \) and \( n_2(e) = 4 \sum_{i=k+1}^{m} (l'_i - m + k) + 3 \).

Thus, the following is obtained by combining the above cases and the def-
inition of Co-PI index.

$$Co - PI(H^0_n) = 4 \sum_{j=1}^{l'_1-1} (4j - 1) - ((4 \sum_{i=2}^{m} l'_i - m + 1) + 4l'_1 - 4j + 3)$$

$$+ 2 \sum_{k=1}^{m-1} \left| (4(\sum_{i=1}^{k-1} l'_i - k + 1) - 1) - (4(\sum_{i=k+1}^{m} l'_i - m + k) + 3) \right|$$

$$+ \sum_{k=2}^{m-1} \sum_{j=2}^{l'_k-1} \left| (4(\sum_{i=1}^{k-1} l'_i - k + 1) + 4j - 1) - (4(\sum_{i=k+1}^{m} l'_i - m + k) + 4l'_k - 4j + 3) \right|$$

$$+ \sum_{k=2}^{m-1} (l'_k + 1) \left| (4(\sum_{i=1}^{k-1} l'_i - k + 1) + 2l'_k + 1) - (4(\sum_{i=k+1}^{m} l'_i - m + k) + 2l'_k + 1) \right| (1)$$

Deduce (1) we obtain the required result.

3 Co – PI Index of Polyomino Chain

Polyomino is a finite 2-connected planar graph and each interior face is surrounded by a square with length 4. Polyomino chain is one class of polyomino such that the connection of centers for adjacent squares constitutes a path $c_1 c_2 \ldots c_n$, where $c_i$ is the center of $i^{th}$ square. Polyomino chain $H^4_n$ is called a linear chain if the subgraph induced by all 3-degree vertices is a graph with $n-2$ squares. Furthermore, polyomino chain $H^4_n$ is called a zig-zag chain if the subgraph induced by all vertices with degree > 2 is path with $n-1$ edges. In what follows, we use $L^4_n$ and $Z^4_n$ to denote linear polyomino chain and zig-zag polyomino chain, respectively. For the structure of $L^4_n$ and $Z^4_n$, see Figure 3.

Use the similar technology raised in Gutman and Klavzar [9] and we define elementary cut as follows. Choose an edge $e$ of the polyomino system and draw a straight line through the center of $e$, orthogonal on $e$. This line will intersect the perimeter in two end points, $P_1$ and $P_2$. The straight line segment $C$ whose end points are $P_1$ and $P_2$ is the elementary cut, intersecting the edge $e$. A fragment $S$ in polyomino chain is just maximal linear chain which includes the squares in start and end vertices. Let $l(S)$ be the length of fragment which denotes the number of squares it contained. Let $H^4_n$ be a polyomino chain with $n$ squares consisting of fragment sequence $S_1, S_2, \ldots, S_m (m \geq 1)$. Denote $l(S_i) = l_i (i = 1, \ldots, m)$. It is not difficult to verify that $l_1 + l_2 + \ldots + l_m = n + m - 1$ and $|V(H^4_n)| = 2n + 2, |E(H^4_n)| = 3n + 1$. For the $k^{th}$ fragment of polyomino chain, the cut of this fragment is the cut which intersects with $l_k + 1$ parallel
edges of squares in this fragment. A fragment is called horizontal fragment if its cut parallels the horizontal direction and called vertical fragment if its cut parallels the vertical direction. Unilateral polyomino chain is a special kind of polyomino chain such that, for each vertical fragment, two horizontal fragments (if exists) are adjacent and it appears in the left and right sides, respectively. Now we obtain the \( \text{co-PI} \) index of unilateral polyomino chain.

**Theorem 3.1.** Let \( H_n^4 \) be a unilateral polyomino chain consisting of \( m \) fragment \( S_1, S_2, \ldots, S_m (m \geq 1) \), and let \( l(S_i) = l_i (i = 1, \ldots, m) \) be the length of each fragment. Then

\[
\text{co-PI}(H_n^4) \leq 3l_1(l_1 - 1) - 4(l_1 - 1)(n + m - 1) + 2(1 - 2m)(l_m - 1) + 2(l_m(l_m + 1) - 2) + 2(m - 2)(l_k - 2) + 3(m - 2)(l_k(l_k - 1) - 2) - 3(l_k - 2)(m(m - 2)) + \cdots
\]
\[
1) - 2) + 2 \sum_{k=2}^{m-1} \sum_{j=2}^{l_k-1} \left( 2 \sum_{i=1}^{k-1} l_i - \sum_{i=k}^{l_m} l_i \right) + 2 \sum_{j=2}^{l_m} \left( 2 \sum_{i=1}^{k-1} l_i - l_m \right) + \sum_{k=2}^{m} \left( 2 \sum_{i=1}^{k-1} l_i - \sum_{i=k+1}^{l_m} l_i \right) + 2(m+1) \sum_{i=1}^{l_k} (l_k + 1) - 4 \sum_{k=2}^{m} \sum_{j=1}^{l_k} (l_k + 1).
\]

**Proof:** The cuts in \( H_4^n \) are divided into two types type I and type II, see Figure 4. An edge is called type I (resp. type II) if it intersects with type I (resp. type II) cut. Now, we consider the following two cases.

**Case 1:** If edge \( e \) is \( I \)-type in \( j \)-th square of \( k \)-th fragment, we observe that there is \( l_k + 1 \) such edges in \( k \)-th fragment.

- If \( k = 1 \), then we have \( n_1(e) = l_1 + 1 \) and \( n_2(e) = 2 \sum_{i=2}^{l_m} l_i - 2m + l_1 + 3 \).

- If \( k = m \), then we obtain \( n_1(e) = 2 \sum_{i=1}^{l_m} l_i - 2m + 3 \) and \( n_2(e) = l_m + 1 \).

- If \( 2 \leq k \leq m - 1 \), then we get \( n_1(e) = 2 \sum_{i=1}^{k-1} l_i - 2k + l_k + 3 \), \( n_2(e) = 2 \sum_{i=k+1}^{l_m} l_i - 2(m-k) + l_k + 1 \).

**Case 2:** If edge \( e \) is type II in \( j \)-th square of \( k \)-th fragment, then we observe the following subcases.

- If \( k = 1 \), then we have \( n_1(e) = 2j \) and \( n_2(e) = 2 \sum_{i=2}^{l_m} l_i - 2m + 2(l_1 - j) + 4 \).

- If \( k = m \), then we have \( n_1(e) = 2 \sum_{i=1}^{l_m} l_i - 2m + 2j + 2 \) and \( n_2(e) = 2l_m - 2j + 2 \).

- If \( 2 \leq k \leq m - 1 \), then we deduce \( n_1(e) = 2 \sum_{i=1}^{k-1} l_i - 2k + 2j + 2 \) and \( n_2(e) = 2 \sum_{i=k+1}^{l_m} l_i - 2(m-k) + 2(l_k - j) + 2 \).

Hence, the following is obtained by combining the above case and the def-
inition of the Co-PI index.

\[ Co - PI(H_n^4) = 2 \sum_{j=1}^{l_1-1} \left| j - (2 \sum_{i=2}^{m} l_i - 2m + 2(l_1 - 1) + 4) \right| \]

+ \[ 2 \sum_{j=2}^{l_m} \left| (2 \sum_{i=1}^{m-1} l_i - 2m + 2j + 2) - (l_m - j + 1) \right| \]

+ \[ 2 \sum_{k=2}^{m-1} \sum_{j=2}^{k-1} \left| (2 \sum_{i=1}^{k} l_i - 2k + 2j + 2) - (\sum_{j=k+1}^{m} l_i - (m - k) + (l_k - j) + 1) \right| \]

+ \[ \sum_{k=1}^{m} (l_k + 1) \left| (2 \sum_{i=1}^{k-1} l_i - 2k + l_k + 3) - (2 \sum_{i=k+1}^{m} l_i - 2(m - k) + l_k + 1) \right| \]

\[ \leq 2 \sum_{j=1}^{l_1-1} (3j - 2m + 2m - 2) + 2 \sum_{j=2}^{l_m} \left( 2 \sum_{i=1}^{m-1} l_i - l_m - 2m + 2j + 1 \right) \]

\[ + 2 \sum_{k=2}^{m-1} \sum_{j=2}^{k-1} \left( 2 \sum_{i=1}^{k} l_i - \sum_{j=k}^{m} l_i - 3k + 3j + 1 \right) \]

+ \[ \sum_{k=1}^{m} (l_k + 1) \left( 2 \sum_{i=1}^{k-1} l_i - 2 \sum_{i=k+1}^{m} l_i - 4k + 2m + 2 \right). \] (2)

Deduce (2), we obtain the required result.

\section{Co – PI Index of V-Phenylenic Nanotubes}

The molecular structures V-phenylenic nanotubes is denoted by \( VPHX[m, n] \), see Figure 5. One can see that the number of vertices of \( VPHX[m, n] \) is \( 6mn \). We calculate the Co – PI index of \( VPHX[m, n] \), we assume that \( E_1, E_2 \) and \( E_3 \) are the set of all vertical, oblique and horizontal edges, respectively.

\textbf{Theorem 4.1.} The Co – PI index of \( VPHX[m, n] \) is

\[ Co – PI(VPHX[m, n]) \leq \alpha - \frac{3mn(n-1)(8m+6n-5)}{2}, \]

where

\[ \alpha = \begin{cases} 
2((m - n) - 1)((2\beta + 1)S_\beta - 6mn) \\
+(2\beta + 1)(6\beta - 3)(|m - n|\frac{|m-n|-1}{2}), \text{ if } m \neq n, \\
(4n + 1)S_n + 24n^2 - 6mn - 6n - 3, \text{ if } m = n.
\]
Proof: Let $G = VPHX[m,n]$. Then

$$co - PI(G) = \sum_{e \in E(G)} |n(u) - n(v)|$$

$$= \left[ \sum_{e \in E_1} + \sum_{e \in E_2} + \sum_{e \in E_3} \right] |n(u) - n(v)|,$$  \hspace{1cm} (3)

where $E_1, E_2$ and $E_3$ are the set of all vertical, oblique and horizontal edges, respectively.

First we obtain $\sum_{e \in E_1} |n(u) - n(v)|$.

$$\sum_{e \in E_1} |n(u) - n(v)| = 2 \sum_{i=1}^{n-1} |6mi - (6mn - 6mi)(2m)|$$

$$\leq 2 \sum_{i=1}^{n-1} (6mi - 12m^2n + 12m^2i)$$

$$= 2(-12m^2n)(n-1) + 2(6m + 12m^2)(\frac{n}{2})$$

$$= (6m + 12m^2)n(n-1) - 2(n-1)(12m^2n).$$  \hspace{1cm} (4)

Next we obtain $\sum_{e \in E_2} |n(u) - n(v)|$.

$$\sum_{e \in E_2} |n(u) - n(v)| = \sum_{i=1}^{n-1} |3mi - (6mn - 3mi)(2n)|$$

$$\leq \sum_{i=1}^{n-1} [3mi - 12mn^2 + 6mni]$$

$$= (3m + 6mn)\binom{n}{2} - 12mn^2(n - 1).$$  \hspace{1cm} (5)
Finally, we calculate $\sum_{e \in E_3} |n(u) - n(v)|$ by considering following two cases.

**Case 1:** If $m \neq n$, then

$$
\sum_{e \in E_3} |n(u) - n(v)| = 2 \sum_{i=1}^{\lfloor m-n \rfloor-1} \left| 2\beta(S_\beta + (6\beta - 3)i) - (6mn - S_\beta - (6\beta - 3)i) \right|
\leq 2 \sum_{i=1}^{\lfloor m-n \rfloor-1} \left( (2\beta S_\beta - 6mn + S_\beta) + (2\beta + 1)(6\beta - 3)i \right)
= 2(|m - n| - 1)(2\beta + 1)S_\beta - 6mn
+ (2\beta + 1)(6\beta - 3)\frac{|m - n| - 1}{2},
$$

(6)

where $S_i = 3 + 9 + \ldots + (6i - 3)$ and $\beta = \min\{m, n\}$.

**Case 2:** If $m = n$, then

$$
\sum_{e \in E_3} |n(u) - n(v)| = |4n(S_n + 6n - 3) - (6mn - S_n - (6n - 3))|
\leq 4nS_n + 24n^2 - 12n - 6mn + S_n + 6n - 3)
= (4n + 1)S_n + 24n^2 - 6mn - 6n - 3).
$$

(7)

Using (4),(5),(6) and (7) in (3), we obtain the required result.

5  **Co – PI Index of V-Phenylenic Nanotori**

The molecular structures $V$–phenylenic nanotorus is denoted by $VPHY[m, n]$, see Figure 3. One can see that the number of vertices of $VPHY[m, n]$ is $6mn$. We calculate the **Co – PI** index of $VPHY[m, n]$, we assume that $E_1, E_2$ and $E_3$ are the set of all vertical, oblique and horizontal edges, respectively.
Theorem 5.1. The \( \text{Co} - \text{PI} \) index of \( VPHY[m,n] \) is

\[
\text{Co} - \text{PI}(VPHY[m,n]) \leq \alpha_2 - 3mn(n-1)(4m+2n-3), \text{ where}
\]

\[
\alpha_2 = \begin{cases} 
4(|m-n|-1)((2\beta + 1)S_\beta - 6mn) \\
+2(2\beta + 1)(6\beta - 3)|m-n|(|m-n|-1), \text{ if } m \neq n, \\
(4n+1)S_n + 24n^2 - 6mn - 6n - 3, \text{ if } m = n.
\end{cases}
\]

Proof: Let \( G = VPHY[m,n] \). Let \( E_1, E_2 \) and \( E_3 \) be the set of all vertical, oblique and horizontal edges, respectively. Then by the definition of Co-PI

\[
\text{co} - \text{PI}(G) = \sum_{e \in E(G)} |n(u)-n(v)|
\]

\[
= \left( \sum_{e \in E_1} + \sum_{e \in E_2} + \sum_{e \in E_3} \right) |n(u)-n(v)|. \tag{8}
\]

First we obtain \( \sum_{e \in E_1} |n(u)-n(v)| \).

\[
\sum_{e \in E_1} |n(u)-n(v)| = 2 \sum_{i=1}^{n} |6mi - (6mn - 6mi)2m|
\]

\[
= 2 \sum_{i=1}^{n} |6mi - 12m^2n + 12m^2i|
\]

\[
= 2 \sum_{i=1}^{n-1} |(6m + 12m^2)i - 12m^2n|
\]

\[
\leq 2 \sum_{i=1}^{n-1} (6m + 12m^2)i - 24m^2n(n-1)
\]

\[
= (6m + 12m^2)n(n-1) - 24m^2n(n-1). \tag{9}
\]

Next we obtain \( \sum_{e \in E_2} |n(u)-n(v)| \).

\[
\sum_{e \in E_2} |n(u)-n(v)| = 2 \sum_{i=1}^{n-1} |3mi - (6mn - 3mi)(2n)|
\]

\[
= 2 \sum_{i=1}^{n-1} |3mi - 12mn^2 + 6mni|
\]

\[
\leq 2 \sum_{i=1}^{n-1} (3m + 6mn)i - 24mn^2(n-1)
\]

\[
= (3m + 6mn)n(n-1) - 24mn^2(n-1). \tag{10}
\]
we calculate \( \sum_{e \in E_2} |n(u) - n(v)| \) according to the relationship between \( m \) and \( n \).

**Case 1:** If \( m \neq n \), then we have

\[
\sum_{e \in E_3} |n(u) - n(v)| = \frac{|m-n|-1}{4} \sum_{i=1}^{\frac{|m-n|-1}{2}} [2\beta(S_\beta + (6\beta - 3)i) - (6mn - S_\beta - (6\beta - 3)i)]
\]

\[
\leq 4(|m - n| - 1)[2\beta S_\beta - 6mn + S_\beta]
+ 4 \sum_{i=1}^{\frac{|m-n|-1}{2}} [(2\beta(6\beta - 3) + (6\beta - 3)i)]
\]

\[
= 4(|m - n| - 1)((2\beta + 1)S_\beta - 6mn)
+ 4(2\beta + 1)(6\beta - 3)\frac{|m - n|}{2}(|m - n| - 1)
\]

\[
= 4(|m - n| - 1)((2\beta + 1)S_\beta - 6mn)
+ 2(2\beta + 1)(6\beta - 3)|m - n|(|m - n| - 1).
\] (11)

**Case 2:** If \( m = n \), then we have

\[
\sum_{e \in E_3} |n(u) - n(v)| = |4n(S_n + 6n - 3) - (6mn - S_n - (6n - 3))|
\]

\[
\leq (4nS_n + 24n^2 - 12n - 6mn + S_n + 6n - 3)
= (4n + 1)S_n + 24n^2 - 6mn - 6n - 3.
\] (12)

Using (9), (10), (11) and (12) in (8), we obtain the required result.

**References**


