On n-Normal Operators

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Abstract

In this paper we introduce n-normal operators on a Hilbert space H. We give some basic properties of these operators. In general an n-normal operators need not be a normal operator, a hyponormal operator.

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1 Introduction

Throughout this paper, B(H) denotes to the algebra of all bounded linear operators acting on a complex Hilbert space H. An operator T is said to be normal if T*T = TT*, (it is well known that normal operators have translation-invariant property, i.e., if T is a normal operator, then (T − λ) is a normal operator for every λ ∈ C); self adjoint if T* = T; positive if T* = T and ⟨Tx, x⟩ ≥ 0 for all x ∈ H; and projection if T^2 = T = T*. For an operator T ∈ H, if ∥Tx∥ = ∥x∥ for all x ∈ H ( or equivalently T*T = I), then T is called an isometry. An onto isometry is called unitary. An operator T ∈ B(H) is called partial isometry if T*T is projection. An operator T on H is called subnormal if there exists a Hilbert space K with H is a subspace of K and a normal operator N on K such that NH ⊆ H and N|H = T; T is hyponormal if T*T ≥ TT*. Let T ∈ B(H) and x ∈ H. The sequence \{T^nx\}_{n=0}^\infty is called orbit of x under T, and is denoted by orb(T, x). If orb(T, x) is dense in H, then x is called a hypercyclic vector for T. An operator T ∈ B(H) is called scalar of order m if it possesses a spectral distribution of order m, i.e., if there is a
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A continuous unital morphism $\phi : C^m_0(C) \to B(H)$ such that $\phi(z) = T$ where $z$ stands for the identity function on $C$ and $C^m_0(C)$ for the space of compactly supported functions on $C$, continuously differentiable of order $m$, $0 \leq m \leq \infty$. An operator $T \in B(H)$ is called subscalar if it is similar to the restriction of a scalar operator to an invariant subspace.

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Definition 2.1 $T \in B(H)$ is called an $n$-normal operator if $T^n T^* = T^* T^n$.

Proposition 2.2 Let $T \in B(H)$. Then $T$ is $n$-normal if and only if $T^n$ is normal where $n \in \mathbb{N}$.

Let $T$ is $n$-normal, $T^n T^* = T^* T^n$. Therefore

$$T^n (T^*)^n = T^* T^n (T^*)^{n-1} = T^* (T^n T^n) (T^*)^{n-2} = (T^*)^2 T^n (T^*)^{n-2} = (T^*)^n T^n.$$ 

Then $T^n$ is normal. Now, let $T^n$ is normal. Since $T^n T = TT^n$, by Fuglede theorem [8], $T^* T^n = T^n T^*$. Therefore $T$ is $n$-normal.

It is clear that a bounded normal operator is $n$-normal for any $n$. The converse is not true. Indeed if $T = \begin{pmatrix} i & 2 \\ 0 & -i \end{pmatrix}$, then $T$ is 2-normal which is not normal. And all nonzero nilpotent operators are $n$-normal operators, for $n \geq k$ where $k$ the index of nilpotance, but they are not normal. It is well known that if $T$ is normal, then it is hyponormal. And if $T$ is normal and $T^k$ is compact for some $k$, then $T$ is compact by [8]. The following example shows that these need not be true in case of $n$-normal operator.

Example 2.3 Let $H = \ell^2$ and $e_1, e_2, \ldots$ be standard orthogonal basis for $\ell^2$.
Define $T$ on $H$ by $Te_i = \begin{cases} e_1, & i = 1 \\ e_{i+1}, & i = 2j, j = 1, 2, \ldots \\ 0, & i = 2j + 1 \end{cases}$. Then $T^2 = P$, where $P$ is the orthogonal projection on the space span by $e_1$. So $T$ is 2-normal but neither $T$ nor $T^*$ is hyponormal.
Now, since $T^2$ is a projection on one-dimensional space, it is compact. However, since range of $T$ contains an infinite orthonormal set $\{e_i, i = 1, 3, 5, \ldots\}$, $T$ is not compact.

The following example shows that there exists an operator which is subnormal but not $n$-normal for any $n \in \mathbb{N}$.

Example 2.4 Let $U$ be unilateral shift on $\ell^2$ (i.e., $U(\alpha_0, \alpha_1, \ldots) = (0, \alpha_0, \alpha_1, \ldots)$). Then $U$ is subnormal but for any $n \in \mathbb{N}$, $U^n$ is not normal.
It is well known that if $T$ is hyponormal and compact, then $T$ is normal. But we note that the nilpotent operator $T = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ an $n$-normal operator, which is compact but not normal. Thus $T$ is not hyponormal.

**Theorem 2.5** The set of all $n$-normal operators on $H$ is closed subset of $B(H)$ which is closed under scalar multiplication.

First if $T$ is $n$-normal, and $\alpha$ is scalar, then $(\alpha T)^n(\alpha T)^* = \alpha^n\overline{\sigma}(T^nT^*) = \overline{\sigma}\alpha^n(T^nT^*)$ and $(\pi T^*)(\alpha^nT^n) = (\alpha T)^*(\alpha T)^n$. Hence $\alpha T$ is $n$-normal. Now, suppose that $(T_k)$ is sequence of $n$-normal operators converging to $T$ in $B(H)$. Now, $\|T^nT^* - T^*T^n\| \leq \|T^nT^* - T_k^n T_k^*\| + \|T_k^n T_k^* - T^*T^n\| \longrightarrow 0$ as $k \longrightarrow \infty$. Hence $T^*T^n = T^nT^*$. Thus $T$ is $n$-normal.

**Proposition 2.6** Let $T \in B(H)$ be $n$-normal. Then

1. $T^*$ is $n$-normal.
2. If $T^{-1}$ exists, then $(T^{-1})$ is $n$-normal.
3. If $S \in B(H)$ is unitary equivalent to $T$, then $S$ is $n$-normal.
4. If $M$ is a closed subspace of $H$ such that $M$ reduces $T$, then $S = T/M$ is an $n$-normal operator.

(1) Since $T$ is $n$-normal, $T^n$ is normal. So $(T^n)^* = (T^*)^n$ is normal, $T^*$ is an $n$-normal operator.
(2) Since $T$ is $n$-normal, $T^n$ is normal. Since $(T^n)^{-1} = (T^{-1})^n$ is normal, $T^{-1}$ is an $n$-normal operator.
(3) Let $T$ be an $n$-normal operator and $S$ be unitary equivalent to $T$. Then there exists unitary operator $U$ such that $S = UTU^*$ so $S^n = UT^n U^*$. Since $T^n$ is normal, $S^n$ is normal. Therefore $S$ is $n$-normal.
(4) Since $T$ is $n$-normal, $T^n$ is normal. So $T^n/M$ is normal. And since $M$ is invariant under $T$, $T^n/M = (T/M)^n$. Thus $(T/M)^n$ is normal. So $T/M$ is $n$-normal.

Now, the following example shows that the class of 2-normal operators may not have the translation-invariant property.

**Example 2.7** Let $T = \begin{pmatrix} 0 & T_1 \\ 0 & 0 \end{pmatrix}$, where $T_1 : H_1 \longrightarrow H$. Then $T$ is 2-normal operator. But $[(T - \lambda)^2, (T - \lambda)^2] = \begin{pmatrix} -4 & |\lambda|^2 T_1 T_1^* \\ 0 & 4 |\lambda|^2 T_1^* T_1 \end{pmatrix}$ not necessarily equal to zero unless $\lambda = 0$. Hence $(T - \lambda)^2$ is not normal. So $(T - \lambda)$ is not necessarily 2-normal operator.
Theorem 2.8 If $S$, $T$ are commuting n-normal operators, then $ST$ is an n-normal operator.

Since $S$, $T$ are commuting n-normal operators, $S^n$, $T^n$ are commuting normal operator. So $S^nT^n$ is a normal operator. Since $S^nT^n = (ST)^n$, $(ST)^n$ is normal. Hence $ST$ is n-normal.

The following example shows that Theorem 2.8 is not necessarily true if $S$, $T$ are not commuting.

Example 2.9 Let $S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $T = \begin{pmatrix} i & 2 \\ 0 & -i \end{pmatrix}$ be operators on the Hilbert space $\mathbb{C}^2$. Then $S$ and $T$ are 2-normal. We note that $ST = \begin{pmatrix} i & 2 \\ 0 & i \end{pmatrix} \neq \begin{pmatrix} i & -2 \\ 0 & i \end{pmatrix} = TS$. But as $(ST)^2 = \begin{pmatrix} -1 & 4i \\ 0 & -1 \end{pmatrix}$ is not normal, $ST$ is not 2-normal.

Corollary 2.10 If $T$ is n-normal, Then $T^m$ is n-normal for any positive integer $m$.

The following example shows that sum of two commuting n-normal operators need not be n-normal.

Example 2.11 Let $S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $S$ and $T$ are commuting 2-normal. But $S + T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $(S + T)^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ is not normal. Thus $S + T$ is not 2-normal. We note here $S$ is a selfadjoint operator.

Proposition 2.12 Let $T$, $S$ be commuting n-normal operator, such that $(S + T)^*$ commutes with $\sum_{k=1}^{n-1} \frac{n}{k} S^{n-k}T^k$. Then $(S + T)$ is an n-normal operator.

Since $(S+T)^n(S+T)^* = (\sum_{k=0}^{n-1} \binom{n}{k} S^{n-k}T^k)(S^*+T^*)$, $(S+T)^n(S+T)^* = S^nS^* + \sum_{k=1}^{n-1} \binom{n}{k} S^{n-k}T^k(S+T)^* + T^nS^* + S^nT^* + T^nT^*$. And since $(S+T)^*$ is commuting with $\sum_{k=1}^{n-1} \binom{n}{k} S^{n-k}T^k$, $(S+T)^n(S+T)^* = S^nS^* + (S + T)^* S^n + (S + T)^* S^n T^* + T^n S^* + T^n T^*$. So $(S+T)^n(S+T)^* = (S+T)^*(S^n+T^n) + (S+T)^*(\sum_{k=1}^{n-1} \binom{n}{k} S^{n-k}T^k)$. Hence $(S+T)^n(S+T)^* = (S+T)^*(\sum_{k=1}^{n-1} \binom{n}{k} S^{n-k}T^k) = (S+T)^*(S+T)^n$. 

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Lemma 2.13 If $S, T \in B(H)$ are 2-normal operators and $ST + TS = 0$, then $T + S$ and $ST$ are 2-normal.

Since $ST + TS = 0$, $S^2T^2 = T^2S^2$. So $(S + T)^2 = S^2 + T^2$ is normal. Thus $(S + T)$ is an 2-normal operator.

Now since $ST + TS = 0$, $(ST)^2 = -S^2T^2 = -T^2S^2$. Hence by Theorem 2.8, $ST$ is a 2-normal operator.

Now we state some well known lemmas which we shall need.

Lemma 2.14 Let $P$, $Q$ be the projections on closed subspaces $M$, $N$ respectively. Then $M \perp N$ if and only if $PQ = 0$.

Lemma 2.15 If $T$ is normal, then $Tx = \lambda x$ if and only if $T^*x = \overline{\lambda}x$.

Lemma 2.16 If $P$ is the projection on a closed subspace $M$ of $H$, then $M$ reduces of $T$ if and only if $TP = PT$.

Theorem 2.17 Let $T$ be an operator on finite dimensional Hilbert space $H$, $\lambda_1, \ldots, \lambda_m$ be eigenvalues of $T$ such that $\lambda_i^* \neq \lambda_j^*$, $i \neq j$, $M_1, \ldots, M_m$ the corresponding eigenspaces, and $P_1, \ldots, P_m$ the projections on $M_1, \ldots, M_m$ respectively. Then $M_i$'s are pairwise orthogonal and they span $H$ if and only if $T$ is $n$-normal operator.

Assume $M_i$'s are pairwise orthogonal and they span $H$. Then for $x \in H$, $x = x_1 + x_2 + \ldots + x_m$, $x_i \in M_i$, $T^n x = T^n x_1 + \ldots + T^n x_m = \lambda_1^n x_1 + \ldots + \lambda_m^n x_m$.

Since $P_i$'s are projection on eigenspace $M_i$'s which are pairwise orthogonal, by lemma 2.14 $P_i x = x_i$. Hence $Ix = x_1 + \ldots + x_m = P_1 x + \ldots + P_m x = (P_1 + \ldots + P_m)x$ for every $x \in H$. Thus $I = \sum_{i=1}^m P_i$. Since $T^n x = \lambda_1^n x_1 + \ldots + \lambda_m^n x_m = \lambda_1^n P_1 x + \ldots + \lambda_m^n P_m x = (\lambda_1^n P_1 + \ldots + \lambda_m^n P_m)x$ for all $x \in H$. So $T^n = \sum_{i=1}^m \lambda_i^n P_i$. Hence $T^{*n} = \sum_{i=1}^m \lambda_i^n P_i$. Since $M_i$'s are pairwise orthogonal, $P_i P_j = \begin{cases} P_i, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$ So $T^n T^{*n} = |\lambda_1|^{2n} P_1 + \ldots + |\lambda_m|^{2n} P_m$ and $T^{*n} T^n = |\lambda_1|^{2n} P_1 + \ldots + |\lambda_m|^{2n} P_m$. Thus $T^n$ is normal, i.e., $T$ is an $n$-normal operator.

Suppose $T$ is an $n$-normal operator. Then $T^n$ is a normal operator. We claim that $M_i$'s are pairwise orthogonal. Let $x_i, x_j$ be vectors in $M_i, M_j$, $(i \neq j)$ such that $T^n x_i = \lambda_i^n x_i$ and $T^n x_j = \lambda_j^n x_j$. Then $\lambda_i^n \langle x_i, x_j \rangle = \langle \lambda_i^n x_i, x_j \rangle = \langle T^n x_i, x_j \rangle = \langle x_i, T^n x_j \rangle = \langle x_i, \lambda_j^n x_j \rangle = \lambda_j^n \langle x_i, x_j \rangle$. So $\langle \lambda_i^n - \lambda_j^n \rangle \langle x_i, x_j \rangle = 0$. Since $\lambda_i^n \neq \lambda_j^n$, $\langle x_i, x_j \rangle = 0$. This shows that $M_i$'s are pairwise orthogonal.

Let $M = M_1 + \ldots + M_m$. Then $M$ is a closed subspace of $H$. Let $P$ be associated projection onto $M$. Then $P = P_1 + \ldots + P_m$. Since $T^n$ is normal, each $M_i$ reduces $T^n$. It follows that $T^n P = P T^n$. Consequently $M^n$ is invariant under $T^n$. Suppose that $M^n \neq \{0\}$. Let $T_1 = T^n / M^n$. Then $T_1$ is an operator on non-trivial finite dimensional complex Hilbert space $M^n$ with empty point spectrum which is impossible. Therefore $M^n = \{0\}$, i.e., $M = H$.
Theorem 2.18 Let $T_1, \ldots, T_m$ be $n$-normal operators in $B(H)$. Then $(T_1 \oplus \ldots \oplus T_m)$ and $(T_1 \otimes \ldots \otimes T_m)$ are $n$-normal operators.

Since $(T_1 \oplus \ldots \oplus T_m)^n(T_1 \oplus \ldots \oplus T_m)^* = (T_1^n \oplus \ldots \oplus T_m^n)(T_1^* \oplus \ldots \oplus T_m^*) = T_1^n T_1^* + \ldots + T_m^n T_m^* = (T_1^n + \ldots + T_m^n)(T_1^* + \ldots + T_m^*) = (T_1 \oplus \ldots \oplus T_m)^n(T_1 \oplus \ldots \oplus T_m)^*$, then $(T_1 \oplus \ldots \oplus T_m)$ is an $n$-normal operator.

Now, for $x_1, \ldots, x_m \in H$, $(T_1 \otimes \ldots \otimes T_m)^n(T_1 \otimes \ldots \otimes T_m)^*(x_1 \otimes \ldots \otimes x_m) = (T_1^n \otimes \ldots \otimes T_m^n)(T_1^* \otimes \ldots \otimes T_m^*)(x_1 \otimes \ldots \otimes x_m) = T_1^n T_1^* x_1 \otimes \ldots \otimes T_m^n T_m^* x_m.$

So $(T_1 \otimes \ldots \otimes T_m)^n(T_1 \otimes \ldots \otimes T_m)^* = (T_1 \otimes \ldots \otimes T_m)^n(T_1 \otimes \ldots \otimes T_m)^*$. Thus $(T_1 \otimes \ldots \otimes T_m)^*$ is normal.

Proposition 2.19 $(T - \lambda)$ is an $n$-normal operator for every $\lambda \in C$ if and only if $T$ is a normal operator.

Since $(T - \lambda)$ is $n$- normal for every $\lambda \in C$, $(T - \lambda)^n(T - \lambda)^* = (T - \lambda)^n(T - \lambda)^*$. Hence $(T^* - \overline{\lambda})(\sum_{k=1}^n (-1)^k \binom{n}{k} T^{n-k} \lambda^k) = (\sum_{k=1}^n (-1)^k \binom{n}{k} T^{n-k} \lambda^k)(T^* - \overline{\lambda})$. So $(\sum_{k=1}^n (-1)^k \binom{n}{k} T^* T^{n-k} \lambda^k)(T^* - \overline{\lambda}) = (\sum_{k=1}^n (-1)^k \binom{n}{k} T^{n-k} T^* \lambda^k) - (\sum_{k=1}^n (-1)^k \binom{n}{k} T^{n-k} \lambda^k) T^* - \overline{\lambda}$. Therefore

Proposition 2.20 Let $T \in B(H)$ with the Cartesian decomposition $T = A + iB$ where $A$ and $B$ are selfadjoint operators. Then $T$ is 2-normal operator if and only if $B^2$ commutes with $A$, and $A^2$ commutes with $B$.

\[ iBA^2 + BAB \] and \[ T^2 T^* = A^3 - AB^2 + iA^2B + iABA - iBA^2 + iB^3 + BAB + B^2A. \]

Since \( B^2A = AB^2 \) and \( A^2B = BA^2 \), \( T^2 T^* = T^* T^2 \). Hence \( T \) is 2-normal.

Now let \( T \) be 2-normal. So \( T^2 T^* = T^* T^2 \). Hence \( -B^2A + iBA^2 - iA^2B + AB^2 = AB^2 + iA^2B - iBA^2 + B^2A, (AB^2 - B^2A) + i(BA^2 - A^2B) = 0. \)

Let \( T_1 = AB^2 - B^2A, T_2 = BA^2 - A^2B \). Then \( T_1^* = -T_1, T_2^* = -T_2 \) (i.e., \( T_1, T_2 \) are skew hermitian) and \( T_1 + iT_2 = 0 \). So \( -T_1 + iT_2 = 0 \). This gives \( T_1 = AB^2 - B^2A = 0 \). Similarly, \( B^2A = AB^2 \). It is clear that a 2-normal operator is a 2\( k \)-normal operator and a 3-normal operator is a 3\( k \)-normal operator. The following examples show that a 2-normal operator need not be 3-normal operator and vice versa.

**Example 2.21** Let \( T = \begin{pmatrix} 2 & 1 \\ 0 & -2 \end{pmatrix} \). Then \( T^2 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \) is a normal operator. But \( T^3 = \begin{pmatrix} 8 & 4 \\ 0 & -8 \end{pmatrix} \) is not normal. So \( T \) is 2-normal but it is not 3-normal.

**Example 2.22** Let \( T = \begin{pmatrix} 2 & 2 \\ -2 & 0 \end{pmatrix} \). Then \( T^3 = \begin{pmatrix} -8 & 0 \\ 0 & -8 \end{pmatrix} \) is a normal operator. But \( T^2 = \begin{pmatrix} 0 & 4 \\ -4 & -4 \end{pmatrix} \) is not normal. So \( T \) is 3-normal but it is not 2-normal.

**Proposition 2.23** Suppose \( T \) is both \( k \)-normal and \((k+1)\)-normal for some positive integer \( k \). Then \( T \) is \((k+2)\)-normal. And hence \( T \) is \( n \)-normal for all \( n \geq k \).

Since \( T \) is \( k \)-normal, \( T^k T^* = T^* T^k \). Hence \( TT^k T^* = TT^* T^k \). So \( T^{k+1} T^* = TT^* T^{k+1} \). Since \( T \) is \((k+1)\)-normal, \( T^* T^{k+2} = T^{k+2} T^* \). Thus \( T \) is \((k+2)\)-normal.

**Corollary 2.24** If \( T \) is 2-normal and 3-normal, then \( T \) is an \( n \)-normal for all \( n \geq 2 \).

The following example shows a 2-normal and 3-normal operator may not be normal.

**Example 2.25** Let \( T = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \) be an operator acting in two-dimensional complex Hilbert space. Then \( T \) is 2-normal, 3-normal, and hence it is \( n \)-normal for all \( n \geq 2 \) but it is not normal.

**Proposition 2.26** Suppose \( T \) is a \( k \)-normal operator for a positive integer \( k \) and it is a partial isometry. Then \( T \) is a \((k+1)\)-normal operator. And hence \( T \) is \( n \)-normal for all \( n \geq k \).
Since $T$ is partial isometry, $TT^*T = T$ by [5, p.250]. Hence $TT^*T^k = T^k$ and $T^kT^*T = T^k$. Since $T$ is $k$-normal, $T^{k+1}T^* = T^k$ and $T^*T^{k+1} = T^k$. Thus $T^{k+1}T^* = T^*T^{k+1}$. Therefore $T$ is $(k+1)$-normal. And hence by Proposition 2.23 $T$ is $n$-normal for all $n \geq k$.

**Corollary 2.27** If $T$ is 2-normal and partial isometry, then $T$ is $n$-normal for all integer $n \geq 2$.

We note that, in Example 2.25 if $a$ equal to 1, then $T$ is a 2-normal operator and a partial isometry but not normal.

**Lemma 2.28** Let $T$ be $k$-normal and $(k+1)$-normal. If either $T$ or $T^*$ is injective, then $T$ is normal.

Since $T$ is $(k+1)$-normal, $T^{k+1}T^* = T^*T^{k+1}$. And since $T$ is $k$-normal, $T^{k+1}T^* = T^kT^*T$. Hence $T^k(TT^* - T^*T) = 0$. Since $T$ is injective, $TT^* - T^*T = 0$. Thus $T$ is normal. In case $T^*$ is injective, since $T^*$ is $k$-normal and $(k+1) - normal$, $T^*$ is normal. Hence $T$ is normal.

**Proposition 2.29** Let $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a, b, c, d \in C$. Then $T$ is 2-normal if and only if $(a + d) = 0$ and $(|b| = |c|$ or $b(d - \pi) = \pi(d - a))$.

Suppose $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is 2-normal. Then $T^2 = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + dc & cb + d^2 \end{pmatrix}$ is normal. Hence $|ab + bc| = |ac + dc|$ and $(ab + bd)((cd + d^2) - (a^2 + bc)) = (ac + dc)((cb + d^2) - (a^2 + bc))$. Since $|b(a+d)| = |c(a+d)|$ and $b(a+d) = (\overline{d} - \pi)(a+d) = \pi(d-a)(d-a)$. Thus $|b| = |c|$ or $|a+d| = 0$ and $b(\overline{d} - \pi) = \pi(d-a)$ or $|a+d|^2 = 0$.

By giving similar arguments that in the last Proposition one can prove the following.

**Proposition 2.30** Let $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a, b, c, d \in C$. Then $T$ is 3-normal if and only if $(a^2 + bc + ad + d^2) = 0$ and $(|b| = |c|$ or $\overline{c}(d-a) = c(d-a))$.

Next, we characterize when a two-dimensional upper triangular complex matrix is $n$-normal.

**Proposition 2.31** For $n \geq 2$ we have $T = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ is $n$-normal if and only if $b(a^{n-1} + a^{n-2}c + \ldots + c^{n-1}) = 0$. 


Let $T = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$. Then $T$ is $n$-normal if and only if

$$T^n = \begin{pmatrix} a^n & b(a^{n-1} + a^{n-2}c + \ldots + c^{n-1}) \\ 0 & c^n \end{pmatrix},$$

is normal if and only if $|b(a^{n-1} + a^{n-2}c + \ldots + c^{n-1})| = 0$ if and only if $b(a^{n-1} + a^{n-2}c + \ldots + c^{n-1}) = 0$.

**Example 2.32** Consider $n = 3$ in the last Proposition. Then $T$ is a $3$-normal operator if and only if $b(a^2 + ac + c^2) = 0$. Take $a = 2$, $b = 1$, and $c = -1 + \sqrt{3}i$. Then $T = \begin{pmatrix} 2 & 1 \\ 0 & -1 + \sqrt{3}i \end{pmatrix}$ is $3$-normal. Note that $T^3 = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$ is normal. Thus $T$ is $3$-normal.

We note that by use the last Proposition we may get an $n$-normal operator but not normal.

**Proposition 2.33** Let $T \in B(H)$, $F = T^n + T^*$, and $G = T^n - T^*$. Then $T$ is an $n$-normal operator if and only if $G$ commutes with $F$.

$$FG = GF$$

if and only if $(T^n + T^*)(T^n - T^*) = (T^n - T^*)(T^n + T^*)$ if and only if $T^{2n} - T^nT^* + T^*T^n - T^{n2} = T^{2n} + T^nT^* - T^*T^n - T^{n2}$ if and only if $T^nT^* - T^*T^n = 0$ if and only if $T$ is an $n$-normal.

**Proposition 2.34** Let $T \in B(H)$, $B = T^nT^*$, $F = T^n + T^*$, and $G = T^n - T^*$. If $T$ is an $n$-normal, then $B$ commutes with $F$ and $G$.

Since $T$ is an $n$-normal, $BF = T^nT^*(T^n + T^*) = T^nT^*T^n + T^nT^*T^* = T^nT^nT^* + T^nT^nT^* = (T^n + T^*)T^nT^* = FB$. By similar way we can prove that $BG = GB$.

**Proposition 2.35** Let $T$ be a weighted shift with nonzero weights $\{\alpha_k\}_{k=0}^\infty$. Then $T$ is $n$-normal if and only if $|\alpha_{k-n}||\alpha_{k-1}||\alpha_k|\ldots|\alpha_{k+n-1}|$ for $k = n, n+1, \ldots$.

Let $\{e_k\}_{k=0}^\infty$ be an orthogonal basis of Hilbert space $H$. Since $T^n e_k = \alpha_k \ldots \alpha_{k+n-1} e_{k+n}$ and $T^* e_k = \alpha_{k-1} \ldots \alpha_{k-n} e_{k-n}$, $T^nT^* e_k = |\alpha_{k-1}|^2 \ldots |\alpha_{k-n}|^2 e_k$ and $T^nT^* e_k = |\alpha_k|^2 \ldots |\alpha_{k+n-1}|^2 e_k$. Thus $T^n$ is normal if and only if $|\alpha_k|^2 \ldots |\alpha_{k+n-1}|^2 = |\alpha_{k-1}|^2 \ldots |\alpha_{k-n}|^2$ for $k = n, n+1, \ldots$.

**Proposition 2.36** Let $T \in B(H)$ be an $n$-normal operator and invertible. Then $T$ and $T^{-1}$ have a common nontrivial closed invariant subspace.
Since $T$ is $n$-normal and invertible, $T^n$ and $(T^{-1})^n$ are normal. Hence by [1, Corollary 4.5] $T^n$ and $(T^{-1})^n$ both have no hypercyclic vector. Thus by [7], $T$ and $T^{-1}$ both have no hypercyclic vector. Therefore by [2], $T$ and $T^{-1}$ have a common nontrivial closed invariant subspace.

Let $\lambda$ be the coordinate in $C$ and $d_\mu(\lambda)$, denotes planar Lebesgue measure.

Let $D$ be a bounded open subset of $C$. We shall denote by $L^2(D, H)$ the Hilbert space of measurable function $f : D \rightarrow H$ such that

$$\|f\|_{2,D} = \left\{ \int_D \|f(\lambda)\|^2 d_\mu(\lambda) \right\}^{\frac{1}{2}} < \infty.$$ 

The space of functions $f \in L^2(D, H)$ that are analytic in $D$ (i.e., $\bar{\partial}f = 0$) is denoted by

$$A^2(D, H) = L^2(D, H) \cap \hat{\mathcal{O}}(U, H).$$

$A^2(D, H)$ is called the Bergman space for $D$.

Let $D$ be a bounded open subset of $D$ and $m$ a fixed non-negative integer. The vector valued Sobolev space $W^m(D, H)$ with respect to $\bar{\partial}$ and of order $m$ will be the space of those functions $f \in L^2(D, H)$ whose derivatives $\bar{\partial}f, ... , \bar{\partial}^m f$ in the sense of distributions also belong to $L^2(D, H)$. Endowed with the norm

$$\|f\|_{W^m} = \sum_{i=0}^m \|\bar{\partial}^i f\|_{2,D}^2.$$ 

$W^m(D, H)$ becomes a Hilbert space contained continuously in $L^2(D, H)$.

**Theorem 2.37** Let $D$ be an arbitrary bounded disk in $C$. If $T \in B(H)$ is 2-normal with the property that $\sigma(T) \cap (-\sigma(T)) = \emptyset$, then the operator

$$\lambda - T : W^2(D, H) \rightarrow L^2(D, H)$$

is one to one.

Let $f \in W^2(D, H)$ such that $(\lambda - T)f = 0$ i.e.,

$$\| (\lambda - T)f \|_{W^2} = 0.$$  

(1)

Then, for $i = 1, 2$, we have

$$\| (\lambda - T)\bar{\partial}^i f \|_{2,D} = 0.$$  

(2)

Hence for $i = 1, 2$, we get $\| (\lambda^2 - T^2)\bar{\partial}^i f \|_{2,D} = 0$. For $i = 1, 2$, Since $T^2$ is normal,

$$\| (\lambda^2 - T^2)\bar{\partial} f \|_{2,D} = 0.$$  

(3)

Since $\lambda - T$ is invertible for $\lambda \in D \setminus \sigma(T)$, the equation 2 implies that $\| \bar{\partial} f \|_{2,D \setminus \sigma(T)} = 0$. Therefore

$$\| (\lambda - T^*)\bar{\partial} f \|_{2,D \setminus \sigma(T)} = 0.$$  

(4)
Since $\sigma(T) \cap (-\sigma(T)) = \emptyset$ and $\sigma(T^*) = \sigma(T)^*$, $\overline{\lambda + T^*}$ is invertible for $\lambda \in \sigma(T)$. Therefore, from equation 3, we have

$$\| (\overline{\lambda} - T^*) \overline{\partial^i f} \|_{2,\sigma(T)} = 0.$$  \hspace{1cm} (5)

Hence from 4 and 5, we get

$$\| (\overline{\lambda} - T^*) \overline{\partial^i f} \|_{2,D} = 0.$$  \hspace{1cm} (6)

By [6, Proposition 2.1], we obtain

$$\| (I - P) f \|_{2,D} = 0,$$  \hspace{1cm} (7)

where $P$ denotes the orthogonal projection of $L^2(D,H)$ onto the Bergman space $A^2(D,H)$. Hence $(\lambda - T) Pf = (\lambda - T) f = 0$. Since $T$ has SVEP, $f = Pf = 0$. Hence $\lambda - T$ is one to one.

**Lemma 2.38** Let $T \in B(H)$ be an 2-normal operator with property for $\sigma(T) \cap (-\sigma(T)) = \emptyset$. If $V$ is an isometry, then the operator $\lambda - VT V^*: W^2(D,H) \rightarrow L^2(D,H)$ is one to one.

Let $f \in W^2(D,H)$ such that $(\lambda - VT V^*) f = 0$. Then $(\lambda - T)V^* f = 0$. Hence for $i = 0, 1, 2$ $(\lambda - T)V^* \overline{\partial^i f} = 0$. By Theorem 2.37, for $i = 0, 1, 2$, $V^* \overline{\partial^i f} = 0$. Hence for $i = 0, 1, 2$, $VT V^* f = 0$. Thus $\lambda \overline{\partial^i f} = 0$ for $i = 0, 1, 2$. By [6, Proposition 2.1] with $T = (0)$, we get $\| (I - P) f \|_{2,D} = 0$, where $P$ denotes the orthogonal projection of $L^2(D,H)$ onto the Bergman space $A^2(D,H)$. Hence $\lambda f = P f = 0$. By [4, Corollary 10.7], there exists a constant $c > 0$ such that

$$c \| Pf \|_{2,D} \leq \| \lambda P f \|_{2,D} = 0.$$  \hspace{1cm} (6)

So $f = Pf = 0$. Thus $\lambda - VT V^*$ is one to one.

**Proposition 2.39** Let $T \in B(H)$ be an $n$-normal operator. If $T$ is quasinilpotent, then $T$ is nilpotent, and hence $T$ is subscalar.

Since $T$ is quasinilpotent, $\sigma(T) = \{0\}$. Hence by the spectral mapping theorem we get $\sigma(T^n) = \sigma(T)^n = \{0\}$. Thus $T^n$ is quasinilpotent and normal. So $T^n = 0$ i.e., $T$ is nilpotent and $T$ is algebraic operator and hence by [3], $T$ is subscalar.

**Proposition 2.40** Let $T \in B(H)$ be a 2-normal Operator with the property that $\sigma(T) \cap (-\sigma(T)) = \emptyset$. Then $T$ is subscalar of order 2.

Consider an arbitrary bounded disk $D \subset C$ which contains $\sigma(T)$ and the quotient space $H(D) = W^2(D,H)/\langle (\lambda - T) W^2(D,H) \rangle$ endowed with the Hilbert space norm. The class of a vector or an operator $A$ on $H(D)$ will be denoted respectively by $\tilde{f}, \tilde{A}$. Let $M$ be the operator of multiplication by $\lambda$ on
On $n$-normal operators

$W^2(D, H)$. Then $M$ is a scalar operator of order 2 and has a spectral distribution $\phi$. Let $S = \tilde{M}$. Since $(\lambda - T)W^2(D, H)$ is invariant under every operator $M_f, f \in C_0^\infty(C)$, we infer that $S$ is a scalar operator of order 2 with spectral distribution $\tilde{\phi}$.

Consider the natural map $V : H \rightarrow H(D)$ denoted by $V h = 1 \otimes h$, for $h \in H$, where $1 \otimes h$ denotes the constant function sending $\lambda \in D$ to $h$. Then $VT = SV$. In particular $R(V)$ is an invariant subspace for $S$. Now we shall prove that $V$ is one to one and has closed range.

Let $\{h_n\}, \{f_n\}$ be sequences respectively in $H, W^2(D, H)$ such that

$$\lim_{n \to \infty} \| (\lambda - T) f_n + 1 \otimes h_n \|_{W^2} = 0. \quad (8)$$

It suffices to show that $\lim_{n \to \infty} h_n = 0$.

By the definition of the norm of Sobolev space 8 implies that

$$\lim_{n \to \infty} \| (\lambda - T) \bar{\partial} f_n \|_{2,D} = 0. \quad (9)$$

$$\lim_{n \to \infty} \| (\lambda - T) \bar{\partial} f_n \|_{2,D} = 0 \quad \text{Since } T^2 \text{ is normal, for } i = 1, 2$$

$$\lim_{n \to \infty} \| (\bar{\lambda}^2 - T^*^2) \bar{\partial} f_n \|_{2,D} = 0. \quad (10)$$

Since $\lambda - T$ invertible for $\lambda \in D \setminus \sigma(T)$, 9 implies that $\lim_{n \to \infty} \| \bar{\partial} f_n \|_{\sigma(T)} = 0$. Therefore

$$\lim_{n \to \infty} \| (\bar{\lambda} - T^*) \bar{\partial} f_n \|_{2,D, \sigma(T)} = 0. \quad (11)$$

Since for $\sigma(T) \cap (-\sigma(T)) = \emptyset$ and $\sigma(T^*) = \sigma(T)^*$, $\lambda + T^*$ is invertible for $\lambda \in \sigma(T)$. Therefore from 10 we have

$$\lim_{n \to \infty} \| (\bar{\lambda} - T^*) \bar{\partial} f_n \|_{2,\sigma(T)} = 0. \quad (12)$$

Hence by 11 and 12 we get

$$\lim_{n \to \infty} \| (\bar{\lambda} - T^*) \bar{\partial} f_n \|_{2,D} = 0. \quad (13)$$

By [6, Proposition 2.1], we obtain

$$\lim \| (I - P) f_n \|_{2,D} = 0, \quad (14)$$

where $P$ denotes the orthogonal projection of $L^2(D, H)$ onto the Bergman space $A^2(D, H)$. Substituting 14 into 8, we get $\lim_{n \to \infty} \| (\lambda - T) Pf_n + 1 \otimes h_n \|_{2,D} = 0$. Let $\Gamma$ be a curve in $D$ surrounding $\sigma(T)$. Then for $\lambda \in \Gamma$

$$\lim_{n \to \infty} \| Pf_n(\lambda) + (\lambda - T)^{-1}(1 \otimes h) \| = 0$$
uniformly. Hence by Riesz-Dunford functional
\[
\lim_{n \to \infty} \left\| \frac{1}{2\pi i} \int \Gamma P f_n(\lambda) d\lambda + h_n \right\| = 0.
\]
But since \( \frac{1}{2\pi i} \int \Gamma P f_n(\lambda) d\lambda = 0 \), by Cauchy’s theorem calculus, \( \lim_{n \to \infty} h_n = 0 \).
Thus \( V \) is one to one and has closed range.

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