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Split Domination in Normal Product of Paths and Cycles

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Abstract

A dominating set $D \subseteq V$ is a split dominating *set of a graph $G = (V, E)$ if the induced subgraph of $\langle V - D \rangle$ is disconnected. The split domination number $\gamma_s(G)$ is the minimum cardinality of a split dominating set of a graph G. In this article, we establish some results on split domination number of $P_m \oplus P_n$, $P_m \oplus C_n$ and $C_m \oplus C_n$.*

Keywords: *Graph, Domination, Split dominating set, Split domination number, Normal product graphs.*

1 Introduction

All graphs considered here are finite, undirected without loops or multiple edges. As usual $p = |V|$ and $q = |E|$ denote the number of vertices and edges of a graph G, respectively. In general we use $\langle X \rangle$ to denote the subgraph induced by the set of vertices X and $N(v)$ and $N[V]$ denote open and closed neighborhoods of a vertex v, respectively. Any undefined term in this paper may be found in Harary [2].

A set D of vertices in a graph G is a *dominating set* if every vertex in $V - D$ is adjacent to some vertex in D. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. For complete review on theory of domination and its related parameters, we refer [3] and [11].

The *Normal product* of two graphs G and H, denoted $G \oplus H$, is a graph with vertex set $V(G \oplus H) = V(G) \times V(H)$, that is, the set $\{(g, h)/\ g \in G, h \in H\}$, and an edge $[(g_1, h_1), (g_2, h_2)]$ exists whenever any of the following conditions holds: (i) $[g_1, g_2] \in E(G)$ and $h_1 = h_2$, (ii) $g_1 = g_2$ and $[h_1, h_2] \in E(H)$, (iii) $[g_1, g_2] \in E(G)$ and $[h_1, h_2] \in E(H)$. The normal product or strong product was first introduced by Sabidussi [9]. For comprehensive details on product graph and its related concepts, we refer [4] and [8].

A dominating set D of of a graph G is a *split dominating* set if the induced subgraph of $\langle V - D \rangle$ is disconnected. The split domination number $\gamma_s(G)$ is the minimum cardinality of split dominating set. The minimum cardinality taken over all split dominating set in a graph G is called *split domination number* $\gamma_s(G)$ of G. The concept of split domination was introduced by Kulli and Janakiram [7]. For more details on split domination, we refer [1], [5], [6] and [10]. A dominating set D of a graph G with $|D| = \gamma(G)$ is called γ-set. Similarly, the other types of dominating set are defined on the same line.

2 Results

Theorem 2.1 *For any non-complete connected graph G,*

$$\gamma_s(G \oplus K_n) = \gamma_s(G) \cdot n.$$

Proof: Let the vertices of a complete graph K_n be labeled as v_1, v_2, $\ldots$, v_n and the vertices of G be labeled as u_1, u_2, $\ldots$, u_m. Since K_n is a complete graph, from the definition of normal product, whenever u_i is adjacent to u_j in G, each vertex (u_i, v_k), $1 \leq k \leq n$ is adjacent to every vertex (u_j, v_l), $1 \leq l \leq n$ in $G \oplus K_n$. Hence, in $G \oplus K_n$, the dominating set S with $\mid S \mid < \gamma_s(G) \cdot n$ does not split the graph $G \oplus K_n$ and the removal of $B = \{(u_i, v_k)/u_i \in A$ and $v_k \in V(K_p)\}$, where A is the γ_s-set of G, splits the graph $G \oplus K_n$. Hence, $\gamma_s(G \oplus K_n) =\mid B \mid= \gamma_s(G) \cdot n$.

3 Normal Product of $P_m \oplus P_n$

Remark 3.1 *If $m = 2$ and $n = 2$, then $P_2 \oplus P_2 \cong K_4$. Hence, γ_s-set does not exists.*

Theorem 3.2 *For $n \geq 3$, $k \geq 1$,*

$$\gamma_s(P_2 \oplus P_n) = \begin{cases} k+1, & \text{if } n = 3k \\ k+2, & \text{if } n = 3k+1 \text{ or } 3k+2. \end{cases}$$

Proof: Let P_2 be labeled as u_1, u_2 and P_n be labeled as $v_1, v_2, \dots v_n$. Then the following cases are arise.

Case 1. $n = 3k$.
In $P_2 \oplus P_n$, the subset $A = \{(u_1, v_{3t-1})/1 \le t \le k\}$ is the γ-set. Clearly no dominating set with cardinality $|A|$ split the graph. The set $A \cup \{(u_2, v_2)\}$ dominates and split the graph $P_2 \oplus P_n$. Hence, $\gamma_s(P_2 \oplus P_n) = k + 1$.

Case 2. $n = 3k + 1$ or $3k + 2$.
In $P_2 \oplus P_n$, the subset $B = A \cup \{(u_1, v_n)\}$ is the γ-set. Clearly no dominating set with cardinality $|B|$ split the graph. The set $B \cup \{(u_2, v_2)\}$ dominates and split the graph. Hence, $\gamma_s(P_2 \oplus P_n) = k + 2$.

Theorem 3.3 *For $m, n \ge 3$ and $k_1, k_2 \ge 1$,*

$$\gamma_s(P_m \oplus P_n) = \begin{cases} k_1k_2 + 2, & \text{if } m = 3k_1 \text{ and } n = 3k_2 \\ k_2(k_1+1) + 2, & \text{if } m = 3k_1 + 1 \text{ or } 3k_1 + 2 \text{ and} \\ & n = 3k_2 \\ k_1(k_2+1) + 2, & \text{if } m = 3k_1 \text{ and } n = 3k_2 + 1 \text{ or} \\ & n = 3k_2 + 2 \\ k_1(k_2+1) + k_2 + 2, & \text{if } m \text{ and } n \text{ are not multiple of 3.} \end{cases}$$

Proof: Let P_m be labeled as $u_1, u_2, \dots u_m$ and P_n be labeled as $v_1, v_2, \dots v_n$. Then the following cases are arise.

Case 1. $m = 3k_1$ and $n = 3k_2$.
In $P_m \oplus P_n$, the subset $A = \{(u_{3t_1-1}, v_{3t_2-1})/1 \le t_1 \le k_1, 1 \le t_2 \le k_2\}$ is the γ-set. Clearly no dominating set with cardinality $|A|$ or $|A| + 1$ split the graph $P_m \oplus P_n$. The set $A \cup \{(u_1, v_2), (u_2, v_1)\}$ dominates and split the graph $P_m \oplus P_n$. Hence, $\gamma_s(P_m \oplus P_n) = k_1k_2 + 2$.

Case 2. $m = 3k_1 + 1$ or $3k_1 + 2$ and $n = 3k_2$.
In $P_m \oplus P_n$, the subset $B = A \cup \{(u_m, v_{3t_2-1})/1 \le t_2 \le k_2\}$ is the γ-set. Clearly no dominating set with cardinality $|B|$ or $|B| + 1$ split the graph $P_m \oplus P_n$. The set $B \cup \{(u_1, v_2), (u_2, v_1)\}$ dominates and split the graph $P_m \oplus P_n$. Hence, $\gamma_s(P_m \oplus P_n) = k_2(k_1 + 1) + 2$.

Case 3. $m = 3k_1$ and $n = 3k_2 + 1$ or $3k_2 + 2$.
In $P_m \oplus P_n$, the subset $B = A \cup \{(u_{3t_1-1}, v_n)/1 \le t_1 \le k_1\}$ is the γ-set. Clearly no dominating set with cardinality $|B|$ or $|B| + 1$ split the graph $P_m \oplus P_n$. The set $B \cup \{(u_1, v_2), (u_2, v_1)\}$ dominates and split the graph $P_m \oplus P_n$. Hence, $\gamma_s(P_m \oplus P_n) = k_1(k_2 + 1) + 2$.

Case 4. $m = 3k_1 + 1$ or $3k_1 + 2$ and $n = 3k_2 + 1$ or $3k_2 + 2$.
In $P_m \oplus P_n$, the subset $B = A \cup \{(u_{3t_1-1}, v_n)/1 \leq t_1 \leq k_1\} \cup \{(u_m, v_{3t_2-1})/1 \leq t_2 \leq k_2\}$ is the γ-set. Clearly no dominating set with cardinality $|B|$ or $|B|+1$ split the graph $P_m \oplus P_n$. The set $B \cup \{(u_1, v_2), (u_2, v_1)\} \cup \{(u_m, v_n)\}$ dominates and split the graph $P_m \oplus P_n$. Hence, $\gamma_s(P_m \oplus P_n) = k_1(k_2+1) + k_2 + 2$.

4 Normal Product of $C_m \oplus P_n$

Theorem 4.1 *For any cycle C_m of length atleast 4,*

$$\gamma_s(C_m \oplus P_2) = \begin{cases} k+2, & \text{if } m = 3k,\ k \geq 1 \\ k+3, & \text{if } m = 3k+1 \text{ or } = 3k+2,\ k \geq 1. \end{cases}$$

Proof: Let C_m be labeled as $u_1, u_2 \ldots u_m$ and P_2 be labeled as v_1, v_2. Then the following cases are arise.

Case 1: $m = 3k$, $k \geq 2$.
In $C_m \oplus P_2$, the subset $A = \{(u_{3t-1}, v_1)/1 \leq t \leq k\}$ is the γ-set. Clearly no dominating set with cardinality $|A|$ or $|A|+1$ split the graph. The set $A \cup \{(u_2, v_2), (u_5, v_2)\}$ dominate and split the graph. Hence, $\gamma_s(C_m \oplus P_2) = k+2$.

Case 2: $m = 3k+1$ or $3k+2$, $k \geq 1$.
The subset $B = A \cup \{(u_1, v_n)\}$ is the γ-set. Clearly no dominating set with cardinality $|B|$ or $|B|+1$ split the graph. The set $B \cup \{(u_2, v_2), (u_5, v_2)\}$ dominates and split the graph. Hence, $\gamma_s(C_m \oplus P_2) = k+3$.

Remark 4.2 *The Theorem 4.1 does not hold true for $m = 3$, since $C_3 \oplus P_2 \cong K_6$. Hence, γ_s-set does not exists.*

Theorem 4.3 *For $n \geq 3$,*

$$\gamma_s(C_3 \oplus P_n) = \begin{cases} k+2, & \text{if } n = 3k,\ k \geq 1 \\ k+3, & \text{if } n = 3k+1 \text{ or } 3k+2,\ k \geq 1. \end{cases}$$

Proof: Let C_3 be labeled as u_1, u_2, u_3 and P_n be labeled as $v_1, v_2 \ldots v_n$. Then the following cases are arise.

Case 1: $n = 3k$, $k \geq 1$.
In $C_3 \oplus P_n$, the subset $A = \{(u_2, v_{3t-1})/1 \leq t \leq k\}$ is the γ-set. Clearly no dominating set with cardinality $|A|$ split the graph. The set $A \cup \{(u_1, v_2), (u_3, v_2)\}$ dominates and split the graph. Hence, $\gamma_s(C_3 \oplus P_n) = k+2$.

Case 2: $n = 3k + 1$ or $3k + 2$, $k \geq 1$.
The subset $B = A \cup \{(u_2, v_n)\}$ is the minimum γ-set. Clearly no dominating set with cardinality $|B|$ or $|B| + 1$ split the graph. The set $B \cup \{(u_1, v_2), (u_3, v_2)\}$ dominates and split the graph. Hence, $\gamma_s(C_3 \oplus P_n) = k + 3$.

Theorem 4.4 *For $n \geq 3$,*

$$\gamma_s(C_4 \oplus P_n) = \begin{cases} 2(k+1), & \text{if } n = 3k,\ k \geq 1 \\ 2(k+2), & \text{if } n = 3k+1 \text{ or } 3k+2,\ k \geq 1. \end{cases}$$

Proof: Let C_4 be labeled as u_1, u_2, u_3, u_4 and P_n be labeled as $v_1, v_2 \ldots v_n$. Then the following cases are arise.

Case 1: $n = 3k$, $k \geq 1$.
In $C_4 \oplus P_n$, the subset $A = \{(u_i, v_{3t-1})/i = 2, 4 \text{ and } 1 \leq t \leq k\}$ is the γ-set. Clearly no dominating set with cardinality $|A|$ or $|A| + 1$ split the graph. The set $B = A \cup \{(u_1, v_2), (u_3, v_2)\}$ dominates and split the graph. Hence $\gamma_s(C_4 \oplus P_n) = 2(k + 1)$.

Case 2: $n = 3k + 1$ or $3k + 2$, $k \geq 1$.
The subset $B = A \cup \{(u_2, v_n), (u_4, v_n)\}$ is the γ-set. Clearly no dominating set with cardinality $|B|$ or $|B| + 1$ split the graph. The set $C = B \cup \{(u_1, v_2), (u_3, v_2)\}$ dominates and split the graph. Hence, $\gamma_s(C_4 \oplus P_n) = 2(k+2)$.

Theorem 4.5 *For $m = 3k_1$, $k_1 \geq 2$,*

$$\gamma_s(C_m \oplus P_n) = \begin{cases} k_1k_2 + 4, & \text{if } n = 3k_2,\ k_2 \geq 1 \\ k_1(k_2+1) + 4, & \text{if } n = 3k_2+1 \text{ or } 3k_2+2,\ k_2 \geq 1\ . \end{cases}$$

Proof: Let C_m be labeled as $u_1, u_2, \ldots u_m$ and P_n be labeled as $v_1, v_2, \ldots v_n$. Then the following cases are arise.

Case 1: $n = 3k_2$, $k_2 \geq 1$.
In $C_m \oplus P_n$, the subset $A = \{(u_{3t_1-1}, v_{3t_2-1})/1 \leq t_1 \leq k_1, 1 \leq t_2 \leq k_2\}$ is the γ-set. Clearly no dominating set with cardinality $|A|$ or $|A| + 1$ or $|A| + 2$ or $|A| + 3$ split the graph $C_m \oplus P_n$. The set $A \cup \{(u_1, v_2), (u_2, v_1), (u_m, v_1), (u_m, v_2)\}$ dominate and split the graph $C_m \oplus P_n$. Hence, $\gamma_s(C_m \oplus P_n) = k_1k_2 + 4$.

Case 2: $n = 3k_2 + 1$ or $3k_2 + 2$, $k_2 \geq 1$.
In $C_m \oplus P_n$, the subset $B = A \cup \{(u_{3t_1-1}, v_n)/1 \leq t_1 \leq k_1\}$ is the γ-set. Clearly no dominating set with cardinality $|B|$ or $|B| + 1$ or $|B| + 2$ or $|B| + 3$ split the graph $C_m \oplus P_n$. The set $B \cup \{(u_1, v_2), (u_2, v_1), (u_m, v_1), (u_m, v_2)\}$ dominates and split the graph $C_m \oplus P_n$. Hence, $\gamma_s(C_m \oplus P_n) = k_1(k_2 + 1) + 4$.

Theorem 4.6 *For $m \neq 4$, $m \neq 3k$, $k \geq 1$*

$$\gamma_s(C_m \oplus P_n) = \begin{cases} k_2(k_1+1)+3, & \text{if } n = 3k_2,\ k_2 \geq 1 \\ k_1k_2+k_1+k_2+3, & \text{if } n = 3k_2+1 \text{ or } 3k_2+2,\ k_2 \geq 1. \end{cases}$$

Proof: Let C_m be labeled as $u_1, u_2, \ldots u_m$ and P_n be labeled as $v_1, v_2, \ldots v_n$. Then the following cases are arise.

Case 1: $m \neq 4$, $m = 3k_1+1$ or $3k_1+2$, $k_1 \geq 1$ and $n = 3k_2$, $k_2 \geq 1$.
In $C_m \oplus P_n$, the subset $A = \{(u_{3t_1-1}, v_{3t_2-1})/1 \leq t_1 \leq k_1, 1 \leq t_2 \leq k_2\} \cup \{(u_m, v_{3t_2-1})/1 \leq t_2 \leq k_2\}$ is the γ-set. Clearly no dominating set with cardinality $|A|$ or $|A|+1$ or $|A|+2$ split the graph. The set $A \cup \{(u_1, v_2), (u_2, v_1), (u_m, v_1)\}$ dominates and split the graph $C_m \oplus P_n$. Hence, $\gamma_s(C_m \oplus P_n) = k_2(k_1+1)+3$.

Case 2: $m \neq 4$, $m = 3k_1+1$ or $3k_1+2$, $k_1 \geq 1$ and $n = 3k_2+1$ or $3k_2+2$, $k_2 \geq 1$.
In $C_m \oplus P_n$, the subset $A = \{(u_{3t_1-1}, v_{3t_2-1})/1 \leq t_1 \leq k_1, 1 \leq t_2 \leq k_2\} \cup \{(u_{3t_1-1}, v_n)/1 \leq t_1 \leq k_1\} \cup \{(u_m, v_{3t_2-1})/1 \leq t_2 \leq k_2\} \cup \{(u_m, v_n)\}$ is the γ-set. Clearly no dominating set with cardinality $|A|$ or $|A|+1$ or $|A|+2$. The set $B = A \cup \{(u_1, v_2), (u_2, v_1), (u_m, v_1)\}$ dominates and split the graph. Hence, $\gamma_s(C_m \oplus P_n) = k_1k_2+k_1+k_2+3$.

5 Normal Product of $C_m \oplus C_n$

Theorem 5.1 *For $n \geq 3$,*

$$\gamma_s(C_3 \oplus C_n) = \begin{cases} k+6, & \text{if } n = 3k,\ k \geq 1 \\ k+7, & \text{if } n = 3k+1 \text{ or } 3k+2,\ k \geq 1. \end{cases}$$

Proof: Let C_3 be labeled as u_1, u_2, u_3 and C_n be labeled as $v_1, v_2 \ldots v_n$. Then the following cases are arise.

Case 1: $n = 3k$, $k \geq 1$.
In $C_3 \oplus C_n$, the subset $A = \{(u_2, v_{3t-1})/1 \leq t \leq k\}$ is the γ-set. Clearly no dominating set with cardinality $|A|$ or $|A|+1$ or $|A|+2$ or $|A|+3$ or $|A|+4$ or $|A|+5$ split the graph. The set $A \cup \{(u_1, v_2), (u_m, v_1), (u_2, v_1),$
$(u_3, v_1), (u_3, v_2), (u_3, v_n)\}$ dominates and split the graph. Hence, $\gamma_s(C_3 \oplus C_n) = k+6$.

Case 2: $n = 3k+1$ or $3k+2$, $k \geq 1$.
The subset $B = A \cup \{(u_2, v_n)\}$ is the γ-set. Clearly no dominating set with cardinality $|B|$ or $|B|+1$ or $|B|+2$ or $|B|+3$ or $|B|+4$ or $|B|+5$ split the graph.

The set $B \cup \{(u_1, v_2), (u_m, v_1), (u_2, v_1), (u_3, v_1), (u_3, v_2), (u_3, v_n)\}$ dominates and split the graph. Hence, $\gamma_s(C_3 \oplus C_n) = k + 7$.

Theorem 5.2 *For $n \geq 3$,*

$$\gamma_s(C_4 \oplus C_n) = \begin{cases} 2(k+2), & \text{if } n = 3k,\ k \geq 1 \\ 2(k+3), & \text{if } n = 3k+1 \text{ or } 3k+2,\ k \geq 1. \end{cases}$$

Proof: Let C_4 be labeled as u_1, u_2, u_3, u_4 and C_n be labeled as $v_1, v_2 \ldots v_n$. Then the following cases are arise.

Case 1: $n = 3k$, $k \geq 1$.
In $C_4 \oplus C_n$, the subset $A = \{(u_i, v_{3t-1})/i = 2, 4 \text{ and } 1 \leq t \leq k\}$ is the γ-set. Clearly no dominating set with cardinality $|A|$ or $|A| + 1$ or $|A| + 2$ or $|A| + 3$ split the graph. The set $B = A \cup \{(u_2, v_1), (u_3, v_2), (u_3, v_n), (u_4, v_1)\}$ dominates and split the graph. Hence $\gamma_s(C_4 \oplus C_n) = 2(k + 2)$.

Case 2: $n = 3k + 1$ or $3k + 2$, $k \geq 1$.
The subset $B = A \cup \{(u_2, v_n), (u_4, v_n)\}$ is the γ-set. Clearly no dominating set with cardinality $|B|$ or $|B| + 1$ or $|B| + 2$ or $|B| + 3$ split the graph. The set $C = B \cup \{(u_2, v_1), (u_3, v_2), (u_3, v_n), (u_4, v_1)\}$ dominates and split the graph. Hence, $\gamma_s(C_4 \oplus C_n) = 2(k + 3)$.

Theorem 5.3 *For $m = 3k_1$, $k_1 \geq 2$,*

$$\gamma_s(C_m \oplus C_n) = \begin{cases} k_1k_2 + 5, & \text{if } n = 3k_2,\ k_2 \geq 1 \\ k_1(k_2+1) + 5, & \text{if } n = 3k_2+1 \text{ or } 3k_2+2,\ k_2 \geq 1. \end{cases}$$

Proof: Let C_m be labeled as $u_1, u_2, \ldots u_m$ and C_n be labeled as $v_1, v_2, \ldots v_n$. Then the following cases are arise.

Case 1: $m = 3k_1$, $k_1 \geq 2$ and $n = 3k_2$, $k_2 \geq 1$.
In $C_m \oplus C_n$, the subset $A = \{(u_{3t_1-1}, v_{3t_2-1})/1 \leq t_1 \leq k_1, 1 \leq t_2 \leq k_2\}$ is the γ-set. Clearly no dominating set with cardinality $|A|$ or $|A|+1$ or $|A|+2$ or $|A|+3$ or $|A|+4$ split the graph $C_m \oplus C_n$. The set $A \cup \{(u_1, v_1), (u_1, v_2), (u_2, v_n), (u_3, v_1), (u_3, v_2)\}$ dominate and split the graph $C_m \oplus C_n$. Hence, $\gamma_s(C_m \oplus C_n) = k_1k_2 + 5$.

Case 2: $n = 3k_2 + 1$ or $3k_2 + 2$, $k_2 \geq 1$.
In $C_m \oplus C_n$, the subset $B = A \cup \{(u_{3t_1-1}, v_n)/1 \leq t_1 \leq k_1\}$ is the γ-set. Clearly no dominating set with cardinality $|B|$ or $|B|+1$ or $|B|+2$ or $|B|+3$ or $|B|+4$ split the graph $C_m \oplus C_n$. The set $B \cup \{(u_1, v_1), (u_1, v_2), (u_2, v_n), (u_3, v_1), (u_3, v_2)\}$ dominates and split the graph $C_m \oplus C_n$. Hence, $\gamma_s(C_m \oplus C_n) = k_1(k_2 + 1) + 5$.

Theorem 5.4 *For $m \neq 4$, $m \neq 3k$, $k \geq 1$,*

$$\gamma_s(C_m \oplus C_n) = \begin{cases} k_2(k_1+1)+5, & \text{if } n = 3k_2,\ k_2 \geq 1 \\ k_1k_2+k_1+k_2+5, & \text{if } n = 3k_2+1 \text{ or } 3k_2+2,\ k_2 \geq 1. \end{cases}$$

Proof: Let C_m be labeled as $u_1, u_2, \ldots u_m$ and C_n be labeled as $v_1, v_2, \ldots v_n$. Then the following cases are arise.

Case 1: $m \neq 4$, $m = 3k_1+1$ or $3k_1+2$, $k_1 \geq 1$ and $n = 3k_2$, $k_2 \geq 1$.
In $C_m \oplus C_n$, the subset $A = \{(u_{3t_1-1}, v_{3t_2-1})/1 \leq t_1 \leq k_1, 1 \leq t_2 \leq k_2\} \cup \{(u_m, v_{3t_2-1})/1 \leq t_2 \leq k_2\}$ is the γ-set. Clearly no dominating set with cardinality $|A|$ or $|A|+1$ or $|A|+2$ or $|A|+3$ or $|A|+4$ split the graph. The set $A \cup \{(u_1, v_1), (u_1, v_2), (u_2, v_n), (u_3, v_1), (u_3, v_2)\}$ dominates and split the graph $C_m \oplus C_n$. Hence, $\gamma_s(C_m \oplus C_n) = k_2(k_1+1)+5$.

Case 2: $m \neq 4$, $m = 3k_1+1$ or $3k_1+2$, $k_1 \geq 1$ and $n = 3k_2+1$ or $3k_2+2$, $k_2 \geq 1$.
In $C_m \oplus C_n$, the subset $A = \{(u_{3t_1-1}, v_{3t_2-1})/1 \leq t_1 \leq k_1, 1 \leq t_2 \leq k_2\} \cup \{(u_{3t_1-1}, v_n)/1 \leq t_1 \leq k_1\} \cup \{(u_m, v_{3t_2-1})/1 \leq t_2 \leq k_2\} \cup \{(u_m, v_n)\}$ is the γ-set. Clearly no dominating set with cardinality $|A|$ or $|A|+1$ or $|A|+2$ or $|A|+3$. The set $B = A \cup \{(u_1, v_2), (u_1, v_n), (u_2, v_1), (u_m, v_1)\}$ dominates and split the graph. Hence, $\gamma_s(C_m \oplus C_n) = k_1k_2+k_1+k_2+5$.

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