



Gen. Math. Notes, Vol. 30, No. 1, September 2015, pp.60-68
ISSN 2219-7184; Copyright ©ICSRs Publication, 2015
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Clustering Using Strong Arcs in Fuzzy Graphs

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(Received: 24-5-14 / Accepted: 18-8-15)

Abstract

The clustering method based on the connectedness concepts in fuzzy graph is discussed. In [8] R.T Yeh and S.Y Bang introduced the concept of ϵ -clusters using reachability matrix of a fuzzy graph. This technique is modified using strong arcs in fuzzy graphs.

Keywords: *Fuzzy graphs, complete fuzzy graph, strong arc, ϵ -clusters.*

1 Introduction

A fuzzy graph [3, 2] is a pair $G : (\sigma, \mu)$ where σ is a fuzzy subset of a set S and μ is a fuzzy relation on σ such that $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$. We assume that S is finite and nonempty, μ is reflexive and symmetric[1]. Also, we denote the underlying crisp graph by $G^* : (\sigma^*, \mu^*)$ where $\sigma^* = \{u \in S : \sigma(u) > 0\}$ and $\mu^* = \{(u, v) \in S \times S : \mu(u, v) > 0\}$. The fuzzy graph $H : (\tau, \nu)$ is called a fuzzy subgraph of $G : (\sigma, \mu)$ induced by P if $P \subseteq S$, $\tau(u) = \sigma(u) \forall u \in P$ and $\nu(u, v) = \mu(u, v) \forall u, v \in P$. A fuzzy graph G is called a complete fuzzy graph if $\mu(u, v) = \sigma(u) \wedge \sigma(v) \forall u, v \in \sigma^*$. Also if G is a complete fuzzy graph then G^* is a complete graph. A path P of length n is a sequence of distinct nodes u_0, u_1, \dots, u_n such that $\mu(u_{i-1}, u_i) > 0, i = 1, 2, \dots, n$ and the degree of membership of a weakest arc is defined as its strength. strength of connectedness between two nodes x and y is defined as the maximum of the strengths of all paths between x , and is denoted by $CONN_G(x, y)$. An $x - y$ path P is called a strongest $x - y$ path if its strength equals $CONN_G(x, y)$ [1,4]. A fuzzy

graph $G : (\sigma, \mu)$ is connected if for every x, y in σ^* , $CONN_G(x, y) > 0$ [2].

An arc of a fuzzy graph is called strong if its weight is at least as great as the strength of connectedness of its end nodes when it is deleted and an $x - y$ path P is called a strong path if P contains only strong arcs[4]. A strong path P from x to y is an $x - y$ geodesic if there is no shorter strong path from x to y and the length of an $x - y$ geodesic is the geodesic distance from x to y denoted by $d_g(x, y)$ [5].

2 Cluster Analysis

Cluster analysis has been employed as an effective tool in scientific inquiry. One of its most useful roles is to generate hypotheses about category structure. Cluster analysis may be used to reveal structure and relations in the data. It is a tool of discovery. The results of cluster analysis can contribute directly to development of classification schemes. In other words it may be possible to reduce a very large body of data to a relatively compact description through cluster analysis. Graph theorists prefer 'partition' as a term describing a collection of clusters.

Fuzzy graph theory has numerous applications in various fields like clustering analysis, database theory, network analysis, information theory etc [2]. fuzzy models can be used in problems handling uncertainty to get more accurate and precise solutions[10,11,12]. As in graphs, connectivity concepts play a key role in applications related with fuzzy graphs [2,7]. R.T Yeh and S.Y Bang introduced different connectivity parameters of a fuzzy graph and discussed their application in the paper titled 'Fuzzy relations, fuzzy graphs and their applications to clustering analysis'[8]. A vertex v is said to be ϵ -reachable, from another vertex u , $0 \leq \epsilon \leq 1$, if there exists a positive integer k such that $\mu^k(u, v) \geq \epsilon$. a fuzzy graph G is called strongly ϵ -connected if every pair of vertices are mutually ϵ -reachable. A maximal strongly ϵ -connected fuzzy subgraph of G is a strongly ϵ -connected fuzzy subgraph not properly contained in any other maximal strongly ϵ -connected fuzzy subgraph. The reachability matrix of a fuzzy graph is denoted by M_{μ^∞} .

In graph theory, there are several ways of defining 'clusters' of vertices. One approach is to call a subset C of S a cluster of order k if the following two conditions hold[2];

1. for all vertices x, y in C , $d(x, y) \leq k$;
2. for all vertices $z \in C, d(z, w) > k$ for some $w' \in C$; where $d(u, v)$ is the length of a shortest path between two vertices u and v .

Thus in a k -cluster C , all pairs of vertices are within distance k of each other, and C is maximal with respect to this property. That is no vertex outside of C is within a distance k of every vertex in C . In[5], Sameena and Sunitha discussed clustering techniques using distance concepts in fuzzy graphs and introduced a method for finding clusters of order k using distance matrix of a fuzzy graph. Note that a k -cluster C is an ordinary subset of S . A new clustering technique based on fuzzy arc connectivity is introduced in [6].

3 Cluster Analysis Using Strong Arcs

This section analysis fuzzy graph from the view point of connectedness. Connectivity concepts are the key in graph clustering and networks. The connectivity parameters in fuzzy graphs introduced by Yeh and Bang [8] and constructed ϵ -clusters using the connectivity reachability matrix of a fuzzy graph. Sunil Mathew and Sunitha introduced clustering techniques based on fuzzy arc connectivity in [6]. This method illustrated with cancer detection problem, which classifies cell clusters in a tissue into different phases of cancer, depending on the distribution, density and the fuzzy connectivity of the cell clusters within the tissue. This process helps in examining the dynamic progress of the cancer qualitatively [6,9].

The algorithm for construction of ϵ -clusters in [8] is given below and a modification of this technique using strong arcs is carried out in this section .

Algorithm 1 [8]: Construction of ϵ - clusters.

1. Compute $\mu, \mu^2, \dots, \mu^\ell$, where ℓ is the smallest integer such that $\mu^\ell = \mu^{\ell+1}$;
2. Construct $F_\epsilon^{\mu^\ell}$ where $F_\epsilon^{\mu^\ell}$ is a family of non fuzzy sets given by

$$F_\epsilon^{\mu^\ell} = \{C \subseteq S : (\forall u \in S)[u \in C \Leftrightarrow (\forall v \in C)[\mu^\ell(u, v) \geq \epsilon]]\}$$

Then, each element in $F_\epsilon^{\mu^\ell}$ is an ϵ -cluster.

Using the above algorithm, ϵ -clusters are obtained in the following fuzzy graph

Example:1

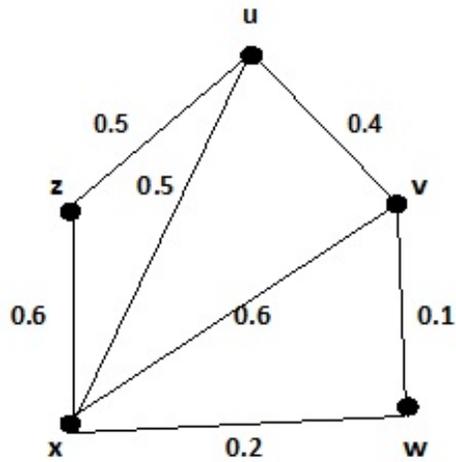


Fig:1

then,

$$M_{\mu}^1 = \begin{bmatrix} 1 & 0.4 & 0 & 0.5 & 0.5 \\ 0.4 & 1 & 0.1 & 0.6 & 0 \\ 0 & 0.1 & 1 & 0.2 & 0 \\ 0.5 & 0.6 & 0.2 & 1 & 0.6 \\ 0.5 & 0 & 0 & 0.6 & 1 \end{bmatrix}$$

$$M_{\mu}^2 = \begin{bmatrix} 1 & 0.4 & 0.1 & 0.5 & 0.5 \\ 0.4 & 1 & 0.2 & 0.6 & 0.6 \\ 0.1 & 0.2 & 1 & 0.2 & 0.2 \\ 0.5 & 0.6 & 0.2 & 1 & 0.6 \\ 0.5 & 0.6 & 0.2 & 0.6 & 1 \end{bmatrix}$$

$$M_{\mu}^3 = \begin{bmatrix} 1 & 0.5 & 0.2 & 0.5 & 0.5 \\ 0.5 & 1 & 0.2 & 0.6 & 0.6 \\ 0.2 & 0.2 & 1 & 0.2 & 0.2 \\ 0.5 & 0.6 & 0.2 & 1 & 0.6 \\ 0.5 & 0.6 & 0.2 & 0.6 & 1 \end{bmatrix}$$

$$M_{\mu}^4 = \begin{bmatrix} 1 & 0.5 & 0.2 & 0.5 & 0.5 \\ 0.5 & 1 & 0.2 & 0.6 & 0.6 \\ 0.2 & 0.2 & 1 & 0.2 & 0.2 \\ 0.5 & 0.6 & 0.2 & 1 & 0.6 \\ 0.5 & 0.6 & 0.2 & 0.6 & 1 \end{bmatrix}$$

$$= M_{\mu}^3$$

Thus the ϵ -clusters are as follows,

$$F_{0.2}^3 = \{u, v, w, x, z\}$$

$$F_{0.5}^3 = \{u, v, x, z\}, \{w\},$$

$$F_{0.6}^3 = \{u\}, \{v, x, z\}, \{w\}$$

$$F_1^3 = \{u\}, \{v\}, \{w\}, \{x\}, \{z\}$$

A modification of this technique using strong arc is discussed as,

Algorithm 2 : Construction of ϵ -clusters using strong arcs:

Let $G : (\sigma, \mu)$ be a fuzzy graph with $\sigma^* = S$.

Step 1 Identify the arcs (u, v) which are not strong in G .

Step 2 Construct $G' : (\sigma, \mu')$ from G where $\mu'(x, y) = \mu(x, y)$ for all strong arcs (x, y) in G , and for all arcs (u, v) which are not strong in G , $\mu'(u, v) = \bigvee_C \{ \wedge \mu(x, y) / (x, y) \text{ is strong in } C \}$ where C is a cycle in G^* which contains the arc (u, v) .

Thus all arcs in G' become strong.

Step 3 If G'^* is complete then put $G' = H$ and go to Step 5.
Otherwise go to Step 4.

Step 4 Construct $H : (\sigma, \nu)$ from G' in such a way that H^* is complete. Also $\nu(x, y) = \mu'(x, y) \forall (x, y) \in G'$ and for all (u, v) not in G' , put $\nu(u, v) = \bigvee_C \{ \wedge \mu'(x, y) / (x, y) \text{ is strong in } C \}$ where C is a cycle in H^* which contains (u, v) .

step 5 An ϵ - cluster C_ϵ is the collection of the set of nodes, such that the induced subgraph $\langle C_\epsilon \rangle$ is a complete subgraph of H^* with $\nu(u, v) \geq \epsilon \forall u, v \in C_\epsilon$.

Using the above algorithm ϵ -clusters are obtained in the following example as,

Example: 2

Consider the fuzzy graphs in Fig: 1, note that (u, v) and (v, w) are not strong in G . Therefore update the weights of these arcs as

$$\mu'(u, v) = \bigvee [\wedge(0.6, 0.5), \wedge(0.6, 0.6, 0.5), \wedge(0.1, 0.2, 0.6, 0.5), \wedge(0.1, 0.2, 0.5)] = 0.5.$$

$$\mu'(v, w) = \bigvee [\wedge(0.2, 0.6), \wedge(0.2, 0.6, 0.5, 0.4), \wedge(0.2, 0.5, 0.4)] = 0.2$$

Note that the arc (u, v) is in the cycles $C_1 = vxuv, C_2 = vxyuv, C_3 = vwxyuv, C_4 = vwxyuv$ in G .

Similarly, (v, w) contained in the cycles $C_1 = wxvw, C_2 = wxyuvw, C_3 = wxuvw$ Then the modified fuzzy graph $G' : (\sigma, \mu')$ as shown in Fig: 2.

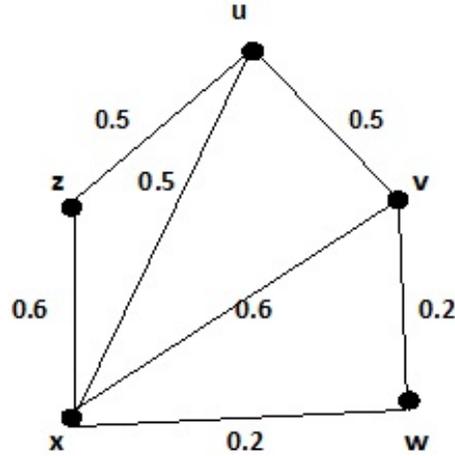


Fig:2

Note that G'^* in Fig:2 is not complete so construct the fuzzy graph $H : (\sigma, \nu)$ from G' in such a way that H^* is complete with $\nu(x, y) = \mu'(x, y) \forall (x, y) \in G'$ and for all (u, v) not in G' , put $\nu(u, v) = \bigvee_C \{ \wedge \mu'(x, y), (x, y) \text{ is strong in } C \}$ where C is a cycle in H^* which contains (u, v) .

Therefore $\nu(u, w) = \bigvee[\wedge(0.2, 0.5), \wedge(0.2, 0.6, 0.5), \wedge(0.2, 0.5)] = 0.2$, where (u, w) is contained in the cycles $C_1 = uvwv$, $C_2 = uvwxyu$ and $C_3 = uwxu$ in H .

Similarly $\nu(v, y) = \bigvee[\wedge(0.5, 0.5), \wedge(0.2, 0.2, 0.6,), \wedge(0.6, 0.6)] = 0.6$, where (v, y) contained in the cycles $C_1 = vyuv$, $C_2 = vwxyv$ and $C_3 = vxyv$ in H .

$\nu(w, y) = \bigvee[\wedge(0.2, 0.6), \wedge(0.2, 0.5, 0.5,), \wedge(0.2, 0.5)] = 0.2$, where (w, y) contained in the cycles $C_1 = wxyw$, $C_2 = wvuyw$, $C_3 = wvyw$ Then the modified fuzzy graph $H : (\sigma, \nu)$ with $\nu(u, v) = 0.5$, $\nu(v, w) = 0.2$, $\nu(w, x) = 0.2$, $\nu(u, x) = 0.5$, $\nu(u, z) = 0.5$, $\nu(v, x) = 0.6$, $\nu(x, z) = 0.6$, $\nu(, w) = 0.2$, $\nu(v, z) = 0.6$ and $\nu(w, z) = 0.2$. as shown in Fig: 3.

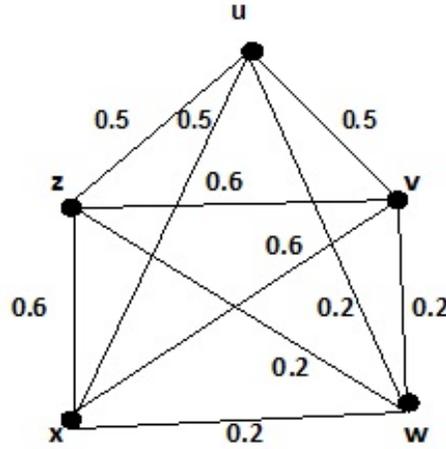


Fig:3

Now an ϵ -cluster C_ϵ is the collection of the set of nodes, such that the induced subgraph $\langle C_\epsilon \rangle$ is a complete subgraph of H^* with $\nu(u, v) \geq \epsilon, u, v \in C_\epsilon$.

There fore the various ϵ -clusters are, $C_{0.2} = \{u, v, w, x, z\}$,
 $C_{0.5} = \{u, v, x, z\}, \{w\}$,
 $C_{0.6} = \{u\}, \{v, x, z\}, \{w\}$ and
 $C_1 = \{u\}, \{v\}, \{w\}, \{x\}, \{z\}$, which are same as those obtained using Algorithm.1.

4 Concluding Remarks

In Algorithm:1 [8] R.T Yeh and S.Y Bang used the reachability matrix for the construction of ϵ -clusters, which take more computational time compared to the technique specified in this paper using strong arc.

Acknowledgements: This work was supported by the Department of Science and Technology , Govt. of India under the Womens Scientist Scheme 2009. I thank to professor M.S Sunitha, Department Of Mathematics, National Institute of Technology Calicut, for her valuable suggestions in improving the quality of this paper.

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