Gen. Math. Notes, Vol. 20, No. 1, January 2014, pp. 52-57
ISSN 2219-7184;
www.i-csrs.org
Available free online at http://www.geman.in

A Common Fixed Point Theorem of Non Continuous Mappings

Q.H. Khan

Department of Mathematics
Z.H. (E) College Delhi University, India
E-mail: qhkhan.ssitm@gmail.com

(Received: 8-9-13 / Accepted: 19-11-13)

Abstract

In this paper we prove a common fixed point theorem for non-continuous non-commuting mapping in metric space for six mappings which generalized the result of M. Kulkarn I , Badshah and many others.

Keywords: *Fixed Point, Coincidently Commuting mappings, Coincidence point.*

1 Introduction:

Every common fixed point theorem generally involved condition on the commutativity and continuity of the involved maps besides a suitable contraction condition and researches in this domain are aimed at weakening on or more these conditions. In recent years many other improve the commutativity condition and introduced the notion of Compatible mappings [4] Compatible mapping type (A) [6] Compatible mappings type (B) [10] Compatible type (C) [9] Compatible mapping type (P) [11] weak compatible mappings type (P) [3]. Recently R.P. Pant attempt to relax continuity requirement in such results and for the work of this kind one can referred to Sing and Mishra [12] and Pant [7, 8]. While proving our result, we utilized the notion of coincidentally commuting mappings which asserts that a pair of self-mappings is said to be coincidentally commuting, if they commute at their coincidence point [2]. Apart from this improvement the numbers of mappings are, raised from three to six whereas the completeness requirement of the space is substituted by a set of alternative weak conditions.

Here enlist the Kulkarni [6] results.

Theorem 1.1: *Let S, A and T be the continuous self- mappings of a complete metric space (X, d) such that the following conditions are satisfies.*

$$ST = TS, SA = AS, S(X) \subset A(X), S(X), \subset T(X)$$

$$[d(Sx, Sy)]^2 \leq a\,[d(Ax, Sx)d(Ty, Sx) + d(Ax, Sy)d(Ty, Sx)]$$
$$+b[d(Ax,Sx)d(Ty,Sy)+d(Ax,Sy)d(Ty,Sy)]$$

For all x, y in X, where a, b are non negative reals satisfying $a+b<1$, $a\geq 0, b\geq 0$.

Thus S, A and T have a unique common fixed point.

Motivated by Kulkarni [6] and Badshah [1] we present yet another extension of Theorem 1.1 by improving the contraction conditions.

2 Main Result

Theorem 2.1: *Let A, B, S, T, I and J be the self- mappings of a complete metric space* (X, d) satisfying $AB(X) \subset J(X), ST(X), \subset I(X)$.

$$[d(ABx, STy)]^2 \leq a\,[d(Ix, ABx)d(Jy, ABx) + d(Ix, STy)d(Jy, ABx)]$$
$$+b[d(Ix, ABx)d(Jy, STy) + d(Ix, STy)d(Jy, STy)] \qquad 2.1.1$$

and for each x, y in X, where a, b are non-negative reals satisfying $a+b\leq 1$, $a\geq 0$, $b\geq 0$. *If one of AB(X), ST(X), I(X) and J(X) is a complete subspace of X, then*

(a) (AB, I) has a coincidence point
(b) (ST, J) has a coincidence point

Further if the point (AB, I) and (ST, J) are coincidentally commuting, then AB, ST, I and J have a unique common fixed point. Moreover if the pair (AB, A), (AB, B), (AB, I), (A, I) (B, I), (ST, S), (ST, T), (S,J) and (T, J) commute at z, then z remain the unique common fixed point A, B, S, T, I and J.

Proof: Let x_o be an arbitrary point in X. Since AB(X) ⊂ J(X), we can find a point x_1 in X, such that $ABx_o = Jx_1$. Also since ST(X) ⊂ I(X), we can choose a point x_2 with $STx_1 = Ix_2$ using this arguments repeatedly one can construct a sequence $\{z_n\}$ such that

$z_n = ABx_{2n} = Jx_{2n+1}$, $z_{2n+1} = STx_{2n+1} = Ix_{2n+2}$, for n=0, 1, 2 …… for the sake of brevity let us put.

$$d^2(z_{2n+1}, z_{2n+2}) = d^2(STx_{2n+1}, ABx_{2n+2})$$

$$<a[d(Ix_{2n+2}, ABx_{2n+2})d(Jx_{2n+1}, ABx_{2n+2}) + d(Ix_{2n+2}, STx_{2n+1})\, d(Jx_{2n+1}, ABx_{2n+2}) +$$

$$b[d(Ix_{2n+2}), ABx_{2n+2})\, d(Jx_{2n+1}, STx_{2n+1}) + d(Ix_{2n+2}, STx_{2n+1})d(Jx_{2n+1}, STx_{2n+1})]$$

$$\leq a[d(z_{2n+2}, z_{2n+2})\, d(z_{2n}, z_{2n+2})] + b[d(z_{2n+2}, z_{2n+1})\, d(z_{2n}, z_{2n+1})]$$

$$[d(z_{2n+1}, z_{2n+2})]^2 \leq \frac{a+b}{(1-a)} [d(z_{2n}, z_{2n+1})]^2$$

Similarly one can show that

$$[d(z_{2n+1}, z_{2n})]^2 \leq \frac{a+b}{(1-a)} [d(z_{2n}, z_{2n-1})]^2$$

Which shows that $\{z_n\}$ is a Cauchy sequence in the complete metric space (X, d) and so has a limit point z in X. Now if follows that the sequence

$$\{ABx_0, ST_{x1}, AB_{x2}, \ldots\ldots\ldots\ldots STx_{2n-1}, ABx_{2n}, STx_{2n+1} \ldots\ldots\ldots\ldots\}$$

is a Cauchy sequency in the compete metric space (X,d) and so has a limit z in X. Hence the sequences $ABx_{2n} = Jx_{2n+1}$, $STx_{2n+1} = Ix_{2n+2}$ which are Subsequences, also converge to the point z.

Now suppose that I(X) is a complete subspace of X, then by observing that the subsequences $\{z_{2n}\}$ which is contained in I(X) must get a limit z in I(X).

Let $u \in I^{-1}z$, then lu = z

To prove that ABu = z, set x = u, and y = x_{2n-1} in (2.1.1), then

$$[d(ABu, STx_{2n-1})]^2 \leq a[d(Iu, ABu)\, d(Jx_{2n-1}, ABu) + d(lu, STx_{2n-1})d(Jx_{2n-1}, ABu)]$$

$$+b[d(Iu, ABu)\, d(Jx_{2n-1}, STx_{2n-1}) + d(Iu, STx_{2n-1})d(Jx_{2n-1}, STx_{2n-1})]$$

Which on leatting n → α, reduce to

$$[d(ABu, z)]^2 \leq a[d(ABu, z)]^2$$

Yielding thereby ABu = z = lu

Since AB(X) ⊂J (X), ABu = z implies that z∈J(X). Let v∈J^{-1} (z), then Jv = z

Again using the earlier arguments, it can be easily shown that STv = z, yielding thereby Jv = STv = z

The remaining two cases pertain essentially to the previous cases. Indeed if ST(X) is complete then z∈ ST(X) ⊂ I(X) and if AB(X) is complete then z∈ AB(X) ⊂ J(X).

Moreover if the pair (AB, I) and (ST, J) are coincidentally commuting at u and v respectively, then

(i) z = ABu = Iu = STv = Jv
(ii) ABz = AB (Iu) = I (ABu) = Iz
(iii) STz = ST(Jv) = J (STv) = Jz

Now we prove that STz = z

$$[d(ABu, STz)]^2 \leq a\,[d\,(lu, ABu)\,d\,(Jz, ABu) + d(lu, STz)\,d(Jz, ABu)]$$

$$+b[d(lu, ABu)\,d(Jz, STz) + d(lu, STz)\,d\,(Jz, STz))]$$

$$[d(z, STz)]^2 \leq a[d(z, STz)]^2$$

Yielding thereby STz = z = Jz similarly we can show that z = ABz = Iz. Then z is a common fixed point of AB, ST, I, and J. the uniqueness of common fixed point follows easily.

Finally we need to show that z is also common fixed point AB, ST, A, B, S, T I and J. For this let z be a common fixed point of the pair (AB, I) then

$$Az = A(ABz) = A(BAz) = AB(Az),\ Az = A(Iz) = I\,(Az)$$

$$Bz = B(BAz) = BA(Bz) = AB(Bz),\ Bz = B(Iz) = I(Bz)$$

Which shows that Az and Bz is a common fixed point of (AB, I) yielding thereby Az = Bz = Iz = ABz = z in view of the uniqueness of the common fixed point of the pair (AB, I).
Similarly we can show by using commutativity of (S, T), (S, J), (T, J), it can show that Sz = z = Tz = Jz = STz. Thus z is the unique common fixed point A,B,S,T,I and J.

3 An Illustrative Example

Now we furnish an example to demonstrate the validity of the hypothesis and degree of generality of Theorem 2.1 over earlier result.

Example 3.1. Consider X = [0, 6] with usual metric define self-mappings A, B, S, T, I and J as

$A0 = 0$, $Ax = 1$, $0 < x \le 6$	
$B0 = 0$, $Bx = 2$ $0 < x \le 6$	
$S0 = 0$, $Sx = 3$, $0 < x \le 6$	$S6 = 0$
$T0 = 0$, $Tx = 4$, $0 < x \le 6$	$T6 = 3$
$I0 = 0$, $Ix = 5$, $0 < x \le 6$	$I6 = 3$
$J0 = 0$, $Jx = 6$, $0 < x \le 6$	$J6 = 1$

One may note that all six maps in this example are discontinuous and even at their unique common fixed points '0'. Also the pairs (AB, I) and (ST, J) commute at o which is their common coincident point clearly AB(X) = {0, 1} ⊂J(x) = {0, 1, 6] and ST(X) = {0, 3} ⊂ I(X) = {0, 3, 5}. Also all needed pair wise commutativity coincidence point '0' is immediate.

By a routine calculation one can verify that contraction condition is satisfied for a = 1/3 and b = 1/5. Thus all the conditions, of Theorem 2.1 satisfied and '0' is the unique common fixed point of A, B, S, T, I and J.

References

[1] V.H. Badash and B. Singh, On common fixed points of commuting mappings, *Vikram Mathematical Journal*, V(1984), 13-16.

[2] B.C. Dhage, On common fixed points of pairs of coincidentally community mappings in D. metric spaces, *India J. Pure Applied Math.*, 30(4) (1999), 395-406.

[3] M. Imdad and Q.H. Khan, A general common fixed point theorem for six mappings, *Alig. Bull. of Maths*, 19(2002), 33-49.

[4] G. Jungck, Compatible mappings and common fixed points, *Internat J. Math. Math. Sci.*, 9(4) (1986), 771-779.

[5] G. Jungck, P.P. Murthy and Y.J. Cho, Compatible mappings of type (A) and common fixed point theorem, *Math. Japan*, 28(1993), 381-390.

[6] M. Kulkarni, Commuting fixed point theorem for the three mappings, *The Mathematical Education*, 2(June) (1993), 129-137.

[7] R.P. Pant, Common fixed point theorem for contractive maps, *J. Math. and Appl.*, 226(1998), 251-258.

[8] R.P. Pant, A common fixed point theorem under a new condition, *Indian J. Pure Appl. Math.*, 30(2) (1999), 147-152.

[9] H.K. Pathak, Y.J. Cho, S.M. Kang and B. Madharia, Compatible mappings of type (C) and common fixed point, *Theorems of Gregus Type*, 30(2) (1998), 499-518.

[10] H.K. Pathak and M.S. Khan, Compatible mappings of type (B) and common fixed point theorems of Gregus type, *Crzech Math. Jour*, 45(120) (1995), 685-698.

[11] H.K. Pathak, Y.J. Cho, S.S. Chang and S.M. Kang, Compatible mappings of type (P) and fixed point theorem in metric spaces and probabilistic metric spaces, *Novi Sad. J. Math*, 26(2) (1996), 87-109.

[12] S.L. Singh and S.N. Mishra, Remarks on Jachymski's fixed point theorem for compatible maps, *Indian J. Pure and Appl. Math.*, 28(5) (1997), 611-615.