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Differential Sandwich Theorems for Integral Operator of Certain Analytic Functions

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Abstract

In the present paper, we obtain some subordination and superordination results involving the integral operator $\Im_{\mu}^{\lambda}$ *for certain normalized analytic functions in the open unit disk. These results are applied to obtain sandwich results.*

Keywords: *Analytic functions, Differential subordination, Superordination, Sandwich theorems, Dominant, Subordinant, Integral operator.*

1 Introduction

Let $H = H(U)$ be the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. For n a positive integer and $a \in \mathbb{C}$. Let $H\,[a\,, n]$ be the subclass of H consisting of functions of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \qquad (a \in \mathbb{C}). \tag{1.1}$$

Also, let T be the subclass of H consisting of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.2}$$

Let $f, g \in H$. The function f is said to be subordinate to g, or g is said to be superordinate to f, if there exists a Schwarz function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z) = g(w(z))$. In such a case we write $f \prec g$ or $f(z) \prec g(z) (z \in U)$. If g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let $p, h \in H$ and $\psi(r, s, t; z): \mathbb{C}^3 \times U \longrightarrow \mathbb{C}$. If p and $\psi(p(z), zp'(z), z^2p''(z); z)$ are univalent functions in U and if p satisfies the second-order differential superordination

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z), \tag{1.3}$$

then p is called a solution of the differential superordination (1.3). (If f is subordinate to g, then g is superordinate to f). An analytic function q is called a subordinant of (1.3),if $q \prec p$ for all the functions p satisfying (1.3). An univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all the subordinants q of (1.3) is called the best subordinant. Miller and Mocanu [6] have obtained conditions on the functions h, q and ψ for which the following implication holds:

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z) \implies q(z) \prec p(z). \tag{1.4}$$

Komatu [4] introduced and investigated a family of integral operator $\mathfrak{I}_\mu^\lambda : T \longrightarrow T$, which is defined as follows:

$$\mathfrak{I}_\mu^\lambda f(z) = \frac{\mu^\lambda}{\Gamma(\lambda) z^{\mu-1}} \int_0^z \varepsilon^{\mu-2} \left(log \frac{z}{\varepsilon}\right)^{\lambda-1} f(\varepsilon) d\varepsilon$$

$$= z + \sum_{n=2}^{\infty} \left(\frac{\mu}{\mu + n - 1}\right)^\lambda a_n z^n \quad (z \in U, \mu > 0, \lambda \geq 0). \tag{1.5}$$

We note from (1.5) that, we have

$$z\left(\mathfrak{I}_\mu^{\lambda+1} f(z)\right)' = \mu \mathfrak{I}_\mu^\lambda f(z) - (\mu - 1)\mathfrak{I}_\mu^{\lambda+1} f(z). \tag{1.6}$$

Ali et al. [1] obtained sufficient conditions for certain normalized analytic functions to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$. Also, Tuneski [9] obtained a sufficient conditions for star likeness of f in terms of the quantity $\frac{f''(z)f(z)}{\left(f'(z)\right)^2}$. Recently, Shanmugam et al. [7,8], Goyal et al. [3] also obtained sandwich results for certain classes of analytic functions.

The main object of the present paper is to find sufficient conditions for certain normalized analytic functions f to satisfy

$$q_1(z) \prec \left(\frac{\Im_\mu^{\lambda+1} f(z)}{z}\right)^\gamma \prec q_2(z),$$

and

$$q_1(z) \prec \left(\frac{t\Im_\mu^{\lambda+1} f(z) + (1-t)\Im_\mu^{\lambda} f(z)}{z}\right)^\gamma \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$.

2 Preliminaries

In order to prove our subordination and superordination results, we need the following definition and lemmas.

Definition 2.1 [5]: *Denote by Q the set of all functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where*

$$\mathrm{E}(f) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty \right\} \tag{2.1}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

Lemma 2.1 [5]: *Let q be univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$ .Set $Q(z) = zq'(z)\phi\left(q(z)\right)$ and $h(z) = \theta\left(q(z)\right) + Q(z)$. Suppose that*

(i) $Q(z)$ is starlike univalent in U,
(ii) $Re\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$ for $z \in U$.

If p is analytic in U, with $p(0) = q(0), p(U) \subset D$ and

$$\theta\left(p(z)\right) + zp'(z)\phi\left(p(z)\right) \prec \theta\left(q(z)\right) + zq'(z)\phi\left(q(z)\right), \tag{2.2}$$

then $p \prec q$ and q is the best dominant of (2.2).

Lemma 2.2 [6]: *Let q be a convex univalent function in U and let $\alpha \in \mathbb{C}, \beta \in \mathbb{C} \setminus \{0\}$ with*

$$Re\left\{1+\frac{zq''(z)}{q'(z)}\right\}>\max\left\{0,-Re(\frac{\alpha}{\beta})\right\}.$$

If p is analytic in U and

$$\alpha p(z)+\beta zp'(z)\prec\alpha q(z)+\beta zq'(z), \tag{2.3}$$

then $p\prec q$ and q is the best dominant of (2.3).

Lemma 2.3 [6]: *Let q be convex univalent in U and let $\beta\in\mathbb{C}$. Further assume that $Re(\beta)>0$. If $p\in H\,[q(0),1]\cap Q$ and $p(z)+\beta zp'(z)$ is univalent in U, then*

$$q(z)+\beta zq'(z)\prec p(z)+\beta zp'(z), \tag{2.4}$$

which implies that $q\prec p$ and q is the best subordinant of (2.4).

Lemma 2.4 [2]: *Let q be convex univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$. Suppose that*

(i) $Re\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\}>0$ for $z\in U$,
(ii) $Q(z)=zq'(z)\phi\big(q(z)\big)$ is starlike univalent in U.

If $p\in H\,[q(0),1]\cap Q$, with $p(U)\subset D,\theta\big(p(z)\big)+zp'(z)\phi\big(p(z)\big)$ is univalent in U and

$$\theta\big(q(z)\big)+zq'(z)\phi\big(q(z)\big)\prec\theta\big(p(z)\big)+zp'(z)\phi\big(p(z)\big), \tag{2.5}$$

then $q\prec p$ and q is the best subordinant of (2.5).

3 Subordination Results

Theorem 3.1: *Let q be convex univalent in U with $q(0)=1,\ 0\neq\eta\in\mathbb{C},\gamma>0$ and suppose that q satisfies*

$$Re\left\{1+\frac{zq''(z)}{q'(z)}\right\}>\max\left\{0,-Re\left(\frac{\gamma}{\eta}\right)\right\}. \tag{3.1}$$

If $f\in T$ satisfies the subordination

$$(1-\mu\eta)\left(\frac{\Im_\mu^{\lambda+1}f(z)}{z}\right)^\gamma+\mu\eta\left(\frac{\Im_\mu^{\lambda+1}f(z)}{z}\right)^\gamma\left(\frac{\Im_\mu^{\lambda}f(z)}{\Im_\mu^{\lambda+1}f(z)}\right)\prec q(z)+\frac{\eta}{\gamma}zq'(z), \tag{3.2}$$

then

$$\left(\frac{\Im_\mu^{\lambda+1}f(z)}{z}\right)^\gamma \prec q(z) \tag{3.3}$$

and q is the best dominant of (3.2).

Proof: Define the function p by

$$p(z) = \left(\frac{\Im_\mu^{\lambda+1}f(z)}{z}\right)^\gamma. \tag{3.4}$$

Differentiating (3.4) with respect to z logarithmically, we get

$$\frac{zp'(z)}{p(z)} = \gamma\left(\frac{z\left(\Im_\mu^{\lambda+1}f(z)\right)'}{\Im_\mu^{\lambda+1}f(z)} - 1\right). \tag{3.5}$$

Now, in view of (1.6), we obtain the following subordination

$$\frac{zp'(z)}{p(z)} = \gamma\mu\left(\frac{\Im_\mu^{\lambda}f(z)}{\Im_\mu^{\lambda+1}f(z)} - 1\right).$$

Therefore,

$$\frac{zp'(z)}{\gamma} = \mu\left(\frac{\Im_\mu^{\lambda+1}f(z)}{z}\right)^\gamma\left(\frac{\Im_\mu^{\lambda}f(z)}{\Im_\mu^{\lambda+1}f(z)} - 1\right).$$

The subordination (3.2) from the hypothesis becomes

$$p(z) + \frac{\eta}{\gamma}zp'(z) \prec q(z) + \frac{\eta}{\gamma}zq'(z).$$

An application of Lemma 2.2 with $\beta = \frac{\eta}{\gamma}$ and $\alpha = 1$, we obtain (3.3).

Putting $q(z) = \left(\frac{1+z}{1-z}\right)^\sigma$ $(0 < \sigma \leq 1)$ in Theorem 3.1, we obtain the following corollary:

Corollary 3.1: Let $0 < \sigma \leq 1, 0 \neq \eta \in \mathbb{C}, \gamma > 0$ and

$$Re\left\{\frac{1 + 2\sigma z + z^2}{1 - z^2}\right\} > \max\left\{0, -Re\left(\frac{\gamma}{\eta}\right)\right\}.$$

If $f \in T$ satisfies the subordination

$$(1-\mu\eta)\left(\frac{\Im_\mu^{\lambda+1}f(z)}{z}\right)^\gamma+\mu\eta\left(\frac{\Im_\mu^{\lambda+1}f(z)}{z}\right)^\gamma\left(\frac{\Im_\mu^{\lambda}f(z)}{\Im_\mu^{\lambda+1}f(z)}\right)$$
$$\prec\left(1+\frac{2\eta\sigma z}{\gamma(1-z^2)}\right)\left(\frac{1+z}{1-z}\right)^\sigma,$$

then

$$\left(\frac{\Im_\mu^{\lambda+1}f(z)}{z}\right)^\gamma\prec\left(\frac{1+z}{1-z}\right)^\sigma$$

and $q(z)=\left(\frac{1+z}{1-z}\right)^\sigma$ is the best dominant.

Theorem 3.2: *Let q be convex univalent in U with $q(0)=1, q(z)\neq 0\ (z\in U)$ and assume that q satisfies*

$$Re\left\{1+\frac{um}{\eta}+\frac{v(m+1)}{\eta}q(z)+(m-1)\frac{zq'(z)}{q(z)}+\frac{zq''(z)}{q'(z)}\right\}>0, \qquad (3.6)$$

where $u,v,m\in\mathbb{C},\eta\in\mathbb{C}\setminus\{0\}$ and $z\in U$.

Suppose that $z\big(q(z)\big)^{m-1}q'(z)$ is starlike univalent in U . If $f\in T$ satisfies

$$\varphi(u,v,\gamma,\lambda,t,m,\mu,\eta;z)\prec\big(u+vq(z)\big)\big(q(z)\big)^m+\eta z\big(q(z)\big)^{m-1}q'(z), \qquad (3.7)$$

where

$$\varphi(u,v,\gamma,\lambda,t,m,\mu,\eta;z)$$

$$=u\left(\frac{t\Im_\mu^{\lambda+1}f(z)+(1-t)\Im_\mu^{\lambda}f(z)}{z}\right)^{\gamma m}+v\left(\frac{t\Im_\mu^{\lambda+1}f(z)+(1-t)\Im_\mu^{\lambda}f(z)}{z}\right)^{\gamma(m+1)}$$

$$+\eta\gamma\mu\left(\frac{t\Im_\mu^{\lambda+1}f(z)+(1-t)\Im_\mu^{\lambda}f(z)}{z}\right)^{\gamma m}\left(\frac{t\Im_\mu^{\lambda}f(z)+(1-t)\Im_\mu^{\lambda-1}f(z)}{t\Im_\mu^{\lambda+1}f(z)+(1-t)\Im_\mu^{\lambda}f(z)}-1\right),$$

$$(0\le t\le 1,\gamma>0,z\in U), \qquad (3.8)$$

then

$$\left(\frac{t\Im_\mu^{\lambda+1}f(z)+(1-t)\Im_\mu^{\lambda}f(z)}{z}\right)^\gamma\prec q(z) \qquad (3.9)$$

and q is the best dominant of (3.7).

Proof: Define the function p by

$$p(z)=\left(\frac{t\Im_\mu^{\lambda+1}f(z)+(1-t)\Im_\mu^{\lambda}f(z)}{z}\right)^\gamma. \qquad (3.10)$$

By setting

$$\theta(w) = (u + vw)w^m \quad and \quad \phi(w) = \eta w^{m-1}, w \neq 0,$$

we see that $\theta(w)$ is analytic in $\mathbb{C}$, $\phi(w)$ is analytic in $\mathbb{C} \setminus \{0\}$ and that $\phi(w) \neq 0, w \in \mathbb{C} \setminus \{0\}$. Also, we get

$$Q(z) = zq'(z)\phi(q(z)) = \eta z(q(z))^{m-1}q'(z)$$

and

$$h(z) = \theta(q(z)) + Q(z) = (u + vq(z))(q(z))^m + \eta z(q(z))^{m-1}q'(z)\,.$$

It is clear that $Q(z)$ is starlike univalent in U,

$$Re\left\{\frac{zh'(z)}{Q(z)}\right\} = Re\left\{1 + \frac{um}{\eta} + \frac{v(m+1)}{\eta}q(z) + (m-1)\frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right\} > 0.$$

By a straightforward computation, we obtain

$$(u + vp(z))(p(z))^m + \eta z(p(z))^{m-1}p'(z) = \varphi(u, v, \gamma, \lambda, t, m, \mu, \eta; z), \quad (3.11)$$

where $\varphi(u, v, \gamma, \lambda, t, m, \mu, \eta; z)$ is given by (3.8).

From (3.7) and (3.11), we have

$$\begin{aligned}&(u + vp(z))(p(z))^m + \eta z(p(z))^{m-1}p'(z)\\ &\prec (u + vq(z))(q(z))^m + \eta z(q(z))^{m-1}q'(z).\end{aligned} \quad (3.12)$$

Therefore, by Lemma 2.1, we get $p(z) \prec q(z)$. By using (3.10), we obtain the result.

Putting $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 3.2, we obtain the following corollary:

Corollary 3.2: *Let* $-1 \leq B < A \leq 1$ *and*

$$Re\left\{\frac{um}{\eta} + \frac{v(m+1)(1+Az)}{\eta(1+Bz)} + \frac{1 + m(A-B)z - ABz^2}{(1+Az)(1+Bz)}\right\} > 0\,,$$

where $u, v, m \in \mathbb{C}, \eta \in \mathbb{C} \setminus \{0\}$ and $z \in U$. If $f \in T$ satisfies

$$\begin{aligned}&\varphi(u, v, \gamma, \lambda, t, m, \mu, \eta; z)\\ &\prec \left(u + v\left(\frac{1+Az}{1+Bz}\right)\right)\left(\frac{1+Az}{1+Bz}\right)^m + \frac{\eta(A-B)(1+Az)^{m-1}z}{(1+Bz)^{m+1}},\end{aligned}$$

where $\varphi(u, v, \gamma, \lambda, t, m, \mu, \eta; z)$ is given by (3.8),

then

$$\left(\frac{t\Im_{\mu}^{\lambda+1}f(z)+(1-t)\Im_{\mu}^{\lambda}f(z)}{z}\right)^{\gamma}\prec\frac{1+Az}{1+Bz}$$

and $q(z)=\frac{1+Az}{1+Bz}$ is the best dominant.

4 Superordination Results

Theorem 4.1: *Let q be convex univalent in U with $q(0)=1,\gamma>0$ and $Re\{\eta\}>0$. Let $f\in T$ satisfies*

$$\left(\frac{\Im_{\mu}^{\lambda+1}f(z)}{z}\right)^{\gamma}\in H\,[q(0),1]\cap Q$$

and

$$(1-\mu\eta)\left(\frac{\Im_{\mu}^{\lambda+1}f(z)}{z}\right)^{\gamma}+\mu\eta\left(\frac{\Im_{\mu}^{\lambda+1}f(z)}{z}\right)^{\gamma}\left(\frac{\Im_{\mu}^{\lambda}f(z)}{\Im_{\mu}^{\lambda+1}f(z)}\right)$$

be univalent in U. If

$$q(z)+\frac{\eta}{\gamma}zq'(z)\prec(1-\mu\eta)\left(\frac{\Im_{\mu}^{\lambda+1}f(z)}{z}\right)^{\gamma}+\mu\eta\left(\frac{\Im_{\mu}^{\lambda+1}f(z)}{z}\right)^{\gamma}\left(\frac{\Im_{\mu}^{\lambda}f(z)}{\Im_{\mu}^{\lambda+1}f(z)}\right),\tag{4.1}$$

then

$$q(z)\prec\left(\frac{\Im_{\mu}^{\lambda+1}f(z)}{z}\right)^{\gamma}\tag{4.2}$$

and q is the best subordinant of (4.1).

Proof: Define the function p by

$$p(z)=\left(\frac{\Im_{\mu}^{\lambda+1}f(z)}{z}\right)^{\gamma}.\tag{4.3}$$

Differentiating (4.3) with respect to z logarithmically, we get

$$\frac{zp'(z)}{p(z)}=\gamma\left(\frac{z\left(\Im_{\mu}^{\lambda+1}f(z)\right)'}{\Im_{\mu}^{\lambda+1}f(z)}-1\right).\tag{4.4}$$

After some computations and using (1.6), from (4.4), we obtain

$$(1-\mu\eta)\left(\frac{\Im_{\mu}^{\lambda+1}f(z)}{z}\right)^{\gamma}+\mu\eta\left(\frac{\Im_{\mu}^{\lambda+1}f(z)}{z}\right)^{\gamma}\left(\frac{\Im_{\mu}^{\lambda}f(z)}{\Im_{\mu}^{\lambda+1}f(z)}\right)=p(z)+\frac{\eta}{\gamma}zp'(z),$$

and now, by using Lemma 2.3, we get the desired result.

Putting $q(z)=\left(\frac{1+z}{1-z}\right)^{\sigma}$ $(0<\sigma\leq 1)$ in Theorem 4.1, we obtain the following corollary:

Corollary 4.1: *Let* $0<\sigma\leq 1, \gamma>0$ *and* $Re\{\eta\}>0$. *If* $f\in T$ *satisfies*

$$\left(\frac{\Im_{\mu}^{\lambda+1}f(z)}{z}\right)^{\gamma}\in H\,[q(0),1]\cap Q$$

and

$$(1-\mu\eta)\left(\frac{\Im_{\mu}^{\lambda+1}f(z)}{z}\right)^{\gamma}+\mu\eta\left(\frac{\Im_{\mu}^{\lambda+1}f(z)}{z}\right)^{\gamma}\left(\frac{\Im_{\mu}^{\lambda}f(z)}{\Im_{\mu}^{\lambda+1}f(z)}\right)$$

be univalent in U. If

$$\left(1+\frac{2\eta\sigma z}{\gamma(1-z^2)}\right)\left(\frac{1+z}{1-z}\right)^{\sigma}$$

$$\prec(1-\mu\eta)\left(\frac{\Im_{\mu}^{\lambda+1}f(z)}{z}\right)^{\gamma}+\mu\eta\left(\frac{\Im_{\mu}^{\lambda+1}f(z)}{z}\right)^{\gamma}\left(\frac{\Im_{\mu}^{\lambda}f(z)}{\Im_{\mu}^{\lambda+1}f(z)}\right),$$

then

$$\left(\frac{1+z}{1-z}\right)^{\sigma}\prec\left(\frac{\Im_{\mu}^{\lambda+1}f(z)}{z}\right)^{\gamma}$$

and $q(z)=\left(\frac{1+z}{1-z}\right)^{\sigma}$ is the best subordinant.

Theorem 4.2: *Let* q *be convex univalent in* U *with* $q(0)=1$, *and assume that* q *satisfies*

$$Re\left\{\frac{um}{\eta}q'(z)+\frac{v(m+1)}{\eta}q(z)q'(z)\right\}>0, \qquad (4.5)$$

where $u,v,m\in\mathbb{C},\eta\in\mathbb{C}\setminus\{0\}$ and $z\in U$.

Suppose that $z\big(q(z)\big)^{m-1}q'(z)$ is starlike univalent in U . Let $f\in T$ satisfies

$$\left(\frac{t\Im_{\mu}^{\lambda+1}f(z)+(1-t)\Im_{\mu}^{\lambda}f(z)}{z}\right)^{\gamma}\in H\,[q(0),1]\cap Q$$

and $\varphi(u,v,\gamma,\lambda,t,m,\mu,\eta;z)$ is univalent in U, where $\varphi(u,v,\gamma,\lambda,t,m,\mu,\eta;z)$ is given by (3.8). If

$$\big(u+vq(z)\big)\big(q(z)\big)^{m}+\eta z\big(q(z)\big)^{m-1}q'(z)\prec\varphi(u,v,\gamma,\lambda,t,m,\mu,\eta;z), \qquad (4.6)$$

then

$$q(z)\prec\left(\frac{t\Im_{\mu}^{\lambda+1}f(z)+(1-t)\Im_{\mu}^{\lambda}f(z)}{z}\right)^{\gamma} \qquad (4.7)$$

and q is the best subordinant of (4.6).

Proof: Define the function p by

$$p(z) = \left(\frac{t\Im_{\mu}^{\lambda+1}f(z) + (1-t)\Im_{\mu}^{\lambda}f(z)}{z}\right)^{\gamma}. \tag{4.8}$$

By setting

$$\theta(w) = (u + vw)w^{m} \quad and \quad \phi(w) = \eta w^{m-1}, w \neq 0,$$

we see that $\theta(w)$ is analytic in $\mathbb{C}$, $\phi(w)$ is analytic in $\mathbb{C} \setminus \{0\}$ and that $\phi(w) \neq 0, w \in \mathbb{C} \setminus \{0\}$. Also, we get

$$Q(z) = zq^{'}(z)\phi\big(q(z)\big) = \eta z\big(q(z)\big)^{m-1}q^{'}(z).$$

It is clear that $Q(z)$ is starlike univalent in U,

$$Re\left\{\frac{\theta^{'}\big(q(z)\big)}{\phi\big(q(z)\big)}\right\} = Re\left\{\frac{um}{\eta}q^{'}(z) + \frac{v(m+1)}{\eta}q(z)q^{'}(z)\right\} > 0.$$

By a straightforward computation, we obtain

$$\varphi(u, v, \gamma, \lambda, t, m, \mu, \eta; z) = \big(u + vp(z)\big)\big(p(z)\big)^{m} + \eta z\big(p(z)\big)^{m-1}p^{'}(z), \tag{4.9}$$

where $\varphi(u, v, \gamma, \lambda, t, m, \mu, \eta; z)$ is given by (3.8).

From (4.6) and (4.9), we have

$$\big(u + vq(z)\big)\big(q(z)\big)^{m} + \eta z\big(q(z)\big)^{m-1}q^{'}(z)$$

$$\prec \big(u + vp(z)\big)\big(p(z)\big)^{m} + \eta z\big(p(z)\big)^{m-1}p^{'}(z). \tag{4.10}$$

Therefore, by Lemma 2.4, we get $q(z) \prec p(z)$. By using (4.8), we obtain the result.

5 Sandwich Results

Concluding the results of differential subordination and superordination, we arrive at the following "sandwich results".

Theorem 5.1: *Let q_1 be convex univalent in U with $q_1(0) = 1$, $Re\{\eta\} > 0$ and let q_2 be univalent in U, $q_2(0) = 1$ and satisfies (3.1). Let $f \in T$ satisfies*

$$\left(\frac{\Im_{\mu}^{\lambda+1}f(z)}{z}\right)^{\gamma} \in H\,[1,1] \cap Q$$

and

$$(1-\mu\eta)\left(\frac{\Im_{\mu}^{\lambda+1}f(z)}{z}\right)^{\gamma}+\mu\eta\left(\frac{\Im_{\mu}^{\lambda+1}f(z)}{z}\right)^{\gamma}\left(\frac{\Im_{\mu}^{\lambda}f(z)}{\Im_{\mu}^{\lambda+1}f(z)}\right)$$

be univalent in U. If

$$q_1(z)+\frac{\eta}{\gamma}zq_1{}'(z)\prec(1-\mu\eta)\left(\frac{\Im_{\mu}^{\lambda+1}f(z)}{z}\right)^{\gamma}+\mu\eta\left(\frac{\Im_{\mu}^{\lambda+1}f(z)}{z}\right)^{\gamma}\left(\frac{\Im_{\mu}^{\lambda}f(z)}{\Im_{\mu}^{\lambda+1}f(z)}\right)$$
$$\prec q_2(z)+\frac{\eta}{\gamma}zq_2{}'(z),$$

then

$$q_1(z)\prec\left(\frac{\Im_{\mu}^{\lambda+1}f(z)}{z}\right)^{\gamma}\prec q_2(z)$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant.

Theorem 5.2: *Let q_1 be convex univalent in U with $q_1(0)=1$ and satisfies (4.5) and let q_2 be univalent in U, $q_2(0)=1$ and satisfies (3.6). Let $f\in T$ satisfies*

$$\left(\frac{t\Im_{\mu}^{\lambda+1}f(z)+(1-t)\Im_{\mu}^{\lambda}f(z)}{z}\right)^{\gamma}\in H\,[1,1]\cap Q$$

and $\varphi(u,v,\gamma,\lambda,t,m,\mu,\eta;z)$ is univalent in U, where $\varphi(u,v,\gamma,\lambda,t,m,\mu,\eta;z)$ is given by (3.8). If

$$\big(u+vq_1(z)\big)\big(q_1(z)\big)^{m}+\eta z\big(q_1(z)\big)^{m-1}q_1{}'(z)\prec\varphi(u,v,\gamma,\lambda,t,m,\mu,\eta;z)$$
$$\prec\big(u+vq_2(z)\big)\big(q_2(z)\big)^{m}+\eta z\big(q_2(z)\big)^{m-1}q_2{}'(z),$$

then

$$q_1(z)\prec\left(\frac{t\Im_{\mu}^{\lambda+1}f(z)+(1-t)\Im_{\mu}^{\lambda}f(z)}{z}\right)^{\gamma}\prec q_2(z)$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant.

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