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The Dynamics of an Eco-Epidemiological Model with (SI), (SIS) Epidemic Disease in Prey

Azhar Abbas Majeed[1] and Inaam Ibrahim Shawka[2]

[1,2]Department of Mathematic, College of Science
University of Baghdad, Baghdad, Iraq
[1]E-mail: azhar_abbas_m@yahoo.com
[2]E-mail: enaam.ib@gmail.com

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Abstract

In this paper, a mathematical model consisting of the prey-predator model with (SI and SIS) infectious diseases in prey which is transmitted within the same species by contact and external source is proposed and analyzed. The existence, uniqueness and boundedness of the solution are discussed. The stability analysis of all possible equilibrium points is studied. Suitable Lyapunov functions are used to study the global dynamics of the proposed model. The effect of the two diseases on the dynamical behavior of the system is discussed using numerical simulation.

Keywords: *Prey-Predator Model, Epidemic Model, Stability Analysis, Lyapunov Function.*

1 Introduction

Mathematical models are divided into two fields the first one describe the dynamical behavior of an interacting species in ecology which is known as ecological models, while the second field, which studying the spread and control of diseases in human or animal population, is known as epidemiological model. Along the years until now many researchers studied the dynamic of mathematical models of ecological and epidemiological independently [1,2,3,4,5,6,7,8,9]. After these pioneering works in two different fields, lots of research works have been done both in theoretical ecology and epidemiology. While Anderson and May [10] were the first who merged the above two fields and formulated Lotka-Volterra predator-prey model with infection disease spread among prey by contact between them and no reproduction in infected prey.

There are many types of epidemic disease such that SI, SIS and SIR. Further, number of ecological models involving SI, SIS and SIR epidemic disease in one species have proposed and studied [11,12], Dahlia [13] studied a prey-predator model with SIS epidemic disease in prey, while [14, 15] studied a prey-predator model involving SI and SIR epidemic disease in prey. Also, many researchers proposed and study eco-epidemiological models with two infected diseases spread in the same population [16, 17, 18].

On the other hand there are many sources to spread the disease among the population while one of the most ways to spread infectious disease is by contact between the susceptible and infected individual. Moreover many diseases are transmitted in the same species not only through contact, but also directly from environment (external source). Ahmed and Hanan [19] studied the effect of external source of disease on epidemic model. Das et al [20] proposed prey-predator model with disease in prey spread by contact and external source, with Holling type II functional response and linear incidence rate. K.Q. Khalaf, A.A. Majeed and R.K. Naji [21] studied prey-predator model involving SIS infectious disease in prey population this disease passed from a prey to predator through attacking of predator to prey and the disease transmitted within the same species by contact and external source.

Recently, Rasha [22] had proposed and studied a prey-predator model involving, in addition to harvest in predator species, two different SIS infectious diseases in prey species; it is assumed that both the disease spread within prey population by contact, between susceptible individuals and infected individuals. Furthermore, she used linear functional response and linear incidence rate to describe spread both diseases.

In this paper, an eco-epidemiological mathematical model consisting of prey-predator model involving SI and SIS epidemic diseases in prey species has been proposed and analyzed. Further, in this model, linear type of functional response as well as linear incidence rate for describing the transition of disease are used.

2 Mathematical Model

In this section, an eco-epidemiological model with two infectious diseases in prey population is proposed for study. The model consists of a prey, whose total population density at time T is denoted by N(T), interacting with predator whose total population density at time T is denoted by Y(T). The following assumptions have been assumed in order to construct our model:

1. In the presence of infectious disease, There are two different epidemic diseases (SI, SIS), divides the prey population in to three classes namely S(T) that represents the density of susceptible prey, I_1 (T) which represents the density of infected prey by first disease and I_2 (T) which represents the density of infected prey by second disease. Therefore, at any time T, we have N (T) = S (T) +I_1 (T) + I_2 (T).

2. It is assumed that only susceptible prey S is capable of reproducing in logistic growth with carrying capacity K > 0 and intrinsic growth rate constant r > 0, the second infected prey I_2 is removed before having the possibility of reproducing. However, the infected prey population I_2 still contributes with S to population growth toward the carrying capacity while the first infected prey I_1 does not grow, recover, reproduce, or compute.

3. The first disease transmitted within the same species by contact with an infected individual at first infection rate constant $\beta_1 > 0$. In addition, there is an external source of disease causes incidence with the first disease within the specific population at a first external infection rate $\alpha_1 > 0$.

4. The second disease transmitted within the same species by contact with an infected individual at second infection rate constant $\beta_2 > 0$.In addition, there is an external source of disease causes incidence with the second disease within the specific population at a second external infection rate $\alpha_2 > 0$, further the second infected prey can recover and become susceptible again at the recovery rate constant $\sigma_1 > 0$.

5. The first and second infected prey faces the natural death due to the effect of the disease at a rates $d_1 > 0$ and $d_2 > 0$, respectively. Moreover, in the absence of the prey the predator face the natural death at natural death rate and $d_3 > 0$.

6. Finally, it is assumed that the predator consumes the first and second infected prey according to Lotka-Volterra type of functional response with attack rates $c_1 > 0$ and $c_2 > 0$, respectively. And conversion rate constants $0 < e_1 < 1$, $0 < e_2 < 1$ with $S(0) \geq 0$, $I_1(0) \geq 0$, $I_2(0) \geq 0$, $Y(0) \geq 0$.

According to the above assumptions, the proposed mathematical model can be represented mathematically by the following set of first order nonlinear differential equations.

$$\frac{dS}{dT} = r\,S\left(1 - \frac{S + I_2}{K}\right) - (\beta_1 I_1 + \alpha_1)S - (\beta_2 I_2 + \alpha_2)S + \sigma_1\, I_2$$

$$\frac{dI_1}{dT} = (\beta_1 I_1 + \alpha_1)S - c_1 I_1 Y - d_1 I_1$$

$$\frac{dI_2}{dT} = (\beta_2 I_2 + \alpha_2)S - c_2 I_2 Y - d_2 I_2 - \sigma_1 I_2 \qquad (2.1)$$

$$\frac{dY}{dT} = e_1 c_1 I_1 Y + e_2 c_2 I_2 Y - d_3 Y$$

Note that the above proposed model has (14) parameters which makes the mathematical analysis of the system difficult. So in order to reduce the number of parameters and determine which parameter represents the control parameter, the following dimensionless variables are used:

$$t = r\,T\,, s = \frac{S}{K}\,, i_1 = \frac{I_1}{K}\,, i_2 = \frac{I_2}{K}\,, y = \frac{c_1 Y}{r}$$

Then system (2.1) can be written in the following dimensionless form:

$$\frac{ds}{dt} = s\big(1 - s - (1 + a_3)\, i_2 - a_1 i_1 - (a_2 + a_4)\big) + a_5 i_2 = f_1(s, i_1, i_2, y)$$

$$\frac{di_1}{dt} = i_1(a_1 s - y - a_6) + a_2 s = f_2(s, i_1, i_2, y)$$

$$\frac{di_2}{dt} = i_2(a_3 s - a_7 y - (a_5 + a_8)) + a_4 s = f_3(s, i_1, i_2, y) \qquad (2.2)$$

$$\frac{dy}{dt} = y(a_9 i_1 + a_{10} i_2 - a_{11}) = f_4(s, i_1, i_2, y)$$

Where:

$$a_1 = \frac{\beta_1 K}{r}\,, a_2 = \frac{\alpha_1}{r}\,, a_3 = \frac{\beta_2 K}{r}\,, a_4 = \frac{\alpha_2}{r}\,, a_5 = \frac{\sigma_1}{r}, a_6 = \frac{d_1}{r}\,,$$

$$a_7 = \frac{c_2}{c_1}\,, a_8 = \frac{d_2}{r}, a_9 = \frac{e_1 c_1 K}{r}\,, a_{10} = \frac{e_2 c_2 K}{r}\,, a_{11} = \frac{d_3}{r}$$

Represent the dimensionless parameters of system (2.2). It is observed that the number of parameters have been reduced from fourteen in the system (2.1) to eleven in the system (2.2).

Since the density of any species cannot be negative, therefore we will solve system (2.2) with the following initial condition $s(0) \geq 0$, $i_1(0) \geq 0$, $i_2(0) \geq 0$ and $y(0) \geq 0$.

It is easy to verify that all the interaction functions f_1, f_2, f_3 and f_4 on the right hand side of system (2.2) are continuous and have continuous partial derivatives on R^4_+ with respect to dependent variables s, i_1, i_2 and y.

Accordingly they are Lipschitzian functions and hence system (2.2) has a unique solution for each non-negative initial condition. Further the boundedness of the system is shown in the following theorem.

Theorem 1: *All the solutions of system (2.2) which initiate in R^4_+ are uniformly bounded.*

Proof: Let $\left(s(t), i_1(t), i_2(t), y(t)\right)$ be any solution of the system (2.2) with non-negative initial condition $\left(s_0, i_{1_0}, i_{2_0}, y_0\right) \in R^4_+$.

Assume that $W(t) = s(t) + i_1(t) + i_2(t) + y(t)$ then taken the time derivative of $W(t)$ along the solution of the system (2) we get:

$$\frac{dW}{dt} = s - s^2 - (1 - a_9)i_1 y - (a_7 - a_{10})i_2 y - s i_2 - a_6 i_1 - a_8 i_2 - a_{11} y$$

Now, since the conversion rate constant from prey population to predator population cannot be exceeding the maximum predation rate constant of predator population to prey population, hence from the biological point of view, always $a_9 < 1$ and $a_{10} < a_7$ hence it is obtained that:

$$\frac{dW}{dt} \le 1 - mW \qquad \text{Where } m = \min\{1, a_6, a_8, a_{11}\}.$$

Then,

$$\frac{dW}{dt} + mW \le 1$$

Now by using the comparison theorem [23] on the above differential inequality with the initial value $W(0) = W_0$, we get that:

$$W(t) \le \frac{1}{m} + \left(W_0 - \frac{1}{m}\right) e^{-mt}$$

Then,

$$\lim_{t\to\infty} W(t) \le \frac{1}{m}$$

So, $0 \le W(t) \le \frac{1}{m}$. Hence all the solutions of system (2.2) are uniformly bounded and the proof is complete. ■

3 Existence of Equilibrium Points

In this section, the existence of all possible equilibrium points of system (2.2) is discussed. It is observed that, system (2.2) has at most three biologically feasible equilibrium points, namely $E_0 = (0,0,0,0)$, $E_1 = (\bar{s}, \bar{\iota_1}, \bar{\iota_2}, 0)$, $E_2 = (s^*, i_1^*, i_2^*, y^*)$.

- The vanishing equilibrium point $E_0 = (0,0,0,0)$ always exist.

- The predator free equilibrium point $E_1 = (\bar{s}, \bar{\iota_1}, \bar{\iota_2}, 0)$ exists if and only if there is a positive solution to the following set of equations:

$$1 - s - (1 + a_3)\, i_2 - a_1 i_1 - (a_2 + a_4) + \frac{a_5 i_2}{s} = 0 \qquad (3.1)$$

$$a_1\, s - a_6 + \frac{a_2\, s}{i_1} = 0 \qquad (3.2)$$

$$a_3\, s - (a_5 + a_8) + \frac{a_4\, s}{i_2} = 0 \qquad (3.3)$$

From equation (3.2) we have,

$$i_1 = \frac{a_2 s}{a_6 - a_1 s} \qquad (3.4)$$

Also, from equation (3.3) we have,

$$i_2 = \frac{a_4 s}{(a_5 + a_8) - a_3 s} \qquad (3.5)$$

Now, by substituting equations (3.4) and (3.5) in equation (3.1) we get:

$$M_1\, s^3 + M_2\, s^2 + M_3\, s + M_4 = 0 \qquad (3.6)$$

Where:

$$M_1 = -\, a_1 a_3 < 0$$

$$M_2 = a_1[\, a_3 + a_4 + a_5 + a_8\,] + a_3 a_6 > 0$$

$$M_3 = -\, a_1[\, a_5 + a_8(1 - a_4)] - a_6[\, a_3\,(1 - a_2) + a_4 + a_5 + a_8\,]$$

$$M_4 = a_6[\,(a_5 + a_8)\,(1 - a_2) - a_4 a_8\,]$$

Note that equation (3.6) has a unique positive root, namely $\bar{s}$ provided that:

$$a_2 < 1 - \frac{a_4 a_8}{a_5 + a_8} \qquad (3.7)$$

$$a_4 < 1 + \frac{a_5}{a_8} \quad (3.8)$$

Substituting the value of $\bar{s}$ in (3.4) and (3.5) yield that $i_1(\bar{s}) = \bar{i_1}$ and $i_2(\bar{s}) = \bar{i_2}$ which are positive if the following condition holds:

$$\bar{s} < \min\left\{\frac{a_6}{a_1}, \frac{a_5 + a_8}{a_3}\right\} \quad (3.9)$$

Consequently, predator free equilibrium point $E_1 = (\bar{s}, \bar{i_1}, \bar{i_2}, 0)$ of system (2.2) exists uniquely in the $Int.R_+^3$ of $si_1i_2 -$ space

- The positive (coexistence) equilibrium point $E_2 = (s^*, i_1^*, i_2^*, y^*)$ exists if and only if there is a positive solution to the following set of equations:

$$1 - s - (1 + a_3)\, i_2 - a_1 i_1 - (a_2 + a_4) + \frac{a_5 i_2}{s} = 0 \quad (3.10)$$

$$a_1\, s - y - a_6 + \frac{a_2\, s}{i_1} = 0 \quad (3.11)$$

$$a_3\, s - a_7\, y - (a_5 + a_8) + \frac{a_4\, s}{i_2} = 0 \quad (3.12)$$

$$a_9\, i_1 + a_{10}\, i_2 - a_{11} = 0 \quad (3.13)$$

From equation (3.13) we have:

$$i_2 = \frac{a_{11} - a_9\, i_1}{a_{10}} \quad (3.14)$$

Substituting equation (3.14) in equation (3.12) we get:

$$y = \frac{(a_{11} - a_9\, i_1)(a_3 s - (a_5 + a_8)) + a_4 a_{10} s}{a_7 (a_{11} - a_9\, i_1)} \quad (3.15)$$

Then by using (3.14) and (3.15) in (3.10) and (3.11) yield the following two isoclines:

$$L_1(s, i_1) = s\left(1 - s - (1 + a_3)\,\frac{a_{11} - a_9\, i_1}{a_{10}} - a_1 i_1 - (a_2 + a_4)\right)$$

$$+ a_5 \frac{a_{11} - a_9\, i_1}{a_{10}} = 0 \quad (3.16)$$

$$L_2(s, i_1) = i_1\left(a_1 s - \frac{(a_{11} - a_9\, i_1)(a_3 s - (a_5 + a_8)) + a_4 a_{10} s}{a_7 (a_{11} - a_9\, i_1)} - a_6\right)$$

$$+ a_2 s = 0 \quad (3.17)$$

Now from equation (3.16) we notice that, when $i_1 \to 0$, then $\boldsymbol{s}$ represents a positive root of the following second order polynomial equation:

$$N_1\, s^2 + N_2\, s + N_3 = 0 \qquad (3.18)$$

Where:

$$N_1 = a_{10}$$
$$N_2 = a_{10}[\,(a_2 + a_4) - 1] + a_{11}(1 + a_3)$$
$$N_3 = -\, a_5 a_{11}$$

Straightforward computation shows that equation (3.18) has a unique positive root namely s_1 .

Moreover, since we have $\frac{ds}{di_1} = -\left(\frac{\partial L_1}{\partial i_1}\right) / \left(\frac{\partial L_1}{\partial s}\right)$. So, $\frac{ds}{di_1} < 0$ and hence the isoclines (3.16) is decreasing passing through s_1 if one set of the following sets of conditions hold:

$$\left(\frac{\partial L_1}{\partial i_1}\right) > 0, \left(\frac{\partial L_1}{\partial s}\right) > 0 \quad OR \quad \left(\frac{\partial L_1}{\partial i_1}\right) < 0, \left(\frac{\partial L_1}{\partial s}\right) < 0. \qquad (3.19)$$

Further, from equation (3.17) we notice that, when $i_1 \to 0$, then $s = 0$,in addition since we have $\frac{ds}{di_1} = -\left(\frac{\partial L_2}{\partial i_1}\right) / \left(\frac{\partial L_2}{\partial s}\right)$. So, $\frac{ds}{di_1} > 0$ and hence the isoclines (3.17) is increasing passing through the origin if one set of the following sets of conditions hold:

$$\left(\frac{\partial L_2}{\partial i_1}\right) > 0, \left(\frac{\partial L_2}{\partial s}\right) < 0 \quad OR \quad \left(\frac{\partial L_2}{\partial i_1}\right) < 0, \left(\frac{\partial L_2}{\partial s}\right) > 0. \qquad (3.20)$$

Then the two isoclines (3.16) and (3.17) intersect at a unique positive point $(\,s^*, {i_1}^*\,)$ in the $Int.\,R_+^2$ of $si_1 -$ plane .Substituting the value of ${i_1}^*$ in (3.14) and the value of s^* and ${i_1}^*$ in (3.15) yield that $i_2(\,{i_1}^*) = {i_2}^*$ and $y(\,s^*, {i_1}^*\,) = y^*$ which are positive if and only if the following conditions hold:

$$i_1^* < \frac{a_{11}}{a_9} \qquad (3.21)$$

$$s^* > \frac{a_5 + a_8}{a_3} \qquad (3.22)$$

Accordingly, the positive equilibrium point E_2 exists uniquely in $Int.\,R_+^4$, if in addition to the conditions $(3.19 - 3.22)$ the isocline $L_1(s, i_1) = 0$ intersect the s-axis at the positive value namely s_1.

4 The Local Stability Analysis

In this section, the local stability analysis of system (2.2) around each of the above equilibrium points is discussed through computing the Jacobian matrix $J(s, i_1, i_2, \mathrm{y})$ and determined the eigen values of system (2.2) at each of them.

Note that, the symbols $\lambda_{is}, \lambda_{ii_1}, \lambda_{ii_2}$ *and* λ_{iy} are used to represent the eigenvalues of Jacobian matrix $J(E_i)$; i = 0, 1, 2 that describe the dynamics in s-direction, i_1-direction, i_2-direction and y-direction respectively, where the Jacobian matrix $J(s, i_1, i_2, \mathrm{y})$ of the system (2.2) at each of them can be written:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial s} & \frac{\partial f_1}{\partial i_1} & \frac{\partial f_1}{\partial i_2} & \frac{\partial f_1}{\partial \mathrm{y}} \\ \frac{\partial f_2}{\partial s} & \frac{\partial f_2}{\partial i_1} & \frac{\partial f_2}{\partial i_2} & \frac{\partial f_2}{\partial \mathrm{y}} \\ \frac{\partial f_3}{\partial s} & \frac{\partial f_3}{\partial i_1} & \frac{\partial f_3}{\partial i_2} & \frac{\partial f_3}{\partial \mathrm{y}} \\ \frac{\partial f_4}{\partial s} & \frac{\partial f_4}{\partial i_1} & \frac{\partial f_4}{\partial i_2} & \frac{\partial f_4}{\partial \mathrm{y}} \end{bmatrix}$$

Where f_i ; 1,2,3,4 are given in system (2.2) and

$$\frac{\partial f_1}{\partial s} = 1 - 2s - (1 + a_3)\, i_2 - a_1 i_1 - (a_2 + a_4), \frac{\partial f_1}{\partial i_1} = -a_1 s\,, \frac{\partial f_1}{\partial i_2} = -(1 + a_3)\, s + a_5$$

$$\frac{\partial f_1}{\partial y} = 0 \;\; , \frac{\partial f_2}{\partial s} = a_1 i_1 + a_2 \;\; , \frac{\partial \mathrm{f}_2}{\partial i_1} = a_1 \mathrm{s} - \mathrm{y} - a_6 \;\; , \qquad \frac{\partial \mathrm{f}_2}{\partial i_2} = 0\, , \frac{\partial \mathrm{f}_2}{\partial \mathrm{y}} = -i_1 \quad ,$$

$$\frac{\partial \mathrm{f}_3}{\partial \mathrm{s}} = a_3 i_2 + a_4\, , \frac{\partial \mathrm{f}_3}{\partial i_1} = 0 \;\; , \frac{\partial \mathrm{f}_3}{\partial i_2} = a_3 \mathrm{s} - a_7 \mathrm{y} - (a_5 + a_8) \;\; , \qquad \frac{\partial \mathrm{f}_3}{\partial \mathrm{y}} = -a_7\, i_2 \quad ,$$

$$\frac{\partial f_4}{\partial s} = 0 \quad , \qquad \frac{\partial f_4}{\partial i_1} = a_9 y \quad , \qquad \frac{\partial f_4}{\partial i_2} = a_{10} y \quad , \qquad \frac{\partial f_4}{\partial y} = a_9\, i_1 + a_{10} i_2 - a_{11}\, .$$

It is easy to verify that, the Jacobian matrix of system (2.2) at E_0 can be written as:

$$J_0 = \begin{bmatrix} 1 - (a_2 + a_4) & 0 & a_5 & 0 \\ a_2 & -a_6 & 0 & 0 \\ a_4 & 0 & -(a_5 + a_8) & 0 \\ 0 & 0 & 0 & -a_{11} \end{bmatrix}$$

Then the characteristic equation of J_0 is given by:

$$[\lambda^2 + B_1\lambda + B_2](-a_6 - \lambda)(-a_{11} - \lambda) = 0$$

Where:

$$B_1 = a_5 + a_8 + [(a_2 + a_4) - 1]$$
$$B_2 = a_5(a_2 - 1) + a_8[(a_2 + a_4) - 1]$$

So, either

$$(-a_6 - \lambda)(-a_{11} - \lambda) = 0$$

which gives two of the eigenvalues of J_0 by:

$$\lambda_{0i_1} = -a_6 < 0 \quad , \quad \lambda_{0y} = -a_{11} < 0$$

Or

$$\lambda^2 + B_1\lambda + B_2 = 0$$

which gives the other two eigenvalues of J_0 by:

$$\lambda_{0S} = \frac{-B_1}{2} + \frac{1}{2}\sqrt{{B_1}^2 - 4B_2} \quad ,$$
$$\lambda_{0i_2} = \frac{-B_1}{2} - \frac{1}{2}\sqrt{{B_1}^2 - 4B_2} \quad .$$

Therefore, if the following condition holds:

$$a_2 > 1 \tag{4.1}$$

The equilibrium point E_0 is locally asymptotically stable in the . R_+^4 .

The Jacobian matrix of system (2.2) at E_1 can be written as:

$$J_1 = [u_{ij}]_{4\times4}$$

Here

$u_{11} = 1 - 2\bar{s} - (1 + a_3)\bar{\iota_2} - a_1\bar{\iota_1} - (a_2 + a_4)$, $\quad u_{12} = -a_1\bar{s} < 0$,

$u_{13} = -(1 + a_3)\bar{s} + a_5$, $u_{14} = 0$, $u_{21} = a_1\bar{\iota_1} + a_2 > 0$, $u_{22} = a_1\bar{s} - a_6$,

$u_{23} = 0, u_{24} = -\bar{\iota_1} < 0, u_{31} = a_3\bar{\iota_2} + a_4 > 0, u_{32} = 0, u_{33} = a_3\bar{s} - (a_5 + a_8)$,

$u_{34} = -a_7\bar{\iota_2} < 0 \quad ,u_{41} = 0 \quad ,u_{42} = 0 \ ,u_{43} = 0 \ ,u_{44} = a_9\,\bar{\iota_1} + a_{10}\bar{\iota_2} - a_{11}$.

Then the characteristic equation of J_1 is given by:

$$(\lambda - u_{44})[\lambda^3 + F_1\,\lambda^2 + F_2\,\lambda + F_3\,] = 0 \qquad (4.2)$$

Where

$$F_1 = -(u_{11} + u_{22} + u_{33})$$

$$F_2 = u_{11}(u_{22} + u_{33}) + u_{22}u_{33} - u_{12}u_{21} - u_{13}u_{31}$$

$$F_3 = -u_{33}(u_{11}u_{22} - u_{12}u_{21}) + u_{13}u_{31}u_{22}$$

It is clear that the eigenvalue in y-direction is given by $\lambda_{1y} = u_{44}$, which is negative provided that:

$$a_9\,\bar{\iota_1} + a_{10}\bar{\iota_2} < a_{11} \qquad (4.3)$$

However by using Routh-Hurwitz criterion all the other eigenvalues, which represent the roots of second part of Eq. (4.2), have negative real parts if and only if $F_1 > 0, F_3 > 0$ and $F_1F_2 - F_3 > 0$.

Straightforward computation shows that condition (3.9) guarantees that $u_{22} < 0, u_{33} < 0$ and hence $F_1 > 0$ provided that:

$$2\bar{s} + (1 + a_3)\,\bar{\iota_2} + a_1\bar{\iota_1} + (a_2 + a_4) > 1 \qquad (4.4)$$

Also, due to conditions (3.9) and (4.4) we obtain that the first and the second term of F_3 are positive provided that:

$$\bar{s} > \frac{a_5}{1 + a_3} \qquad (4.5)$$

And hence $F_3 > 0$. Further, it is easy to check that:

$$F_1F_2 - F_3 = -{u_{11}}^2(u_{22} + u_{33}) - {u_{22}}^2(u_{11} + u_{33}) - {u_{33}}^2(u_{11} + u_{22})$$
$$-2u_{11}u_{22}u_{33} + u_{12}u_{21}(u_{11} + u_{22}) + u_{13}u_{31}(u_{11} + u_{33})$$

Clearly, the first five terms are positive under conditions (3.9) and (4.4) while the last term is positive under condition (4.5), and hence $F_1F_2 - F_3 > 0$.

So, all the eigenvalues of J_1 have negative real part under the given conditions and hence E_1 is locally asymptotically stable.

The Jacobian matrix of system (2.2) at E_2 can be written as:

$J_2 = [\, z_{ij} \,]_{4\times 4}$

Here

$z_{11} = 1 - 2s^* - (1 + a_3)\, {i_2}^* - a_1 {i_1}^* - (a_2 + a_4)$, $z_{12} = -a_1 s^* < 0$,

$z_{13} = -(1 + a_3)\, s^* + a_5$, $\quad z_{14} = 0$, $\quad z_{21} = a_1 {i_1}^* + a_2 > 0,$

$z_{22} = a_1 s^* - y^* - a_6, z_{23} = 0, z_{24} = -{i_1}^* < 0, z_{31} = a_3 {i_2}^* + a_4 > 0,$

$$z_{32} = 0, \qquad z_{33} = -\frac{a_4 a_{10}\, s^*}{a_{11} - a_9 {i_1}^*} < 0, \qquad z_{34} = -a_7\, {i_2}^* < 0,$$

$z_{41} = 0, \qquad z_{42} = a_9 y^* > 0, \qquad z_{43} = a_{10} y^* > 0, \qquad z_{44} = 0.$

Then the characteristic equation of J_2 is given by:

$$\lambda^4 + D_1\lambda^3 + D_2\, \lambda^2 + D_3\, \lambda + D_4 = 0 \qquad (4.6)$$

Where

$D_1 = -(\mu_1 + z_{33})$

$D_2 = z_{33}\mu_1 + \mu_7 - \mu_3 - \mu_4 - \mu_6$

$D_3 = \mu_1\mu_4 + \mu_2\mu_3 + z_{22}\mu_6 - z_{33}\mu_7$

$D_4 = -(\mu_4\mu_7 + \mu_3\mu_8 + \mu_9)$

With

$\mu_1 = z_{11} + z_{22}$, $\mu_2 = z_{11} + z_{33}$, $\mu_3 = z_{24}z_{42} < 0, \mu_4 = z_{34}z_{43} < 0$,

$\mu_5 = z_{12}z_{21} < 0, \qquad \mu_6 = z_{13}z_{31}, \qquad \mu_7 = z_{11}z_{22} - \mu_5,$

$\mu_8 = z_{11}z_{33} - \mu_6, \qquad \mu_9 = (z_{43}z_{12}z_{24}z_{31} + z_{42}z_{13}z_{21}z_{34}) > 0.$

Now by using Routh-Hurwitz criterion all the eigenvalues, which represent the roots of Eq. (4.6), have negative real parts if and only if $D_1 > 0$, $D_3 > 0$, $D_4 > 0$ and $\Delta = (D_1 D_2 - D_3)D_3 - D_1^2\, D_4 > 0$.

Clearly we have $D_1 > 0$ and $D_3 > 0$ provided that:

$$2s^* + (1 + a_3){i_2}^* + a_1 {i_1}^* + (a_2 + a_4) > 1 \qquad (4.7)$$

$$\frac{a_5}{1+a_3} < s^* < \frac{(a_{11} - a_9 i_1{}^*)(a_6 a_7 - (a_5 + a_8))}{(a_{11} - a_9 i_1{}^*)(a_1 a_7 - a_3) + a_4 a_{10}} \qquad (4.8)$$

Also, due to conditions (4.7) and(4.8) we obtain that $D_4 > 0$ are positive provided that:

$$-\mu_4\mu_7 > \mu_9 + \mu_3\mu_8 > 0 \qquad (4.9)$$

Further, it is easy to check that:

$$\Delta = [z_{33}\mu_1 D_1 + \mu_2\mu_6 + z_{22}\mu_3]D_3 - \mu_1\mu_7(\mu_2\mu_3 + z_{22}\mu_6 - z_{33}\mu_7)$$

$$+z_{33}\mu_4[\mu_1(\mu_4 + 2\mu_7) + \mu_2\mu_3 + z_{22}\mu_6] + (\mu_3\mu_8 + \mu_9)D_1^2$$

Clearly, the first two terms are positive under conditions (4.7) and (4.8) while in addition to conditions (4.7) and (4.8) the third term is positive provided that:

$$\mu_7 < -\frac{\mu_4}{2} \qquad (4.10)$$

And the fourth term is positive under conditions (4.7) and (4.9) and hence $(D_1 D_2 - D_3)D_3 - D_1^2\, D_4 > 0$.

So, all the eigenvalues of J_2 have negative real part under the given conditions and hence E_2 is locally asymptotically stable.

5 The Global Stability Analysis

In this section the global stability analysis for the equilibrium points, which are locally asymptotically stable, of system (2.2) is studied analytically by use the suitable of Lyapunov method as shown in the following theorems.

Theorem 2: *Assume that the vanishing equilibrium point $E_0 = (0, 0, 0, 0)$ of system (2.2) is locally asymptotically stable in the R_+^4. Then E_0 is globally asymptotically stable in the region $\Psi_0 \subset R_+^4$, where $\Psi_0 = \{(s, i_1, i_2, y) \in R_+^4 : s > 1\}$.*

Proof: Consider the following function

$$V_0(s, i_1, i_2, y) = s + i_1 + i_2 + y$$

Clearly $V_0\colon R_+^4 \to R$ is a C^1 positive definite function.

Now by differentiating V_0 with respect to time t and doing some algebraic manipulation, gives that:

$$\frac{\mathrm{dV_0}}{\mathrm{dt}} = s - s^2 - (1 - a_9)i_1 y - (a_7 - a_{10})i_2 y - s i_2 - a_6 i_1 - a_8 i_2 - a_{11} y$$

Now, due to the fact $a_9<1$, $a_{10}< a_7$ that is mentioned in theorem (1), then it is obtained that:

$\frac{dV_0}{dt} \le s(1-s)$, hence $\frac{\mathrm{dV_0}}{\mathrm{dt}} < 0$ in the region Ψ_0, and then V_0 is strictly Lyapunov function. Therefore, E_0 is a globally asymptotically stable in the region $\Psi_0 \subset R^4_+$, and the proof is complete. ■

According to the above theorem, it is easy to concludes that the basin of attraction of the vanishing equilibrium point E_0 is

$$B(E_0) = \Psi_0 = \{ (s, i_1, i_2, y) \in R^4_+ : s > 1 \}.$$

Theorem 3: *Assume that the equilibrium point* $E_1 = (\bar{s}, \bar{i_1}, \bar{i_2}, 0)$ *of system (2.2) is locally asymptotically stable in the* R^4_+*.Then* E_1 *is globally asymptotically stable on any region* $\Psi_1 \subset R^4_+$ *that satisfies the following conditions:*

$$\left(\frac{a_5}{s} + \frac{a_4}{i_2} - 1\right)^2 < 2\left(1 + \frac{a_5 \bar{i_2}}{s\bar{s}}\right)\left(\frac{a_4 \bar{s}}{i_2 \bar{i_2}}\right) \tag{5.1}$$

$$\left(\frac{a_2}{i_1}\right)^2 < 2\left(1 + \frac{a_5 \bar{i_2}}{s\bar{s}}\right)\left(\frac{a_2 \bar{s}}{i_1 \bar{i_1}}\right) \tag{5.2}$$

$$(\bar{i_1} + a_7 \bar{i_2}) y < \delta_1 + \delta_2 \tag{5.3}$$

Where

$$\delta_1 = \left[\frac{1}{\sqrt{2}}\sqrt{1 + \frac{a_5\,\bar{i_2}}{s\bar{s}}}\,(s - \bar{s}) - \sqrt{\frac{a_4 \bar{s}}{i_2 \bar{i_2}}}\,(i_2 - \bar{i_2})\right]^2$$

$$\delta_2 = \left[\frac{1}{\sqrt{2}}\sqrt{1 + \frac{a_5\,\bar{i_2}}{s\bar{s}}}\,(s - \bar{s}) - \sqrt{\frac{a_2 \bar{s}}{i_1 \bar{i_1}}}\,(i_1 - \bar{i_1})\right]^2$$

Proof: Consider the following function

$$V_1(s, i_1, i_2, y) = \left(s - \bar{s} - \bar{s}\ \ln\frac{s}{\bar{s}}\right) + \left(i_1 - \bar{i_1} - \bar{i_1}\ \ln\frac{i_1}{\bar{i_1}}\right) + \left(i_2 - \bar{i_2} - \bar{i_2}\ \ln\frac{i_2}{\bar{i_2}}\right) + y$$

Clearly $V_1 : R^4_+ \to R$ is a C^1 positive definite function.

Now by differentiating V_1 with respect to time t and doing some algebraic manipulation, gives that

$$\frac{dV_1}{dt} \leq -\left(1+\frac{a_5\bar{i_2}}{s\bar{s}}\right)(s-\bar{s})^2+\left(\frac{a_5}{s}+\frac{a_4}{i_2}-1\right)(s-\bar{s})(i_2-\bar{i_2})-\left(\frac{a_4\bar{s}}{i_2\bar{i_2}}\right)(i_2-\bar{i_2})^2+\left(\frac{a_2}{i_1}\right)(s-\bar{s})(i_1-\bar{i_1})-\left(\frac{a_2\bar{s}}{i_1\bar{i_1}}\right)(i_1-\bar{i_1})^2-(1-a_9)i_1y-(a_7-a_{10})i_2y+(\bar{i_1}+a_7\bar{i_2})y$$

Now, according to the conditions (5.1) and (5.2) we obtain that:

$$\frac{dV_1}{dt} < -(\delta_1+\delta_2)+(\bar{i_1}+a_7\bar{i_2})y$$

Then $\frac{dV_1}{dt} < 0$ in the region Ψ_1 due to conditions (5.3) and hence V_1 is strictly Lyapunov function. Therefore, E_1 is a globally asymptotically stable in the region $\Psi_1 \subset R^4_+$, and hence the proof is complete ■

Theorem 4: *Assume that the positive (coexistence) equilibrium point* $E_2 = (s^*, i_1{}^*, i_2{}^*, y^*)$ *of system (2.2) is locally asymptotically stable in the* R^4_+. *Then* E_2 *is globally asymptotically stable on any region* $\Psi_2 \subset R^4_+$ *that satisfies the following conditions:*

$$\left(\frac{a_5}{s}+\frac{a_4}{i_2}-1\right)^2 < 2\left(1+\frac{a_5 i_2{}^*}{ss^*}\right)\left(\frac{a_4 s^*}{i_2 i_2{}^*}\right) \quad (5.4)$$

$$\left(\frac{a_2}{i_1}\right)^2 < 2\left(1+\frac{a_5 i_2{}^*}{ss^*}\right)\left(\frac{a_2 s^*}{i_1 i_1{}^*}\right) \quad (5.5)$$

$$(i_1{}^*+a_7 i_2{}^*)y+(i_1+a_7 i_2)y^* < \Omega_1+\Omega_2 \quad (5.6)$$

Where

$$\Omega_1 = \left[\frac{1}{\sqrt{2}}\sqrt{1+\frac{a_5\, i_2{}^*}{ss^*}}\,(s-s^*)-\sqrt{\frac{a_4 s^*}{i_2 i_2{}^*}}\,(i_2-i_2{}^*)\right]^2$$

$$\Omega_2 = \left[\frac{1}{\sqrt{2}}\sqrt{1+\frac{a_5\, i_2{}^*}{ss^*}}\,(s-s^*)-\sqrt{\frac{a_2 s^*}{i_1 i_1{}^*}}\,(i_1-i_1{}^*)\right]^2$$

Proof: Consider the following function

$$V_2(s, i_1, i_2, y) = \left(s-s^*-s^*\ \ln\frac{s}{s^*}\right)+\left(i_1-i_1{}^*-i_1{}^*\ \ln\frac{i_1}{i_1{}^*}\right)$$

$$+\left(i_2 - {i_2}^* - {i_2}^* \ \ln\frac{i_2}{{i_2}^*}\right) + \left(y - y^* - y^* \ \ln\frac{y}{y^*}\right)$$

Clearly $\mathrm{V}_2: \mathrm{R}_+^4 \to \mathrm{R}$ is a C^1 positive definite function.

Now by differentiating V_2 with respect to time t and doing some algebraic manipulation, gives that:

$$\begin{aligned}\frac{\mathrm{dV}_2}{\mathrm{dt}} \le &-\left(1+\frac{a_5 {i_2}^*}{ss^*}\right)(s-s^*)^2 + \left(\frac{a_5}{s}+\frac{a_4}{i_2}-1\right)(s-s^*)(i_2-{i_2}^*)\\ &-\left(\frac{a_4 s^*}{i_2 {i_2}^*}\right)(i_2-{i_2}^*)^2 + \left(\frac{a_2}{i_1}\right)(s-s^*)(i_1-{i_1}^*)\\ &-\left(\frac{a_2 s^*}{i_1 {i_1}^*}\right)(i_1-{i_1}^*)^2 - (1-a_9)(i_1 y + {i_1}^* y^*) + ({i_1}^* + a_7 {i_2}^*)y\\ &+ (i_1 + a_7 i_2)y^*\end{aligned}$$

Now, according to the conditions (5.4) and (5.5) we obtain that:

$$\frac{\mathrm{dV}_2}{\mathrm{dt}} < -(\Omega_1+\Omega_2) + ({i_1}^* + a_7 {i_2}^*)y + (i_1 + a_7 i_2)y^*$$

Then $\frac{\mathrm{dV}_2}{\mathrm{dt}} < 0$ in the region Ψ_2 due to conditions (5.6) and hence V_2 is strictly Lyapunov function. Therefore, E_2 is a globally asymptotically stable in the region $\Psi_2 \subset \mathrm{R}_+^4$, and hence the proof is complete ■

6 Numerical Simulation

In this section, the dynamical behavior of system (2.2) is studied numerically for different sets of parameters and different sets of initial points. The objectives of this study are: first investigate the effect of varying the value of each parameter on the dynamical behavior of system (2.2) and second confirm our obtained analytical results. It is observed that, for the following set of hypothetical parameters that satisfies stability conditions of the positive equilibrium point, system (2.2) has a globally asymptotically stable positive equilibrium point as shown in Fig. (1).

$$\left.\begin{aligned}&a_1 = 0.5 \ , a_2 = 0.3 \ , a_3 = 0.3 \ , a_4 = 0.3 \, , a_5 = 0.2 \ , a_6 = 0.3 \ ,\\ &a_7 = 0.5 \ , a_8 = 0.2 \, , a_9 = 0.5 \, , a_{10} = 0.4 \, , a_{11} = 0.2 \, .\end{aligned}\right\} \quad (6.1)$$

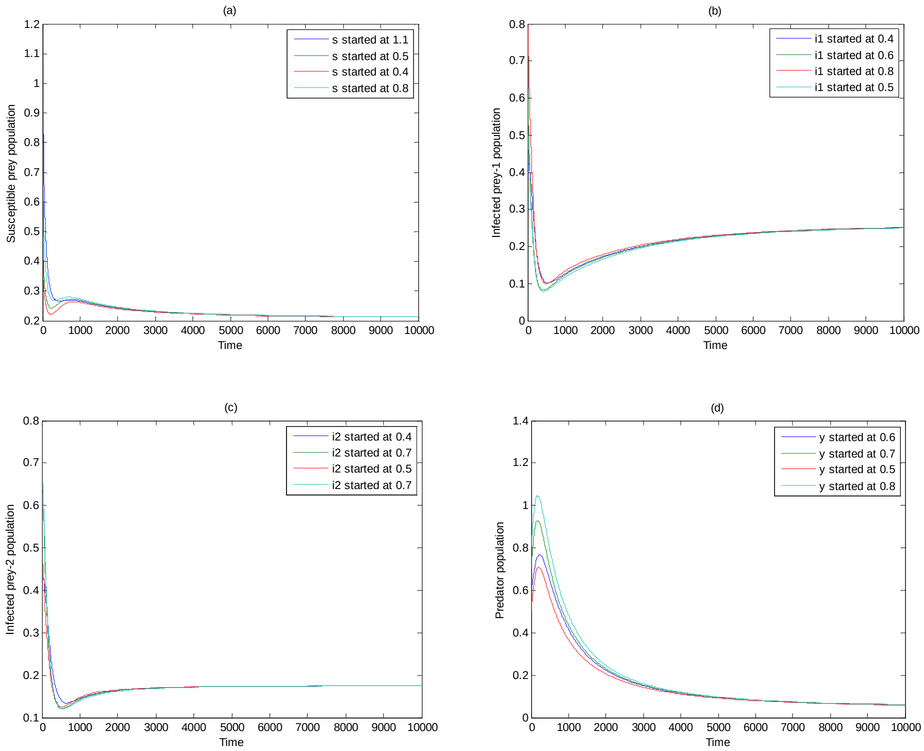


Fig. 1: Time series of the solution of system (2.2) that started from four different initial point (1.1, 0.4, 0.4, 0.6), (0.5, 0.6, 0.7, 0.7), (0.4, 0.8, 0.5, 0.5) and (0.8, 0.5, 0.7, 0.8) for the data given in (6.1). (a) Trajectories of $\boldsymbol{s}$ as a function of time, (b) Trajectories of $\boldsymbol{i_1}$ as a function of time, (c) Trajectories of $\boldsymbol{i_2}$ as a function of time and (d) Trajectories of $\boldsymbol{y}$ as a function of time.

Clearly, figure (1) shows that system (2.2) has a globally asymptotically stable as the solution of system (2.2) approaches asymptotically to the positive equilibrium point $E_2 = (0.212, 0.251, 0.174, 0.059)$ starting from four different initial points and this is confirming our obtained analytical results.

Now, in order to discuss the effect of the parameters values of system (2.2) on the dynamical behavior of the system, the system is solved numerically for the data given in (6.1) with varying one parameter at each time and the obtained results are given below.

The effect of varying the first infection rate in the range $0 < a_1 < 1.47$ keeping other parameters as data given in (6.1) is studied, it is observed that system (2.2) still approach asymptotically to the positive equilibrium point, however increasing this parameter further $1.47 \leq a_1$ causes extinction in the predator and the system will approach the predator free equilibrium point as shown in the following figure.

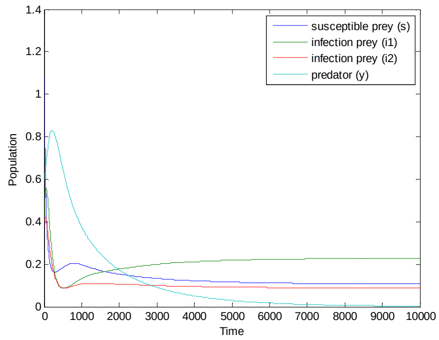
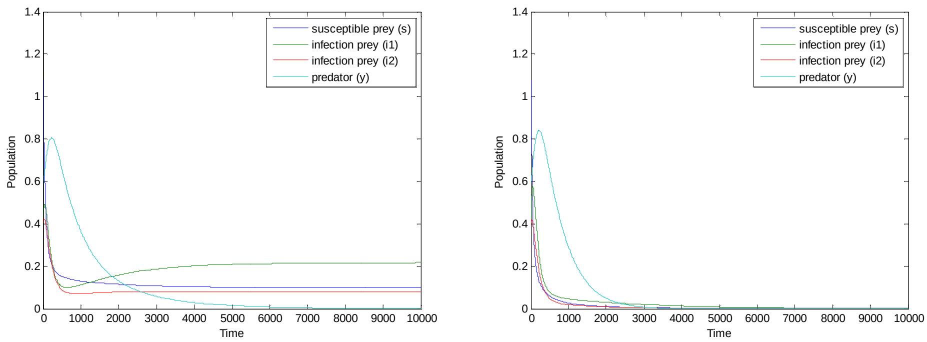


Fig. 2: Time series of the solution of system (2.2) approaches asymptotically the predator free equilibrium point $E_1 = (0.107,\ 0.228, 0.087, 0)$ for the data given in (6.1) with $a_1 = 1.5$.

The effect of varying the first external infection rate in the range $0.001 \leq a_2 < 0.51$ keeping other parameters as data given in (6.1) is studied, it is observed that system (2.2) still approach asymptotically to the positive equilibrium point, however increasing this parameter further in the range $0.51 \leq a_2 \leq 0.84$ causes extinction in the predator and the system will approach the predator free equilibrium point as shown in fig. (3a), while increasing this parameter further $0.89 \leq a_2$ then the system will approach the vanishing equilibrium point E_0 as shown in fig. (3b).



(a) (b)

Fig. 3: Time series of the solution of system (2.2) approaches asymptotically:
(a) The predator free equilibrium point $E_1 = (0.099,\ 0.217, 0.08, 0)$ for the data given in (6.1) with $a_2 = 0.55$, (b) $E_0 = (0\ , 0\ , 0\ , 0)$ for the data given in (6.1) with $a_2 = 0.9$.

On the other hand varying the second infection rate in the range $0 < a_3 < 1.57$ keeping other parameters as data given in (6.1) is studied, it is observed that system (2.2) still approach asymptotically to the positive equilibrium point, however increasing this parameter further $1.57 \leq a_3$ causes extinction in the predator and the system will approach the predator free equilibrium point in $Int. R_+^3$ of $si_1i_2 -$ space.

Moreover, varying the second external infection rate in the range $0.004 \leq a_4 < 0.54$ keeping other parameters as data given in (6.1) is studied, it is observed that system (2.2) still approach asymptotically to the positive equilibrium point, however increasing this parameter further in the range $0.54 \leq a_4 \leq 1.4$ causes extinction in the predator and the system will approach the predator free equilibrium point in $Int. R_+^3$ of $si_1i_2 -$ space, while increasing this parameter further $1.5 \leq a_4$ then the system will approach the vanishing equilibrium point $E_0 = (0, 0, 0, 0)$.

Now, the effect of varying the recovery rate in the range $0.04 \leq a_5 \leq 1$ keeping other parameters as data given in (6.1) is studied, it is observed that system (2.2) still approach asymptotically to the positive equilibrium point, however decreasing this parameter further in the range $0 < a_5 < 0.04$ causes extinction in the predator and the system will approach the predator free equilibrium point as shown in fig. (4).

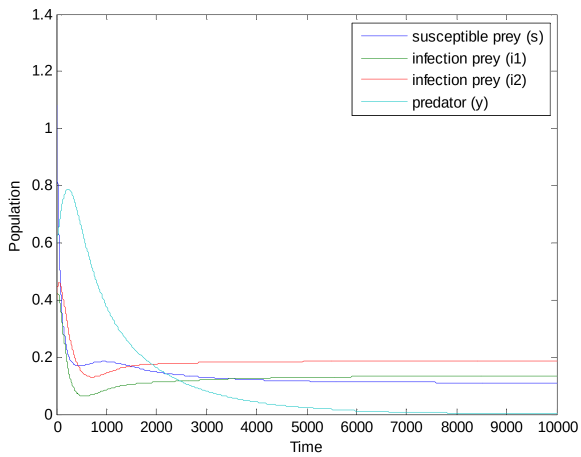


Fig. 4: Time series of the solution of system (2.2) approaches asymptotically the predator free equilibrium point $E_1 = (0.11, 0.133, 0.185, 0)$ for the data given in (6.1) with $a_5 = 0.01$.

The effect of varying the death rate of the infected prey by first disease, in the range $0 < a_6 < 0.64$ keeping other parameters as data given in (6.1) is studied, it is observed that system (2.2) still approach asymptotically to the positive equilibrium point, however increasing this parameter further $0.64 \leq a_6 \leq 1$ causes extinction in the predator and the system will approach the predator free equilibrium point in $Int. R_+^3$ of $si_1i_2 -$ space as shown in fig. (5).

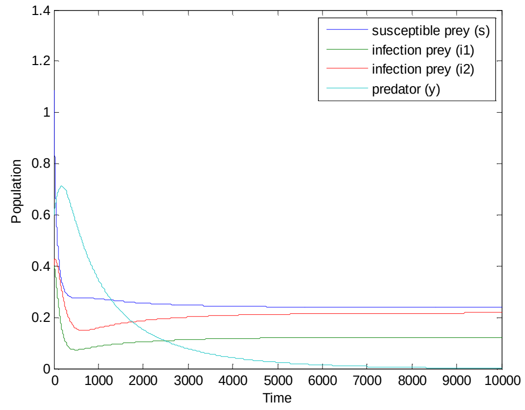


Fig. 5: Time series of the solution of system (2.2) approaches asymptotically the predator free equilibrium point $E_1 = (0.238, 0.123, 0.217, 0)$ for the data given in (6.1) with $a_6 = 0.7$.

An investigation to the effect of varying the conversion rate of the infected prey by first disease from predator, in the range $0.3 \leq a_9 < 1$ keeping other parameters as data given in (6.1) is done and the following result is observed, that system (2.2) still approach asymptotically to the positive equilibrium point, however decreasing this parameter in the range $0 < a_9 < 0.3$ causes extinction in the predator and the system will approach the predator free equilibrium point as shown in fig.(6).

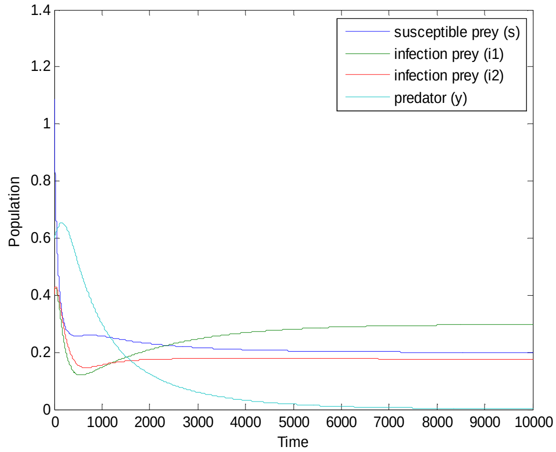


Fig. 6: Time series of the solution of system (2.2) approaches asymptotically the predator free equilibrium point $E_1 = (0.199,\ 0.297, 0.176, 0)$ for the data given in (6.1) with $a_9 = 0.25$.

Similarly, an investigation to the effect of varying the conversion rate of the infected prey by second disease from predator, in the range $0.09 \leq a_{10} < 0.5$ keeping other parameters as data given in (6.1) is done and the following result is observed, that system (2.2) still approach asymptotically to the positive equilibrium point, however decreasing this parameter in the range $0 < a_{10} < 0.09$ causes extinction in the predator and the system will approach the predator free equilibrium point in $Int.R_+^3$ of $si_1i_2 -$ space.

Finally, varying the death rate of the predator, in the range $0.017 \leq a_{11} < 0.256$ keeping other parameters as data given in (6.1) is studied, it is observed that system (2.2) still approach asymptotically to the positive equilibrium point, however increasing this parameter further in the range $0.256 \leq a_{11} \leq 1$ causes extinction in the predator and the system will approach the predator free equilibrium point in $Int.R_+^3$ of $si_1i_2 -$ space.

7 Conclusions and Discussions

In this paper, a prey-predator model, with SI, SIS epidemic disease in prey species, is proposed and analyzed. It is assumed that the disease is transmitted within the individuals of prey by two ways, through contact as well as from an external source. The uniqueness and boundedness of solution of the system are discussed, the existence of all possible equilibrium points are investigated, it is observed that system (2.2) has at most three nonnegative equilibrium points in R_+^4. The dynamical behavior of system (2.2) has been investigated locally as well as globally. Further, it is observed that the vanishing equilibrium point E_0 always exist, and it is locally asymptotically stable point if and only if condition (4.1) hold, in addition to that it is globally in the region $\Psi_0 \subset R_+^4$. The predator free equilibrium point E_1 exist under the conditions (3.7-3.9), it is locally asymptotically stable point if and only if conditions (3.9) and (4.3-4.5) hold, as well as it is globally if the conditions (5.1-5.3) hold. The positive equilibrium point E_2 of system (2.2) exist provided that the conditions (3.19-3.22) are hold and the isocline $L_1(s, i_1) = 0$. Intersect the s-axis at the positive value namely s_1. It is locally asymptotically stable point if and only if conditions (4.7-4.10) hold; in addition it is globally if the conditions (5.4-5.6) hold.

To understand the effect of varying each parameter on the global dynamics of system (2.2) and to confirm our above analytical results, system (2.2) has been solved numerically and the following results are obtained:

1. For the set of hypothetical parameters values given in (6.1), system (2.2) has only one type of attractor in $Int.R_+^4$, approaches to globally stable point.
2. For the set of hypothetical parameters values given in (6.1), system (2.2) approaches asymptotically to a globally stable point $E_2 = (\,0.212,\ 0.251,\ 0.174,\ 0.059\,)$.
3. Varying the attack rate parameter value and the death rate parameter value for the second infected prey a_7 *and* a_8, respectively at each time keeping other parameters fixed as data given in (6.1) do not have any effect on the dynamical behavior of system (2.2) and the solution of the system still approaches to positive equilibrium point $E_2 = (\,s^*, i_1{}^*, i_2{}^*, y^*\,)$.

4. Increasing each of infection rate parameters value and external infection rate parameters value (a_1, a_2, a_3, a_4) at each time keeping other parameters fixed as data given in (6.1) causes extinction in the predator and the solution of the system (2.2) will approaches asymptotically to the predator free equilibrium point $E_1 = (\bar{s}, \bar{\iota_1}, \bar{\iota_2}, 0)$.
5. Further, Increasing the parameter $a_2 \geq 0.89$ and $a_4 \geq 1.5$ at each time keeping other parameters fixed as data given in (6.1) the solution of the system (2.2) will approaches asymptotically to the vanishing equilibrium point $E_0 = (0, 0, 0, 0)$.
6. Decreasing recovery rate parameter value $a_5 < 0.04$ keeping other parameters fixed as data given in (6.1) causes extinction in the predator and the solution of the system (2.2) will approaches asymptotically to the predator free equilibrium point $E_1 = (\bar{s}, \bar{\iota_1}, \bar{\iota_2}, 0)$.
7. While, decreasing conversion rate parameters value $a_9 < 0.3$ and $a_{10} < 0.09$ at each time keeping other parameters fixed as data given in (6.1) causes extinction in the predator and the solution of the system (2.2) will approaches asymptotically to the predator free equilibrium point $E_1 = (\bar{s}, \bar{\iota_1}, \bar{\iota_2}, 0)$.
8. Increasing death rate parameter value $a_6 \geq 0.64$ for the first infected prey at each time keeping other parameters fixed as data given in (6.1) causes extinction in the predator and the solution of the system (2.2) will approaches asymptotically to the predator free equilibrium point $E_1 = (\bar{s}, \bar{\iota_1}, \bar{\iota_2}, 0)$.
9. While, Increasing death rate parameter value $a_{11} \geq 0.256$ of predator at each time keeping other parameters fixed as data given in (6.1) causes extinction in the predator and the solution of the system (2.2) will approaches asymptotically to the predator free equilibrium point $E_1 = (\bar{s}, \bar{\iota_1}, \bar{\iota_2}, 0)$.

References

[1] H.W. Hethcote, The mathematics of infectious diseases, *SIAM Review*, 42(4) (2000), 599-653.

[2] S. Hsu and T. Hwang, Global stability for a class of predator-prey systems, *SIAM Journal on Applied Mathematics*, 55(3) (1995), 763-783.

[3] S. Hsu, T. Hwang and Y. Kuang, A ratio-dependent food chain model and its applications to biological control, *Mathematical Biosciences*, 181(2003), 55-83.

[4] T. Hwang and Y. Kuang, Deterministic extinction effect of parasites on host populations, *Journal Mathematical Biology*, 46(2003), 17-30.

[5] M. Li, H. Smith and L. Wang, Global dynamics of an SEIR epidemic model with vertical transmission, *SIAM Journal on Applied Mathematics*, 62(1) (2001), 58-69.

[6] R.M. May, Qualitative stability in model ecosystem, *Ecology*, 54(3) (May) (1973), 638-641.

[7] J.D. Murray, *Mathematical Biology: An Introduction (Third Edition)*,

Berlin Heidelberg, Springer-Verlag, (2002).

[8] S.B. Hsu and T.W. Huang, Global stability for a class of predator-prey systems, *SIAM Journal on Applied Mathematics*, 55(3) (June) (1995), 763-783.

[9] A.A. Majeed and R.R. Saadi, The persistence of prey-predator model with competition hosts, *International Journal of Applied Mathematical Research*, 4(2) (2015), 351-361.

[10] R.M. Anderson and R.M. May, The invasion, persistence and spread of infectious diseases within animal and plant communities, *Phil. Trans. R. Soc. Lond. B.*, 314(1986), 533-570.

[11] A.A. Muhseen and I.A. Aaid, Stability of a prey-predator model with SIS epidemic disease in predator involving holling type II functional response, *IOSR Journal of Mathematics*, 11(2) (Ver. III) (2015), 38-53.

[12] Y. Xiao and L. Chen, Modeling and analysis of a predator-prey model with disease in the prey, *Math. Biosci.*, 171(1) (2001), 59-82.

[13] B. Kh Dahlia, Stability of a prey-predator model with SIS epidemic model disease in prey, *Iraq Journal of Science*, 52(4) (2011), 484-493.

[14] R.K. Naji and A.N. Mustafa, The dynamics of an eco-epidemiological model with nonlinear incidence rate, *Journal of Applied Mathematics*, Article ID 852631(2012), 24 pages.

[15] G.P. Samanta, Analysis of a delay non autonomous predator-prey system with disease in the prey, *Nonlinear Analysis: Modelling and Control*, 15(1) (2010), 97-108.

[16] R.M. Yaseen, A predator-prey model with transition two infectious diseases in prey population and harvesting of the predator population, *Journal of Mathematical Theory and Modeling*, 4(10) (2014), 1-27.

[17] E. Elena, M. Grammauro and E. Venturino, Predator’s alternative food sources do not support eco epidemics with two strains-diseased prey, *Network Biology*, 3(1) (2013), 29-44.

[18] F. Roman, F. Rossotto and E. Venturino, Eco epidemics with two strains: Diseased predators, *WSEAS Transactions on Biology and Biomedicine*, 8(3) (2011), 73-85.

[19] A.A. Mohsen and H. Kasim, The effect of external source of disease on epidemic model, *Int. J. Adv. Appl. Math. and Mech.*, 2(4) (2015), 53-63.

[20] K.P. Das, S. Roy and J. Chattopadhyay, The effect of disease-selective predation on prey infected by contact and external sources, *Bio Systems*, 95(2009), 188-199.

[21] K.Q. Khalaf, A.A. Majeed and R.K. Naji, The dynamics of an SIS epidemic disease with contact and external source, *Journal of Mathematical Theory and Modeling*, 5(2015), 184-197.

[22] R.M. Yaseen, The effect of two infectious disease and harvesting in the eco-epidemiological model, *Journal of Al-Nahrain University*, 18(1) (2015), 162-171.

[23] J.K. Hale, *Ordinary Differential Equation*, Wiley-Interscience, New York, (1969).