Gen. Math. Notes, Vol. 34, No. 1, May 2016, pp.56-63
ISSN 2219-7184; Copyright ©ICSRS Publication, 2016
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S-Essential Submodules

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(Received: 8-3-16 / Accepted: 17-5-16)

Abstract

Let M_R be a module with $S = End(M_R)$. We say that a submodule K of a module M_R is s-essential if $K \cap X = 0$, X a submodule of M_R, implies that $D_S(X) = 0$ where $D_S(X) = \{\phi \in S \mid Im\phi \subseteq X\}$. The intersection $\overline{A}_R(M)$ of all such submodules of M_R is contained in $Soc(M)$. If M_R is finitely cogenerated, then $\overline{A}_R(M)$ is the unique smallest s-essential submodule of M_R. We study $\overline{A}_R(M)$ and $\overline{K}_S(M)$ in this paper. Conditions when $\overline{A}_R(M)$ is s-essential and $\overline{K}_S(M) = J(S) = Tot(M, M)$ are given.

Keywords: *Essential submodule, S-essential submodule, Annihilator, Endomorphism ring.*

1 Introduction

Throughout this paper all rings are associative with identity and modules are unitary right modules. Let M_R be any module. S will indicate the endomorphism ring End(M) of the right R-module M. The notation $N \leq_e M$ denotes that a submodule N of M is essential in the module M_R (i.e. $\forall 0 \neq L \leq M, L \cap N \neq 0$); $N \ll M$ means that N is small in M (i.e. $\forall L \lneq M, L + N \neq M$). The left annihilator of any submodule X of M is denoted by $\ell_S(X)$ while the right annihilator of any endomorphism f of M is denoted by $r_M(f)$, namely the kernel of f. We also denote $D_S(N) = \{\phi \in S \mid \text{Im}\phi \subseteq N\}$, for $N \subseteq M$.

In [4], Nicholson and Zhou defined annihilator-small right (left) ideals. In [1], Amouzegar-Kalati and Keskin-Tütüncü introduced annihilator-small submodules of any right R-module M. Let M_R be a module and $K \subseteq M_R$. They defined that K is an *annihilator-small* submodule of M_R if $K + X = M$, X a submodule of M_R, implies that $\ell_S(X) = 0$. Motivated by [1], we introduce dual notion of annihilator-small submodules, this of s-essential submodules. A submodule K of a module M_R is called *s-essential* if $K \cap X = 0$, X a submodule of M_R, implies that $D_S(X) = 0$ where $S = \text{End}(M)$. It is obvious that every essential submodule is a s-essential submodule. In Propositions 2.3, 2.4 and 2.5, we give conditions which every s-essential submodule to be essential. Let M_R be a semi-projective module and $k \in S$. Then we prove the following:

$r_M(k) \leq_{s-e} M_R$ iff $r_M(b) \subsetneqq r_M(kb)$ for all $0 \neq b \in S$ iff $r_S(1_S - sk) = 0$ for all $s \in S$ iff $r_S(1_S - ks) = 0$ for all $s \in S$ iff $r_S(k - ksk) = r_S(k)$ for all $s \in S$ (see Lemma 2.8).

We investigate when the equalities $J(S) = \overline{K}_S(M) = \text{Tot}(M, M)$ are satisfied. As we state in the abstract we study $\overline{A}_R(M)$ which is the intersection of all s-essential submodules of M_R.

2 Main Results

Definition 2.1 *We say that a submodule K of a module M_R is* s-essential *if $K \cap X = 0$, X a submodule of M_R, implies that $D_S(X) = 0$ where $S = End(M)$. In this case, we write $K \leq_{s-e} M$.*

Note that a right ideal I of a ring R is s-essential if and only if it is essential.

It is clear that every essential submodule is s-essential, but the converse is not true in general (see Example 2.2).

Example 2.2 *Take $M = \mathbb{Z}_p$ (the $\mathbb{Z}$-module of p-adic integers) and $K = \mathbb{Z} \subset \mathbb{Z}_p$. Then K is not essential: take $K' = \mathbb{Z}x$ with $\{1, x\}$ $\mathbb{Q}$-linearly independent, then $K \cap K' = 0$. (In fact, the $\mathbb{Q}$-dimension of ($\mathbb{Q}$ tensor $\mathbb{Z}_p$) is infinite, and this suffices to show that K is not essential.) But K is s-essential. The point is: $\mathbb{Z}_p$ coincides with its endomorphism ring. So for every nonzero endomorphism f, its image $Im(f)$ contains $p^n\mathbb{Z}_p$ for some (large) n. Now assume $K \cap K' = 0$. Then since $K = \mathbb{Z}$, we have that K' can not contain p^n for any n. Thus $\{f \in End(\mathbb{Z}_p) : Im(f) \subset K'\} = 0$, and by definition, K is s-essential.*

An R-module M_R is called *retractable* if, for any nonzero submodule K of M, there exists a nonzero homomorphism $f : M \to K$.

Proposition 2.3 *Let M_R be a retractable module. If $K \leq_{e-s} M$, then $K \leq_e M$.*

Proof: Let $K \cap X = 0$ for an $X \leq M$. By hypothesis, $D_S(X) = 0$. But M_R is retractable, thus $X = 0$, and so $K \leq_e M$.

Proposition 2.4 *Let M_R be a cyclic and π-projective module. Then K is a s-essential submodule of M if and only if K is an essential submodule of M.*

Proof: Let $M = mR$ for some $m \in M$. Let K be a s-essential submodule of M_R and X a nonzero submodule of M_R. Take $0 \neq x \in X$, then there exists $0 \neq a \in R$ such that $x = ma$. It is clear that $M = mR = maR + m(1-a)R$. Since M is π-projective, there exists $f \in End(M)$ with $\mathrm{Im} f \subseteq maR \subseteq X$ and $\mathrm{Im}(1-f) \subseteq m(1-a)R$. Thus $D_S(X) \neq 0$. As K is s-essential in M_R, $K \cap X \neq 0$. Therefore K is essential in M. The converse is clear.

Proposition 2.5 *Let R be a commutative ring and M be a cyclic R-module. Then K is a s-essential submodule of M_R if and only if K is an essential submodule of M_R.*

Proof: Let $M = mR$ for some $m \in M$. Let K be a s-essential submodule of M_R and X a nonzero submodule of M_R. Take $0 \neq x \in X$, then there exists $0 \neq a \in R$ such that $x = ma$. Define the nonzero homomorphism $\phi : mR \to mR$ with $\phi(mr) = mra$. It is obvious that $\mathrm{Im}\phi \subseteq X$. Thus $D_S(X) \neq 0$. Since K is s-essential in M_R, $K \cap X \neq 0$. Therefore K is essential in M. The converse is clear.

Lemma 2.6 *Let M_R be a module. Then the following statements hold:*

(1) *If $N \leq K \leq M$ and $N \leq_{s-e} M$, then $K \leq_{s-e} M$.*

(2) *If K is a s-essential submodule of a module M_R and L is an essential submodule of M_R, then so is $K \cap L$ is s-essential of M_R.*

Proof: (1) Clear.

(2) Let $K \cap L \cap X = 0$ where X is a submodule of M_R. Since $L \leq_e M$, $K \cap X = 0$. So $D_S(X) = 0$.

Lemma 2.7 *If T is a submodule of M_R and $D_S(T) \subseteq_e S_S$, then $D_S(T)M \leq_{s-e} M_R$. In particular, $T \leq_{s-e} M_R$.*

Proof: Let $D_S(T)M \cap X = 0$. Then $D_S(T) \cap D_S(X) = 0$, so $D_S(X) = 0$ since $D_S(T) \subseteq_e S_S$. The last observation is by Lemma 2.6 since $D_S(T)M \subseteq T \subseteq M$ always holds.

Note that the converse of Lemma 2.7 is true if $(D_S(T) \cap bS)M = D_S(T)M \cap bM$ holds for all submodules T of M_R and all $b \in S$. To see this, let $D_S(T) \cap bS = 0$ for an element b of S. Then $D_S(T)M \cap bM = 0$, so $D_S(bM) = 0$ since $D_S(T)M \leq_{s-e} M_R$. Hence $b = 0$ because $bS \subseteq D_S(bM) = 0$, proving that $D_S(T) \subseteq_e S_S$.

Following Wisbauer [5, p. 261], an R-module M_R is called *semi-injective* if for any $f \in S$,

$$Sf = \ell_S(ker(f)) = \ell_S(r_M(f))$$

(equivalently, for any monomorphism $f : N \to M$, where N is a factor module of M_R, and for any homomorphism $g : N \to M$, there exists $h : M \to M$ such that $hf = g$).

Lemma 2.8 *Consider the following conditions for a right R-module M and $k \in S$:*

(1) $r_M(k) \leq_{s-e} M_R$.
(2) $r_M(b) \subsetneqq r_M(kb)$ *for all* $0 \neq b \in S$.
(3) $r_S(1_S - sk) = 0$ *for all* $s \in S$.
(4) $r_S(1_S - ks) = 0$ *for all* $s \in S$.
(5) $r_S(k - ksk) = r_S(k)$ *for all* $s \in S$.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5). *If* M_R *is semi-injective, then* (5) $\Rightarrow$ (1).

Proof: (1) $\Rightarrow$ (2) Assume that $b \in S$ and $r_M(b) = r_M(kb)$. It is easy to see that $r_M(k) \cap bM = 0$. Since $r_M(k) \leq_{s-e} M$, $D_S(bM) = 0$. As $bS \subseteq D_S(bM) = 0$, $b = 0$.

(2) $\Rightarrow$ (3) Let $s \in S$ and $b \in r_S(1_S - sk)$. Then $b = skb$ implies that $r_M(kb) \subseteq r_M(skb) = r_M(b)$. By (2), $b = 0$.

(3) $\Rightarrow$ (4) Let $s \in S$ and $b \in r_S(1_S - ks)$. Then $(1_S - ks)b = 0$ implies that $(1_S - sk)sb = (s - sks)b = s(1_S - ks)b = 0$. Hence $sb = 0$ by (3), and so $b = ksb = 0$.

(4) $\Rightarrow$ (5) Let $s \in S$ and $b \in r_S(k - ksk)$. By (4), $kb = 0$. Hence $b \in r_S(k)$. The other inclusion always holds.

(5) $\Rightarrow$ (1) Assume that M_R is semi-injective. Let $r_M(k) \cap X = 0$ for a submodule X of M_R. Let $b \in D_S(X)$. Then $bM \cap r_M(k) = 0$ and so $r_M(b) = r_M(kb)$. Since M_R is semi-injective, there exists a homomorphism $s \in S$ such that $b = skb$. Note that $(k - ksk)b = 0$. Hence $b \in r_S(k - ksk) = r_S(k)$. Therefore $kb = 0$, and hence $b = 0$.

Let us define $\overline{K}_S(M) = \{s \in S \mid \mathrm{Ker}s = r_M(s) \leq_{s-e} M_R\}$ for any module M_R.

Corollary 2.9 *Let M_R be a module and $k \in \overline{K}_S(M)$. Then $Sk \subseteq \overline{K}_S(M)$. If M_R is semi-injective, then $kS \subseteq \overline{K}_S(M)$.*

Proof: By Lemma 2.6, $Sk \subseteq \overline{K}_S(M)$. Now assume that M_R is semi-injective. Let $s \in S$. We want to show that $r_M(ks) \leq_{s-e} M_R$. Let $g \in S$. Then $r_S(1_S - ksg) = 0$ since $r_M(k) \leq_{s-e} M_R$, by Lemma 2.8(4). Again by Lemma 2.8(4), $r_M(ks) \leq_{s-e} M_R$. Hence $kS \subseteq \overline{K}_S(M)$.

Corollary 2.10 *We have $\overline{K}_S(M) \subseteq \ell_S(Soc(_SS))$. Moreover, $J(S) \subseteq \overline{K}_S(M)$ if M_R is semi-injective.*

Proof: Let $s \in \overline{K}_S(M)$. We want to show that $s\mathrm{Soc}(_SS) = 0$. Let $0 \neq t \in \mathrm{Soc}(_SS)$. Then $t \in S_1 \oplus S_2 \oplus \cdots \oplus S_n$, where $S_1, \cdots, S_n$ are the simple left ideals of S. Assume $st \neq 0$ and $t = t_1 + t_2 + \cdots t_n$ where $t_i \in S_i$. Then $st_i \neq 0$ for some $i \in \{1, \cdots, n\}$. Since S_i is simple $Sst_i = S_i$. Now, $t_i = \alpha st_i$ for some $\alpha \in S$. Then $t_i \in r_S(1_S - \alpha s)$. Since $r_M(s) \leq_{s-e} M$, $r_S(1_S - \alpha s) = 0$ by Lemma 2.8, hence $t_i = 0$, a contradiction. Thus $st = 0$. Therefore $\overline{K}_S(M) \subseteq \ell_S(\mathrm{Soc}(_SS))$.

Now let $k \in J(S)$. We want to show that $k \in \overline{K}_S(M)$. Let $s \in S$. Take $\alpha \in r_S(1_S - sk)$. Then $(1_S - sk)\alpha = 0$. Since $1_S - sk$ is invertible, $\alpha = 0$. Thus $r_S(1_S - sk) = 0$ for all $s \in S$. By Lemma 2.8, $k \in \overline{K}_S(M)$.

Corollary 2.11 *Let M_R be a module and $f \in S$. If $r_M(f) = Kerf \leq_{s-e} M_R$, then $Sf \ll_a {_SS}$. The converse is true if M_R is semi-injective.*

Proof: First, assume that $\mathrm{Ker}f \leq_{s-e} M$. Let $S = Sf + I$ where I is a left ideal of S. Then $1_S = sf + g$, $s \in S$, $g \in I$. Hence $\mathrm{Ker}g \cap \mathrm{Ker}f = 0$. Since $\mathrm{Ker}f \leq_{s-e} M$, $D_S(\mathrm{Ker}g) = 0$. Thus $D_S(r_M(I)) = 0$, and so $r_S(I) = 0$. Therefore $Sf \ll_a {_SS}$. Conversely, let $Sf \ll_a {_SS}$. By [1, Corollary 2.8], $r_S(f - fsf) = r_S(f)$ for all $s \in S$. By Lemma 2.8, $\mathrm{Ker}f \leq_{s-e} M_R$.

Corollary 2.12 *Let M_R be any module. If $f^2 = f \in \overline{K}_S(M)$, then $f = 0$.*

Proof: Since $r_M(f) \leq_{s-e} M_R$, $r_S(1_S - f) = 0$ by Lemma 2.8(4). Hence $f = 0$ because $f \in r_S(1_S - f)$.

Corollary 2.13 *Let M_R be any module. The following are equivalent for a maximal right ideal I of $S = End(M)$:*

(1) *$IM \leq_{s-e} M_R$;*

(2) *$I \subseteq_e S_S$.*

Proof: (1) $\Rightarrow$ (2) Let $IM \leq_{s-e} M_R$. Assume that I is not essential in S_S. Then there exists a nonzero right ideal J of S such that $I \cap J = 0$. Since I is a maximal right ideal of S, then I is a direct summand of S_S. So, there exists an idempotent $e \in S$ such that $I = eS$. Hence $IM = eM = \mathrm{Ker}(1_S - e) \leq_{s-e} M$. Then $1 - e \in \overline{K}_S(M)$. By Corollary 2.12, $e = 1$, a contradiction.

(2) $\Rightarrow$ (1) Let $I \subseteq_e S_S$. Let $IM \cap X = 0$ for a submodule X of M_R. Then $0 = D_S(0) = D_S(IM) \cap D_S(X)$. Hence $I \cap D_S(X) = 0$. Since I is essential in S_S, $D_S(X) = 0$.

Let f be an element in S. Then f is said to be *partially invertible* if, fS (equivalently, Sf) contains a nonzero idempotent.

For an R-module M_R, the total of M_R is defined as

$$\text{Tot}(S) = \text{Tot}(M, M) = \{f \in S \mid f \text{ is not partially invertible}\}.$$

The total may not be closed under addition. In fact, if 0 and 1 are the only idempotents in S, then total of M_R is the set of non-isomorphisms.

Proposition 2.14 *If M_R is a module, then $\overline{K}_S(M) \subseteq Tot(M, M)$.*

Proof: If $f \in \overline{K}_S(M)$ but $f \notin \text{Tot}(M, M)$, then f is partially invertible. So, there exists $0 \neq e^2 = e \in Sf$. By Corollary 2.9, $e \in \overline{K}_S(M)$, that contradicts to Corollary 2.12.

If I is a subset of a ring R, then R is said to be *I-semipotent* if every right (equivalently, left) ideal not contained in I contains a nonzero idempotent, equivalently if every element $a \notin I$ has a partial inverse. R is called *semipotent* if R is $J(R)$-semipotent.

Lemma 2.15 *Let I be a subset of $S = End(M_R)$. Then the following are equivalent:*
(1) *S is I-semipotent;*
(2) *$Tot(M, M) \subseteq I$.*

Proof: By [4, Lemma 20].

Proposition 2.16 *Let $S = End(M_R)$ for any module M_R. Then S is semipotent if and only if $J(S) = Tot(M, M)$.*

Proof: By [4, Theorem 21].

Proposition 2.17 *Let $S = End(M_R)$ for any semi-injective module M_R. Then $J(S) = \overline{K}_S(M) = Tot(M, M)$ if S is semipotent.*

Proof: By Corollary 2.10, $J(S) \subseteq \overline{K}_S(M)$. Let $s \in \overline{K}_S(M)$. If $s \notin J(S)$, then since S is $J(S)$-semipotent, $\overline{K}_S(M)$ have a nonzero idempotent which is a contradiction (see Corollary 2.12). Thus $J(S) = \overline{K}_S(M)$. By Proposition 2.14, $\overline{K}_S(M) \subseteq \text{Tot}(M, M)$. On the other hand, S is $\overline{K}_S(M)$-semipotent since $J(S) = \overline{K}_S(M)$. So by Lemma 2.15, $\text{Tot}(M, M) \subseteq \overline{K}_S(M)$.

Proposition 2.18 *Let $S = End(M_R)$ for any semi-injective module M_R in which $r_S(a) = 0$, $a \in S$, implies $Sa = S$, then $\overline{K}_S(M) = J(S)$.*

Proof: $J(S) \subseteq \overline{K}_S(M)$ by Corollary 2.10. Let $k \in \overline{K}_S(M)$. Then $\text{Ker}k \leq_{s-e} M$, so $r_S(1_S - sk) = 0$ for all $s \in S$ by Lemma 2.8. Hence $S(1_S - sk) = S$ by hypothesis. Thus $k \in J(S)$.

A ring R is called *left Kasch* if each simple left R-module embeds in R; equivalently, if $r_R(T) \neq 0$ for every (maximal) left ideal T of R. Call R a right C_2 ring if every right ideal that is isomorphic to a direct summand of R_R is itself a direct summand of R_R.

Example 2.19 *In each of the following cases we have $J(S) = \overline{K}_S(M)$ for a semi-injective module M_R:*
(1) *S is semipotent.*
(2) *S is left Kasch.*
(3) *S is a right C_2 ring.*

Proof: (1) By Proposition 2.17.
(2) Let $a \in S$, $r_S(a) = 0$. If $Sa \neq S$, then $r_S(Sa) \neq 0$ by (2); that is $r_S(a) \neq 0$, a contradiction. Thus by Proposition 2.18, $J(S) = \overline{K}_S(M)$.
(3) Let $a \in S$ and $r_S(a) = 0$. Then $aS \cong S$. By (3), aS is a direct summand of S. Then $a = aba$ for some element b of S. But then $0 = r_S(a) = r_S(ab) = (1_S - ab)S$. Then $ab = 1_S$ and so $S = Sa$. By Proposition 2.18, $\overline{K}_S(M) = J(S)$.
Let M_R be a module. We define

$$\overline{A}_R(M) = \cap\{K \leq M_R \mid K \leq_{s-e} M\}.$$

It is clear that $\overline{A}_R(M) \subseteq Soc(M)$.

Proposition 2.20 *Let M_R be a retractable module. Then $\overline{A}_R(M) = Soc(M)$. Moreover, if M_R is semi-projective, then $\overline{A}_R(M) = Soc(M) = Soc(S_S)M$.*

Proof: By Proposition 2.3, $\overline{A}_R(M) = Soc(M)$. Now assume that M_R is semi-projective. Then by [3, Proposition 2.4], $\overline{A}_R(M) = Soc(M) = Soc(S_S)M$.

Proposition 2.21 *Let M_R be a module. Consider the following conditions:*
(1) *$\overline{A}_R(M) \leq_{s-e} M$.*
(2) *If $K \leq_{s-e} M$ and $L \leq_{s-e} M$, then $K \cap L \leq_{s-e} M$.*
Then (1) $\Rightarrow$ (2) holds. If M is finitely cogenerated, then (2) $\Rightarrow$ (1) holds.
Moreover, if M is finitely cogenerated and one of the above conditions holds, then we have:
(a) $\overline{A}_R(M)$ is the unique smallest s-essential submodule of M.
(b) $\overline{A}_R(M) = \sum\{L \leq \overline{A}_R(M) \mid L$ is minimal in M \}.

Proof: (1) $\Rightarrow$ (2) Let $K \leq_{s-e} M$ and $L \leq_{s-e} M$. Then $\overline{A}_R(M) \subseteq K \cap L$. By Lemma 2.6, $K \cap L \leq_{s-e} M$.
(2) $\Rightarrow$ (1) Let M be finitely cogenerated and $\overline{A}_R(M) \cap X = 0$ for a submodule X of M_R. So $K_1 \cap K_2 \cap \cdots \cap K_n \cap X = 0$ for some $K_i \leq \overline{A}_R(M)$ and $n \in \mathbb{N}$. By (1), $D_S(X) = 0$.
Finally, (a) is clear by (1) and (b): Since $\overline{A}_R(M) \subseteq \mathrm{Soc}(M)$, $\overline{A}_R(M)$ is semisimple.

Corollary 2.22 *Let M_R be finitely cogenerated. If the intersection of any two s-essential submodules is s-essential, then $Soc(\overline{A}_R(M)) = \overline{A}_R(M)$.*

Proof: It follows from (b) of Proposition 2.21.

Proposition 2.23 *Let M_R be a finitely cogenerated module. If $A_R(M) = Soc(M)$, then the intersection of any two s-essential submodules is s-essential.*

Proof: Let $K \leq_{s-e} M_R$ and $L \leq_{s-e} M_R$. Then $\overline{A}_R(M) \subseteq K \cap L$. By Lemma 2.6, $K \cap L \leq_{s-e} M$.

Acknowledgements: The author is thankful to Professor Cornelius Greither for providing Example 2.2.

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