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Superstability and Stability of Approximate n-Ring Homomorphisms between Quasi-Banach Algebras

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Abstract

Let $n \geq 2$, A and B be complex algebras. A mapping $T : A \longrightarrow B$ is called n-ring homomorphism if for all elements $a_1, \cdots, a_n, x, y \in A$, we have

$$T(x+y) = T(x) + T(y) \quad \textit{and} \quad T(a_1 \cdots a_n) = T(a_1) \cdots T(a_n).$$

In this paper, we prove the superstability of n-ring homomorphisms on quasi-normed algebras. In addition, we establish the stability of n-ring homomorphisms in quasi-Banach algebras.

Keywords: *n-Ring homomorphism, Hyers-Ulam stability, Approximate n-ring homomorphism, Quasi-Banach algebras, Superstability.*

1 Introduction

Let X be a complex linear space. A quasi-norm is a complex valued function on X satisfying the following conditions:

(i) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|\lambda x\| = |\lambda| \|x\|$ for all $\lambda \in \mathbb{C}$ and $x \in X$;

(iii) there is a constant $K \geq 1$ such that $\|x + y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|.\|)$ is called a quasi-normed space if $\|.\|$ is a quasi-norm on $\|X\|$. The smallest possible K is called the modulus of cancavity of $\|.\|$. A quasi-norm $\|.\|$ is called a p-norm $(0 < p \leq 1)$ if $\|x+y\|^p \leq \|x\|^p + \|y\|^p$ for all $x, y \in X$. By the Aoki-Rolewicz theorem [3], each quasi-norm is equivalent to some p-norm. The p-norm $\|.\|$ induces a metric topology on X defined by $d(x, y) = \|x - y\|^p$. X is called a quasi-Banach space if X is complete for this metric. The quasi-norm space $(A, \|.\|)$ is called a quasi-normed algebra if A is an algebra and there is $M > 0$ such that $\|xy\| \leq M\|x\|\|y\|$ for all $x, y \in A$. A quasi-Banach algebra is a complete quasi-normed algebra. If the quasi-norm $\|.\|$ is a p-norm, then the quasi-Banach algebra $(A, \|.\|)$ is called a p-Banach algebra. Suppose that $n \geq 2$ is a integer and A and B are complex algebras. A mapping $T : A \longrightarrow B$ is called n-multiplicative if $T(a_1 \cdots a_n) = T(a_1) \cdots T(a_n)$ for all elements $a_1, \cdots, a_n \in A$. If T is also linear, it is called an n-homomorphism. Moreover, T is called n-additive if $T(a_1 + \cdots + a_n) = T(a_1) + \cdots + T(a_n)$ for all $a_1, \cdots, a_n$. It is easy to see that T is n-additive if and only if T is additive. If T is n-multiplicative and additive, then T is called n-ring homomorphism. For further details on the above concepts in normed algebras one can refer, for example to [5] and [7]. Let A and B be quasi-normed algebras, $\delta > 0$ and $T : A \longrightarrow B$ be a map. We say that T is (δ, n)-multiplicative if

$$\|T(x_1 \cdots x_n) - T(x_1) \cdots T(x_n)\| \leq \delta \|x_1\| \cdots \|x_n\| \quad (x_1, \cdots, x_n \in A).$$

In the case that $n = 2$, T is called δ-multiplicative. Also, we say that T is (ε, n)-additive for some $\varepsilon > 0$ if

$$\|T(x_1 + \cdots + x_n) - T(x_1) - \cdots - T(x_n)\| \leq \varepsilon(\|x_1\| + \cdots + \|x_n\|) \quad (x_1, \cdots, x_n \in A).$$

Clearly, every (ε, n)-additive function is $(\varepsilon, 2)$-additive, or in brief, ε-additive function. The map T is called approximate n-ring homomorphism, if T is (δ, n)-multiplicative and (ε, n)-additive for some positive numbers ε, δ. In 1979, Baker et al. [1] introduced that every approximately exponential functional is either an exponential function or bounded. This concept is known as the superstability. Later, Baker [2] generalized this famous result as follows: Let $(G, .)$ be a semigroup and $\delta > 0$. If $T : G \longrightarrow \mathbb{C}$ satisfies the inequality

$$|T(xy) - T(x)T(y)| \leq \delta \|x\|\|y\| \qquad (x, y \in G),$$

then either $|T(x)| \leq \frac{1+\sqrt{1+4\delta}}{2}$ for all $x \in G$, or T is multiplicative function. In section two of this paper, we prove that every approximate n-ring homomorphism T on quasi-normed algebra A is n-ring homomorphism or there exists $r > 0$ such that $|Tx| \leq r\|x\|$ for all $x \in A$.

The stability problem of functional equations originated from a question of Ulam, posed in 1940, concerning the stability of group homomorphism:

Given a group G_1, a metric group (G_2, d) and a positive number ε, does there exists a $\delta > 0$ such that if a function $f : G_1 \longrightarrow G_2$ satisfies the condition $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $T : G_1 \longrightarrow G_2$ such that $d(f(x), T(x)) < \varepsilon$ for all $x \in G_1$. A year later, Hyers [6] gave a partial affirmative answer to the question of Ulam in the context of Banach spaces. After Hyerss result, many mathematicians have extended the Ulams problem to other functional equations and replaced Banach spaces with other spaces. Recently, Park and Rassias [9] proved the Hyers-Ulam stability of isometric additive functional equations in quasi-Banach spaces. Also Park [10] proved stability of homomorphisms in quasi-Banach algebra. In the third section, we establish stability of n-ring homomorphisms in quasi-Banach algebras.

2 Superstability of Approximate n-Ring Homomorphisms on Quasi-Normed Algebras

Theorem 2.1. *Let $p \geq 0$, A be a quasi-normed algebra and $T : A \longrightarrow \mathbb{C}$ satisfies*

$$|T(x_1 \cdots x_n) - T(x_1) \cdots T(x_n)| \leq \delta \|x_1\|^p \cdots \|x_n\|^p$$

for all $x_1, \cdots, x_n \in A$. Then at least one of the following holds:

(i) T is n-multiplicative,

(ii) there is a constant r such that $|T(x)| \leq r\|x\|^p$ for all $x \in A$.

Proof: Suppose that T is not n-multiplicative, then there exist $a_1, \cdots a_n \in A$ such that $T(a_1 \cdots a_n) \neq T(a_1) \cdots T(a_n)$. For every element $x \in A$, we have

$$\begin{aligned}|T(x)|^{n-1}|T(a_1 \cdots a_n) - T(a_1) \cdots T(a_n)| = &|T(x)^{n-1}T(a_1 \cdots a_n)\\ &- T(x)^{n-1}T(a_1) \cdots T(a_n) \pm \\ &T(x^{n-1}a_1 \cdots a_n) \pm T(x^{n-1}a_1)T(a_2) \cdots T(a_n)|\\ \leq &|T(x)^{n-1}T(a_1 \cdots a_n) - T(x^{n-1}a_1 \cdots a_n)| + \\ &|T(x^{n-1}a_1 \cdots a_n) - T(x^{n-1}a_1)T(a_2) \cdots T(a_n)| + \\ &|T(x^{n-1}a_1)T(a_2) \cdots T(a_n) - T(x)^{n-1}T(a_2) \cdots T(a_n)|\\ \leq &\delta\|x\|^{(n-1)p}\|a_1 \cdots a_n\|^p + \delta\|x^{n-1}a_1\|^p\|a_2\|^p \cdots \|a_n\|^p\\ &+\delta\|x\|^{(n-1)p}\|a_1\|^p|T(a_2) \cdots (a_n)|.\end{aligned}$$

Therefore, if

$$r = \left(\frac{2M^{(n-1)p}\delta\|a_1\|^p \cdots \|a_n\|^p + \delta\|a_1\|^p|T(a_2) \cdots T(a_n)|}{|T(a_1 \cdots a_n) - T(a_1) \cdots T(a_n)|}\right)^{\frac{1}{(n-1)}},$$

then $|T(x)| \leq r\|x\|^p$ for all $x \in A$.

Corollary 2.2. *Let A be a quasi-normed algebra and $T : A \longrightarrow \mathbb{C}$ be an approximate multiplicative map. Then either T is n-multiplicative map or there exists a constant r such that $|T(x)| \leq r\|x\|$ for all $x \in A$.*

Theorem 2.3. *Let $p \geq 0$, A be a quasi-normed algebra and $T : A \longrightarrow \mathbb{C}$ be a mapping satisfies the following conditions:*

(i) $|T(x+y) - T(x) - T(y)| \leq \varepsilon(\|x\|^p + \|y\|^p) \quad (x, y \in A)$,

(ii) $|T(x_1 \cdots x_n) - T(x_1) \cdots T(x_n)| \leq \varepsilon \|x_1\|^p \cdots \|x_n\|^p \quad (x_1, \cdots, x_n \in A)$.

Then at least one of the following results holds:

(1) T is additive and n-multiplicative,

(2) there exists a constant r such that $|T(x)| \leq r\|x\|^p$ for each $x \in A$.

Proof: If T is not n-multiplicative, then by Corollary 2.2, the result follows. If T is not additive, there exist $a, b \in A$ such that $T(a+b) \neq T(a)+T(b)$. For all $x \in A$, we have

$$\begin{aligned}
|T(x)|^{n-1}|T(a+b)-T(a) - T(b)| = & |T(x)^{n-1}T(a+b) - T(x)^{n-1}T(a) \\
& - T(x)^{n-1}T(b) \pm T(x^{n-1}(a+b)) \pm T(x^{n-1}a) \pm T(x^{n-1}b)| \\
\leq & |T(x)^{n-1}T(a+b) - T(x^{n-1}(a+b))| + \\
& |T(x^{n-1}a + x^{n-1}b) - T(x^{n-1}a) - T(x^{n-1}b)| + \\
& |T(x)^{n-1}T(a) - T(x^{n-1}a)| + |T(x^{n-1}b) - T(x)^{n-1}T(b)| \\
\leq & \varepsilon\|x\|^{(n-1)p}\|a+b\|^p + \varepsilon(\|x^{n-1}a\|^p + \|x^{n-1}b\|^p) + \\
& \varepsilon\|x\|^{(n-1)p}\|a\|^p + \varepsilon\|x\|^{(n-1)p}\|b\|^p \\
\leq & \varepsilon\|x\|^{(n-1)p}\|a+b\|^p + \varepsilon\|x\|^{(n-1)p}M^{(n-1)p}(\|a\|^p + \|b\|^p) + \\
& \varepsilon\|x\|^{(n-1)p}\|a\|^p + \varepsilon\|x\|^{(n-1)p}\|b\|^p.
\end{aligned}$$

Hence if

$$r = \left[\frac{\varepsilon \left[\|a+b\|^p + M^{(n-1)p}(\|a\|^p + \|b\|^p) + \|a\|^p + \|b\|^p \right]}{|T(a+b) - T(a) - T(b)} \right]^{\frac{1}{n-1}},$$

then $|T(x)| \leq r\|x\|^p$ for all $x \in A$.

3 Stability of Approximate n-Ring Homomorphisms between Quasi-Banach Algebras

In this section, we extend the theorem of Hyers-Ulam-Rassias of n- ring homomorphisms in quasi-Banach algebras. We first state the following theorem, which has been proved by Tabor [11].

Theorem 3.1. *Let $r > 1$, $\delta > 0$, A be a quasi-norm algebra and B be a p-Banach algebra. Let $\phi : A \longrightarrow B$ be a mapping such that*

$$\|\phi(x+y) - \phi(x) - \phi(y)\| \leq \delta(\|x\|^r + \|y\|^r) \quad (x, y \in A). \tag{1}$$

Then there exists a unique additive mapping $\psi : A \longrightarrow B$ such that

$$\|\psi(x) - \phi(x)\| \leq \frac{2\delta}{(2^{pr} - 2^p)^{\frac{1}{p}}}\|x\|^r \quad (x \in A). \tag{2}$$

Theorem 3.2. *Let $r > 1$, ε, δ be positive numbers, A be a quasi-norm algebra and B be a p-Banach algebra. Assume that $\phi : A \longrightarrow B$ satisfies the conditions (1) and*

$$\|\phi(x_1 \cdots x_n) - \phi(x_1) \cdots \phi(x_n)\| \leq \varepsilon\|x_1\|^r \cdots \|x_n\|^r \quad (x_1, \cdots, x_n \in A). \tag{3}$$

Then there exists a unique n-ring homomorphism $\psi : A \longrightarrow B$ such that satisfying (2).

Proof: Tabor proved there exists the unique additive mapping $\psi : A \longrightarrow B$ such that $\psi(x) = \lim_{m\to\infty} 2^m \phi\left(\frac{x}{2^m}\right)$ and satisfies (2).
Hence $\psi(x) = \lim_{m\to\infty} 2^{mn} \phi\left(\frac{x}{2^{mn}}\right)$ for all $x \in A$. Thus for each $x_1, \cdots, x_n \in A$,

$$\begin{aligned}\|\psi(x_1 \cdots x_n) - \psi(x_1) \cdots \psi(x_n)\| &= \lim_{m\to\infty} 2^{mn} \left\|\phi\left(\frac{x_1 \cdots x_n}{2^{mn}}\right) - \phi\left(\frac{x_1}{2^m}\right) \cdots \left(\frac{x_n}{2^m}\right)\right\| \\ &\leq \lim \frac{2^{mn}}{2^{mnr}} \varepsilon \|x_1\|^r \cdots \|x_n\|^r = 0.\end{aligned}$$

So ψ is n-multiplicative, and the result follows.

Theorem 3.3. *Let A be a quasi-normed algebra, B be a p- Baqnach algebra, $r < 1$ and ε, δ, ν, and γ are positive numbers. Let $\phi : A \longrightarrow B$ be a mapping such that*

$$\|\phi(x_1 + \cdots + x_n) - \phi(x_1) - \cdots - \phi(x_n)\| \leq \varepsilon(\|x_1\|^r + \cdots + \|x_n\|^r + \delta), \tag{4}$$

and

$$\|\phi(x_1 \cdots x_n) - \phi(x_1) \cdots \phi(x_n)\| \leq \gamma\|x_1\|^r \cdots \|x_n\|^r + \nu \tag{5}$$

for all $x_1, \cdots, x_n \in A$. Then there exists a unique n-ring homomorphism $\psi : A \longrightarrow B$ such that

$$\|\phi(x) - \psi(x)\| \le \varepsilon \left[\frac{n^p}{n^p - n^{rp}}\|x\|^{rp} + \frac{\delta^p}{n^p - 1}\right]^{\frac{1}{p}} \quad (x \in A). \tag{6}$$

Proof: Letting $x = y = n^m x$ in (4), we get

$$\|\phi(n^{m+1} - n\phi(n^m x)\| \le \varepsilon(n\|n^m x\|^r + \delta)$$

for all $m \in \mathbb{N}$ and $x \in A$. So

$$\left\|\frac{\phi(n^{m+1}x)}{n^{m+1}} - \frac{\phi(n^m x)}{n^m}\right\| \le \varepsilon\left(\frac{\|x\|^r}{n^{m(1-r)}} + \frac{\delta}{n^{m+1}}\right).$$

Since B is a p-Banach algebra ($0 < p \le 1$), we conclude that

$$\begin{aligned}\left\|\frac{\phi(n^s x)}{n^s} - \frac{\phi(n^t x)}{n^t}\right\|^p &\le \sum_{s=1}^{t-1}\left\|\frac{\phi(n^{m+1}x)}{n^{m+1}} - \frac{\phi(n^m x)}{n^m}\right\|^p \\ &\le \varepsilon^p \sum_{n=s}^{t-1}\left(\frac{\|x\|^r}{n^{m(1-r)}} + \frac{\delta}{n^{m+1}}\right)^p \\ &\le \varepsilon^p\left(\sum_{m=s}^{t-1}\frac{\|x\|^{rp}}{n^{m(1-r)p}} + \sum_{m=s}^{t-1}\frac{\delta^p}{n^{(m+1)p}}\right) \end{aligned} \tag{7}$$

for all $t > s$ and $x \in A$. Hence $\left\{\frac{\phi(n^m x)}{n^m}\right\}_{m=1}^{\infty}$ is a Cauchy sequence for each $x \in A$, so there exists a limit function $\psi(x) = \lim_{m\to\infty}\dfrac{\phi(n^m x)}{n^m}$. We will now show that ψ is additive. It is sufficient to show that ψ is n-additive. By (4) for all $x_1, \cdots, x_n \in A$, we have

$$\begin{aligned}\|\psi(x_1 + \cdots + x_n) - \psi(x_1) - \cdots - \psi(x_n)\| &= \lim_{m\to\infty}\frac{1}{n^m}\|\phi(n^m x_1 + \cdots + n^m x_n) \\ &\quad - \phi(n^m x_1) - \cdots - \phi(n^m x_n)\| \\ &\le \lim \frac{\varepsilon}{n^m}[n^{mr}(\|x_1\|^r + \cdots + \|x_n\|^r) + \delta] = 0.\end{aligned}$$

Then ψ is n-additive and so ψ is additive. Also, if $s = 0$ and passing the limit $t \longrightarrow \infty$ in (7), then we get (6). To prove the uniqueness property of ψ, assume that $\varphi : A \longrightarrow B$ is another additive mapping satisfying (6). Then for $m \in \mathbb{N}$ and $x \in A$, we have

$$\begin{aligned}m\|\psi(x) - \varphi(x)\| =& \|\psi(mx) - \varphi(mx)\| \\ \le& K(\|\psi(mx) - \phi(mx)\| + \|\phi(mx) - \varphi(mx)\|) \\ \le& 2K\varepsilon\left[\frac{n^p}{n^p - n^{rp}}\|mx\|^{rp} + \frac{\delta^p}{n^p - 1}\right]^{\frac{1}{p}},\end{aligned}$$

where K is the modulus of concavity of $\|.\|_B$. Then

$$\|\psi(x)-\varphi(x)\| \leq 2K\varepsilon\left[\frac{n^p}{n^p-n^{rp}}\frac{\|x\|^{rp}}{m^{p(1-r)}}+\frac{\delta^p}{m^p(n^p-1)}\right]^{\frac{1}{p}}.$$

Letting the limit $m\to\infty$, we get $\varphi=\psi$.
Now we will show that ψ is n-multiplicative. By (5), we have

$$\begin{aligned}\|\psi(x_1\cdots x_n)-\psi(x_1)\cdots\psi(x_n)\| &= \lim_{m\to\infty}\frac{1}{(n^n)^m}\|\phi(n^{nm}x_1\cdots x_n)\\ &\quad -\phi(n^m x_1)\cdots\phi(n^m x_n)\|\\ &\leq \lim \frac{\gamma}{n^{mn(1-r)}}\|x_1\|^r\cdots\|x_n\|^r+\frac{\nu}{n^{mn}}=0\end{aligned}$$

for all $x_1\cdots x_n\in A$. So ψ is n-multiplicative. Then the result follows.

Remark 3.4. *Let A and B be real algebras in the Theorems 3.2 and 3.3, and let B^* (the dual space of B) separate points of B. If the function $\mathbb{R}\ni t\longmapsto\phi(tx)$ is continuous for all $x\in A$, then by a method similar to [8], one can show ψ is linear.*
We notice that, the hypothesis of separating dual is required here because there exists quasi-Banach space with trivial dual [4].

Theorem 3.5. *Let $r>1$, $\varepsilon>0$ and A be a unital real quasi-normed algebra, and let B be a unital real p-Banach algebra such that B^* separates points of B.*
Suppose that $\phi: A\longrightarrow B$ is an n-multiplicative satisfying (1). If the function $\mathbb{R}\ni t\longmapsto\phi(tx)$ is continuous for all $x\in A$ and $\lim 2^m\phi(\frac{1}{2^m})=1$, then ϕ is n-homomorphism.

Proof: Since ϕ is n-multiplicative and ϕ satisfies (1), by Theorem 3.2 and Remark 3.4, there exists an n-homomorphism $\psi: A\longrightarrow B$ satisfying (2), such that $\psi(x)=\lim_{m\to\infty}2^m\phi\left(\frac{x}{2^m}\right)$ and

$$\begin{aligned}\psi(x)=\psi(1^{n-1}x) &= \lim 2^m\phi(\frac{1^{n-1}x}{2^m})=\lim 2^{(n-1)m}\phi(\frac{1^{n-1}x}{2^{(n-1)m}})\\ &= \lim 2^{(n-1)m}\phi^{n-1}(\frac{1}{2^m})\phi(x)=1^{n-1}\phi(x)=\phi(x)\end{aligned}$$

for all $x\in A$. Then $\phi=\psi$ and the result follows.

Theorem 3.6. *Let $r<1$, A be a unital real quasi-normed algebra, B be a unital real p-Banach algebra such that B^* separates points of B and suppose that ε, δ are positive numbers. Let $\phi: A\longrightarrow B$ be an n-multiplicative map satisfying (4). If the function $\mathbb{R}\ni t\longmapsto\phi(tx)$ is continuous for all $x\in A$ and $\lim_{m\to\infty}\frac{\phi(n^m 1)}{n^m}=1$, then ϕ is n-homomorphism.*

Proof: By Theorem 3.3 and Remark 3.4, there exists an n-homomorphism $\psi : A \longrightarrow B$ satisfying (6), such that $\psi(x) = \lim_{m\to\infty} \frac{\phi(n^m x)}{n^m}$ for all $x \in A$. Then by modifying the proof of Theorem 3.5, the result follows.

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