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Some New Sequence Spaces Defined by Musielak-Orlicz Functions on a Real *n*-Normed Space

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Abstract

The purpose of this paper is to introduce the sequence space $E_n^q(B, \mathcal{M}, p, s, \|.,\ldots,.\|)$ defined by using an infinite matrix and Musielak-Orlicz function. We also study some topological properties and prove some inclusion relations involving this space.

Keywords: *Paranorm, Infinite matrix, n-norm, Musielak-Orlicz functions, Euler transform.*

1 Introduction and Preliminaries

The concept of 2-normed spaces was initially developed by Gähler [1] in the mid-1960s, while one can see that of n-normed spaces in Misiak [2]. Since then, many others have studied this concept and obtained various results; see Gunawan [3, 4] and Gunawan and Mashadi [5]. Let n be a non-negative integer and X be a real vector space of dimension d, where $d \geq n \geq 2$. A real-valued function $\|.,...,.\|$ on X^n satisfying the following conditions:

(1) $\|(x_1, x_2, ..., x_n)\| = 0$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent.

(2) $\|(x_1, x_2, ..., x_n)\|$ is invariant under permutation,

(3) $\|\alpha x_1, x_2, ..., x_n\| = |\alpha| \|(x_1, x_2, ..., x_n)\|$, for any $\alpha \in R$,

(4) $\|(x_1 + \bar{x}, x_2, ..., x_n)\| \leq \|(x_1, x_2, ..., x_n)\| + \|(\bar{x}, x_2, ..., x_n)\|$

is called an $n-$norm on X and the pair $(X, \|.,...,.\|)$ is called an $n-$normed space.

A trivial example of an $n-$normed space is $X = R^n$, equipped with the Euclidean $n-$norm $\|(x_1, x_2, ..., x_n)\|_E =$ volume of the n-dimensional parallelepiped spanned by the vectors $x_1, x_2, ..., x_n$ which may be given explicitly by the formula

$$\|(x_1, x_2, ..., x_n)\|_E = \left|\det(x_{ij})\right| = abs\left(\det\left(\langle x_i, x_j \rangle\right)\right) \tag{1}$$

where $x_i = (x_{i1}, x_{i2}, ..., x_{in}) \in R^n$ for each $i = 1, 2, 3 ..., n$. Let $(X, \|.,...,.\|)$ be an $n-$normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, ..., a_n\}$ be a linearly independent set in X. Then the function $\|.,...,.\|_\infty$ on X^{n-1} is defined by

$$\|(x_1, x_2, ..., x_n)\|_\infty = \max_{1 \leq i \leq n} \left\{\|x_1, x_2, ..., x_{n-1}, a_i\|\right\} \tag{2}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, ..., a_n\}$ and this is known as the derived $(n-1)$-norm.

The standard n-norm on X a real inner product space of dimension $d \geq n$ is as follows:

$$\|(x_1, x_2, ..., x_n)\|s = \left[\det\left(\langle x_i, x_j \rangle\right)\right]^{\frac{1}{2}},$$

where $\langle \, , \, \rangle$ denotes the inner product on X. If we take $X = R^n$ then this n-norm is exactly the same as the Euclidean n-norm $\|(x_1, x_2, ..., x_n)\|_E$

mentioned earlier. For $n = 1$ this n-norm is the usual norm $\| x_1 \| = \sqrt{\langle x_1, x_1 \rangle}$ for further details (see Gunawan [4]).

We first introduce the following definitions:

A sequence (x_k) in an n-normed space $(X, \| ., ..., . \|)$ is said to be convergent to some $L \in X$ if

$$\lim_{k \to \infty} \| x_k - L, z_1, z_2, ..., z_{n-1} \| = 0, \text{ for every } z_1, z_2, ..., z_{n-1} \in X. \quad (3)$$

A sequence (x_k) in an n-normed space $(X, \| ., ..., . \|)$ is said to be Cauchy if

$$\lim_{\substack{k \to \infty \\ p \to \infty}} \| x_k - x_p, z_1, z_2, ..., z_{n-1} \| = 0, \text{ for every } z_1, z_2, ..., z_{n-1} \in X. \quad (4)$$

If every Cauchy sequence space in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. A complete n-normed space is said to be a n- Banach space.

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ as $x > 0$ a $M(x) \to \infty$, as $x \to \infty$.

Lindenstrauss and Tzafriri [6] studied some Orlicz type sequence spaces defined as follows:

$$\ell_M = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho} \right) < \infty, \ for \ some \ \rho > 0 \right\}. \quad (5)$$

The space ℓ_M with the norm

$$\| x \| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho} \right) \le 1 \right\} \quad (6)$$

becomes a Banach space which is called an Orlicz sequence space. The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(t) = |t|^p$, for $1 \le p < \infty$. An Orlicz function M is said to satisfy $\Delta_2 - condition$ for all values of u, if there exists a constant K such that $M(2u) \le K M(u), u \ge 0$ (see [7]).

A sequence space $\mathrm{M}=(M_k)$ of Orlicz functions is called a Musielak-Orlicz function see ([8], [9]). A sequence space $\mathrm{N}=(N_k)$ defined by

$$N_k(v)= \sup\{|v|u - M_k(u) : u \geq 0\}, \; k = 1, 2, ... \tag{7}$$

is called the complimentary function of a Musielak-Orlicz function M. For a given Musielak-Orlicz function M, the Musielak-Orlicz sequence space function t_{M} and its subspace h_{M} are defined as follows

$$t_{\mathrm{M}} = \{ x \in w : I_{\mathrm{M}}(cx) < \infty \; \textit{for some} \; c > 0 \}, \tag{8}$$

$$h_{\mathrm{M}} = \{ x \in w : I_{\mathrm{M}}(cx) < \infty \; \textit{for all} \; c > 0 \},$$

where I_{M} is a convex modular defined by

$$I_{\mathrm{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), \; x = (x_k) \in t_M . \tag{9}$$

We consider t_{M} equipped with the Luxemberg norm

$$\|x\| = \inf \left\{ k > 0 : I_{\mathrm{M}}\left(\frac{x}{k}\right) \leq 1 \right\} \tag{10}$$

or equipped with the Orlicz norm

$$\|x\|^0 = \inf \left\{ \frac{1}{k} \left(1 + I_{\mathrm{M}}(kx)\right) : k > 0 \right\}. \tag{11}$$

Let X be a linear metric space. A function $p : X \to R$ is called a paranorm, if

(1) $p(x) \geq 0$, for all $x \in X$;
(2) $p(-x) = p(x)$, for all $x \in X$;
(3) $p(x + y) \leq p(x) + p(y)$, for all $x \; y \in X$;
(4) If (σ_n) is a sequence of scalars with $\sigma_n \to \sigma$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\sigma_n x_n - \sigma x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm, and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [10], Theorem 10. 4.2, P-183). For more details about sequence spaces see [11-24] and the references therein.

Let (s_k) denotes the sequence of partial sums of the infinite series $\sum_{k=0}^{\infty} a_k$ and q be any positive real number. The Euler transform (E,q) of the sequence $s=(s_n)$ is defined by

$$E_n^q(s)=\frac{1}{(1+q)^n}\sum_{v=o}^{n}\binom{n}{v} q^{n-v}\, s_v\,. \tag{12}$$

The series $\sum_{n=0}^{\infty} a_n$ is said to be summable (E,q) to the number s if

$$E_n^q(s)=\frac{1}{(1+q)^n}\sum_{v=o}^{n}\binom{n}{v} q^{n-v}\, s_v \to s \quad as \;\; n\to\infty \tag{13}$$

and is said to be absolutely summable (E,q) or summable $|E,q|$ if

$$\sum_k \left| E_k^q(s) - E_{k-1}^q(s)\right| < \infty\,. \tag{14}$$

Let $x=(x_k)$ be a sequence of scalars we write $N_n(x)=E_n^q(x)-E_{n-1}^q(x)$, where $E_n^q(x)$ is defined by (12). After applications of Abel's transformation, we have

$$N_n(x)=-\frac{1}{(1+q)^{n-1}}\sum_{k=o}^{n-2} x_{k+1}A_k+\frac{s_{n-1}A_{n-1}}{(1+q)^{n-1}}+\frac{s_n}{(1+q)^n}-\frac{q^{n-1}}{(1+q)^n}s_0\,, \tag{15}$$

where

$$A_k=\sum_{i=0}^{k}\left[\frac{q}{1+q}\binom{n}{i}-\binom{n-1}{i}\right]q^{n-i-1}. \tag{16}$$

Note that for any sequences $x=(x_n),\, y=(y_n)$ and scalar λ, we have

$N_n(x+y)=N_n(x)+N_n(y)$ and $N_n(\lambda x)=\lambda N_n(x)$.

Let $\mathcal{M}=(M_k)$ be a sequence of Musielak-Orlicz functions, $p=(p_k)$ be a bounded sequence of positive real numbers, "$B=b_{nk}$" be an infinite matrix, and $(X, \|., ...,.\|)$ be an $n-$normed space, we define the sequence space:

$$E_n^q\left(B,\mathrm{M},p\,,s,\|\,.,\ldots,.\|\right)=$$
$$\left\{\begin{array}{l} x=(x_k):\displaystyle\sum_{k=1}^{\infty}\frac{b_{nk}}{k^s}\left[M_k\left(\left\|\frac{N_k(x)}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_k}<\infty, s\geq 0, \\ \text{for some } \rho>0 \text{ and for every } z_1,z_2,\ldots z_{n-1}\in X \end{array}\right\}. \quad (17)$$

If we take $p=p_k=1$ for all $k\in N$, we have

$$E_n^q\left(B,\mathrm{M},s,\|\,.,\ldots,.\|\right)=$$
$$\left\{\begin{array}{l} x=(x_k):\displaystyle\sum_{k=1}^{\infty}\frac{b_{nk}}{k^s}\left[M_k\left(\left\|\frac{N_k(x)}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]<\infty, s\geq 0, \\ \text{for some } \rho>0, \text{and for every } z_1,z_2,\ldots z_{n-1}\in X \end{array}\right\}. \quad (18)$$

If we take $s=0$, we have

$$E_n^q\left(B,\mathrm{M},p,\|\,.,\ldots,.\|\right)=$$
$$\left\{\begin{array}{l} x=(x_k):\displaystyle\sum_{k=1}^{\infty}b_{nk}\left[M_k\left(\left\|\frac{N_k(x)}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_k}<\infty, \\ \text{for some } \rho>0, \text{ and for every } z_1,z_2,\ldots z_{n-1}\in X \end{array}\right\}. \quad (19)$$

The following well known inequality will be used throughout the article. Let $p=(p_k)$ be any sequence of positive real numbers with $0\leq p_k\leq\sup_k p_k=H$, $D=\max\{1,2^{H-1}\}$ then

$$|a_k+b_k|^{p_k}\leq D\left(|a_k|^{p_k}+|b_k|^{p_k}\right) \quad (20)$$

for all $k\in N$ and $a_k,b_k\in C$. Also $|a|^{p_k}\leq\max\{1,|a|^H\}$ for all $a\in C$ (see [25]).

The main object of the paper is to examine some topological properties and inclusion relations between the above defined sequence spaces.

2 Some Properties of the Sequence Space $E_n^q(B, \mathcal{M}, p, s, \|.,\ldots,.\|)$

Theorem 2.1: *Let* $\mathcal{M}=(M_k)$ *be a Musielak-Orlicz function and* $p=(p_k)$ *be a bounded sequence of positive real numbers, then the space* $E_n^q(B, \mathcal{M}, p, s, \|.,\ldots,.\|)$ *is linear over the real field.*

Proof: Let $x, y \in E_n^q(B, \mathcal{M}, p, s, \|.,\ldots,.\|)$ and $\alpha, \beta \in \Re$ (the set of real numbers). Then there exists numbers ρ_1 and ρ_2 such that

$$\sum_{k=1}^{\infty} \frac{b_{nk}}{k^s}\left[M_k\left(\left\|\frac{N_k(x)}{\rho_1}, z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]^{p_k} < \infty,$$

and

$$\sum_{k=1}^{\infty} \frac{b_{nk}}{k^s}\left[M_k\left(\left\|\frac{N_k(y)}{\rho_2}, z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]^{p_k} < \infty. \qquad (21)$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$.

Since $\mathcal{M}=(M_k)$ is non-decreasing, convex and so by using inequality (20), we have

$$\sum_{k=1}^{\infty} \frac{b_{nk}}{k^s}\left[M_k\left(\left\|\frac{N_k(\alpha x+\beta y)}{\rho_3}, z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]^{p_k}$$

$$\leq \sum_{k=1}^{\infty} \frac{b_{nk}}{k^s}\left[M_k\left(\left\|\frac{N_k(\alpha x)}{\rho_3}, z_1, z_2, \ldots, z_{n-1}\right\|\right) + \left(\left\|\frac{N_k(\beta y)}{\rho_3}, z, z, \ldots, z_{n-1}\right\|\right)\right]^{p_k} \qquad (22)$$

$$\leq D\sum_{k=1}^{\infty} \frac{b_{nk}}{k^s}\left[M_k\left(\left\|\frac{N_k(x)}{\rho_3}, z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]^{p_k} +$$

$$D\sum_{k=1}^{\infty} \frac{b_{nk}}{k^s}\left[M_k\left(\left\|\frac{N_k(y)}{\rho_3}, z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]^{p_k} < \infty.$$

Therefore, $\alpha x+\beta y\in E_n^q\left(B, \mathrm{M}, p, s, \|\cdot,\ldots,\cdot\|\right)$.

Hence, $E_n^q\left(B, \mathrm{M}, p, s, \|\cdot,\ldots,\cdot\|\right)$ is a linear space.

Theorem 2.2: *Let* $\mathrm{M}=(M_k)$ *be a sequence of Musielak-Orlicz functions,* $p=(p_k)$ *be a bounded sequence of positive real numbers. Then the space* $E_n^q\left(B, \mathrm{M}, p, s, \|\cdot,\ldots,\cdot\|\right)$ *is a paranormed space with the paranorm defined by*

$$g(x)=\inf\left\{\rho^{\frac{p_n}{H}}:\left(\sum_{k=1}^{\infty}\frac{b_{nk}}{k^s}\left[M_k\left(\left\|\frac{N_k(x)}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}}\leq 1,\ n=1,2,3,\ldots\right\}, \tag{23}$$

where $H=\max\left(1,\sup_k p_k\right)$.

Proof: It is clear that $g(x)=g(-x)$ and $g(x+y)\leq g(x)+g(y)$. Since $M_k(0)=0$, we get $\inf\left\{\rho^{p_n/H}\right\}=0$ for $x=0$. Finally, we prove that multiplication is continuous. Let $\lambda\neq 0$ be any complex number, then by definition, we have

$$g(\lambda x)=\inf\left\{\rho^{\frac{p_n}{H}}:\left(\sum_{k=1}^{\infty}\frac{b_{nk}}{k^s}\left[M_k\left(\left\|\frac{\lambda N_k(x)}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}}\leq 1,\ n=1,2,3,\ldots\right\}. \tag{24}$$

Thus, we have

$$g(\lambda x)=\inf\left\{(|\lambda| s)^{\frac{p_n}{H}}:\left(\sum_{k=1}^{\infty}\frac{b_{nk}}{k^s}\left[M_k\left(\left\|\frac{N_k(x)}{s},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}}\leq 1,\ n=1,2,3,\ldots\right\}, \tag{25}$$

where $s=\frac{\rho}{|\lambda|}$. Since $|\lambda|^{p_k}\leq\max\left(1,|\lambda|^H\right)$, we have

$$g(\lambda x)\le\left(\max\left(1,|\lambda|^{H}\right)\right)^{1/H}\inf\left\{(s)^{\frac{p_n}{H}}:\left(\sum_{k=1}^{\infty}\frac{b_{nk}}{k^s}\left[M_k\left(\left\|\frac{N_k(x)}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}}\le 1,\ n=1,2,3,\ldots\right\},$$

(26)

and therefore, $g(\lambda x)$ converges to zero when $g(x)$ converges to zero in $E_n^q(B,\mathrm{M},p,s,\|.,\ldots,.\|)$. Now, suppose that $\lambda_n\to 0$ as $n\to\infty$ and x is in $E_n^q(B,\mathrm{M},p,s,\|.,\ldots,.\|)$. For arbitrary $\varepsilon>0$, let n_0 be a positive integer such that

$$\sum_{k=n_0+1}^{\infty}\frac{b_{nk}}{k^s}\left[M_k\left(\left\|\frac{N_k(x)}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_k}<\frac{\varepsilon}{2}\tag{27}$$

for some $\rho>0$. This implies that

$$\left(\sum_{k=n_0+1}^{\infty}\frac{b_{nk}}{k^s}\left[M_k\left(\left\|\frac{N_k(x)}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{1/H}\le\frac{\varepsilon}{2}.\tag{28}$$

Let $0<|\lambda|<1$, then using convexity of (M_k), we get

$$\sum_{k=n_0+1}^{\infty}\frac{b_{nk}}{k^s}\left[M_k\left(\left\|\frac{\lambda N_k(x)}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_k}$$

$$<|\lambda|\sum_{k=n_0+1}^{\infty}\frac{b_{nk}}{k^s}\left[M_k\left(\left\|\frac{N_k(x)}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_k}<\left(\frac{\varepsilon}{2}\right)^{H}.\tag{29}$$

Since (M_k) is continuous everywhere on $[0,\infty)$, then

$$h(t)=\sum_{k=1}^{n_0}\frac{b_{nk}}{k^s}\left[M_k\left(\left\|\frac{tN_k(x)}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\tag{30}$$

is continuous at 0. So there is $0<\delta<1$ such that $|h(t)|<\varepsilon/2$ for $0<t<\delta$. Let K be such that $|\lambda_n|<\delta$ for $n>K$, we have

$$\left(\sum_{k=1}^{n_0}\frac{b_{nk}}{k^s}\left[M_k\left(\left\|\frac{\lambda_n N_k(x)}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{1/H}<\frac{\varepsilon}{2}.\tag{31}$$

Thus,

$$\left(\sum_{k=1}^{\infty}\frac{b_{nk}}{k^{s}}\left[M_{k}\left(\left\|\frac{\lambda_{n}N_{k}(x)}{\rho},z_{1},z_{2},\ldots,z_{n-1}\right\|\right)\right]^{p_{k}}\right)^{1/H}<\varepsilon,\quad for\, n>k. \tag{32}$$

Hence $g(\lambda x)\rightarrow 0$ *as* $\lambda\rightarrow 0$. This completes the proof of the theorem.

Theorem 2.3: *If* $\mathrm{M}'=(M'_{k})$ *and* $\mathrm{M}''=(M''_{k})$ *are two sequences of Musielak-Orlicz functions and* s, s_1, s_2 *are nonnegative real numbers, then*

(i) $E_{n}^{q}(B,\mathrm{M}',p,s,\|.,\ldots,.\|)\cap E_{n}^{q}(B,\mathrm{M}'',p,s,\|.,\ldots,.\|)$
$\subseteq E_{n}^{q}(B,\mathrm{M}'+M'',p,s,\|.,\ldots,.\|)$.
(ii) If $s_1\leq s_2$, then $E_{n}^{q}(B,\mathrm{M}',p,s_1,\|.,\ldots,.\|)\subseteq E_{n}^{q}(B,\mathrm{M}',p,s_2,\|.,\ldots,.\|)$.

Proof: It is obvious, so we omit the details.

Theorem 2.4: *Suppose that* $0<r_k\leq p_k<\infty$, *for each* $k\in N$. *Then*

$$E_{n}^{q}(B,M,r,s,\|.,\ldots,.\|)\subseteq E_{n}^{q}(B,M,p,s,\|.,\ldots,.\|).$$

Proof: Let $x\in E_{n}^{q}(B,\mathrm{M},r,s,\|.,\ldots,.\|)$. Then there exists some $\rho>0$ such that

$$\sum_{k=1}^{\infty}\frac{b_{nk}}{k^{s}}\left[M_{k}\left(\left\|\frac{N_{k}(x)}{\rho_{1}},z_{1},z_{2},\ldots,z_{n-1}\right\|\right)\right]^{r_{k}}<\infty. \tag{33}$$

this implies that , $M_{k}\left(\left\|N_{k}(x)/\rho,z_{1},z_{2},\ldots,z_{n-1}\right\|\right)\leq 1$ for sufficiently large value of k, say $k\geq k_0$, for some fixed $k_0\in N$. Since (M_k) is non decreasing, we get

$$\sum_{k\geq k_0}^{\infty}\frac{b_{nk}}{k^{s}}\left[M_{k}\left(\left\|\frac{N_{k}(x)}{\rho_{1}},z_{1},z_{2},\ldots,z_{n-1}\right\|\right)\right]^{p_{k}}\leq$$

$$\sum_{k\geq k_0}^{\infty}\frac{b_{nk}}{k^{s}}\left[M_{k}\left(\left\|\frac{N_{k}(x)}{\rho_{1}},z_{1},z_{2},\ldots,z_{n-1}\right\|\right)\right]^{r_{k}}<\infty. \tag{34}$$

Hence $x\in E_{n}^{q}(B,\mathrm{M},p,s,\|.,\ldots,.\|)$.

Theorem 2.5:

(i) *If* $0 < p_k \leq 1$ *for each* k, *then* $E_n^q\left(B, M, p, s, \|.,\ldots,.\|\right) \subseteq E_n^q\left(B, M, s, \|.,\ldots,.\|\right)$.

(ii) *If* $p_k \geq 1$ *for all* k, *then* $E_n^q\left(B, M, s, \|.,\ldots,.\|\right) \subseteq E_n^q\left(B, M, p, s, \|.,\ldots,.\|\right)$.

Proof: It is easy to prove by using Theorem 2.4, so we omit the details.

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