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A Parametric Study on Multi-Objective Integer Quadratic Programming Problems Under Uncertainty

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Abstract

This paper presents a parametric study on multi-objective integer quadratic programming problem under uncertainty. The proposed procedure presents a quadratic multi-objective integer programming problem with a stochastic parameters in the right hand sides, and all constraints occurs under certain probability. We consider all random variables are normally distributed. We shall be essentially concerned with three basic notions: the set of feasible parameters; the solvability set and the stability set of the first kind (SSK1). An algorithm to clarify the developed theory as well as an illustrative example are presented.

Keywords: *Quadratic programming, integer programming, stochastic programming, Stability optimization.*

1 Introduction

Deterministic optimization approaches have been well developed and widely used in the process industry to accomplish off-line and on-line process optimization.

The challenging task for the academic research is currently to address large-scale, complex optimization problems under various uncertainties. Therefore, investigations on the development of Chance-constrained optimization approaches are required. Chance-constrained optimization is an important method for managing risk arising from random variations in natural resource systems, but the probabilistic formulations often pose mathematical programming problems that cannot be solved with exact methods. A heuristic estimation method for these problems is presented that combines a formulation for order statistic observations with the sample average approximation method as a substitute for chance constraints ([5], [6], [11]). The effectiveness of evolutionary computation methodologies in the solution of multi-objective optimization problems has generated significant research interest in recent years. A number of evolutionary multiobjective optimization methodologies have been developed and are being continuously improved in order to achieve better performance.

These techniques have illustrated their competency against traditional multiobjective optimization techniques in the solution of this type of problems and are now considered to be a robust optimization tool in the hands of researchers and practitioners ([1], [3], [4], [7]). The methodological development of integer programming has grown by leaps and bounds in the past four decades, with its main focus on nonlinear integer programming. However, the past few years have also witnessed certain promising theoretical and methodological achievements in linear integer programming. These recent developments have produced applications of nonlinear (mixed) integer programming across a variety of various areas of scientific computing, engineering, management science and operations research. Its prominent applications include, for examples, portfolio selection, capital budgeting, production planning, resource allocation, computer networks, reliability networks and chemical engineering([1], [5], [10]). The aim of this paper is to solve a quadratic multi-objective integer programming problem with a stochastic parameters in the right hand sides, and all constraints occurs under certain probability, problem formulation and solution concept (section 2), a parametric study of problem (CHMOIQP) (section 3), utilization of Kuhn-Tucker necessary optimality conditions for $P_a'(\varepsilon)$ (section 4), an algorithm of determination of the set $T(x^*)$ (section 5).

2 Problem Formulation and Solution Concept

The chance-constrained multi-objective integer quadratic programming problem with random parameters in the right-hand side of the constraints can be stated as follows:

(CHMOIQP):
$$\begin{aligned} &\max F(x), \\ &\text{subject to} \\ &x \in X. \end{aligned} \tag{2.1}$$

Where

$$X = \left\{ x \in R^n \middle| P\left\{ g_i(x) \middle| \sum_{j=1}^{n} a_{ij} x_j \le b_i \right\} \ge \alpha_i, i = 1, 2, ..., m, x_j \ge 0 \text{ and integer}, j = 1, 2, ..., n \right\}. \qquad (2.2)$$

Here x is the vector of integer decision variables and F(x) is a vector of k-quadratic real-valued objective functions to be maximized. Furthermore, P means probability and α_i is a specified probability value. This means that the linear constraints may be violated some of the time and at most 100(1-α_i) % of the time. For the sake of simplicity, we assume that the random parameters b_i, (i =1, 2,…,m) are distributed normally with known means E{b_i} and variances Var {b_i} and independently of each other.

Definition 2.1. *A point* $x^* \in X$ *is said to be an efficient solution for problem (CHMOIQP) if there does not exist another* $x \in X$ *such that* $F(x) \ge F(x^*)$ *and* $F(x) \ne F(x^*)$ *with*

$$P\left\{ g_i(x^*) \middle| \sum_{j=1}^{n} a_{ij} x_j^* \le b_i \right\} \ge \alpha_i, (i = 1, 2, ..., m).$$

The basic idea in treating problem (CHMOIQP) is to convert the probabilistic nature of this problem into a deterministic form. Here, the idea of employing deterministic version will be illustrated by using the interesting technique of chance-constrained programming ([5], [8], [11]). In this case, the set of constraints X of problem (CHMOIQP) can be rewritten in the deterministic form as:

$$X' = \left\{ x \in R^n \middle| \sum_{j=1}^{n} a_{ij} x_j \le E\{b_i\} + K_{\alpha_i} \sqrt{Var\{b_i\}}, i = 1, 2, ..., m, x_j \ge 0 \text{ and integer}, j = 1, 2, ..., n \right\} \qquad (2.3)$$

Where K_{α_i} *is the standard normal value such that* $\Phi(K_{\alpha_i}) = 1 - \alpha_i$ *; and* $\Phi(a)$ *represents the "cumulative distribution function" of the standard normal distribution evaluated at a. Thus, problem (CHMOIQP) can be understood as the following deterministic version of a multi-objective integer quadratic programming (MOIQP) problem :*

(MOIQP): $\max\ [f_1(x), f_2(x), \ldots, f_k(x)]$, *(2.4)*
subject to
$x \in X'$.

In what follows, an equivalent multi-objective quadratic linear programming (MOQLP) problem associated with problem (2.1)-(2.2) can be stated with the help of cutting-plane technique ([1], [7], [10]) together with Balinski algorithm [2]. This equivalent MOQLP can be written in the following form:

$$\text{(MOIQP):} \quad \max\,[f_1(x), f_2(x), \ldots, f_k(x)], \quad \text{subject to} \quad x \in [X'], \tag{2.5}$$

where $[X']$ *is the convex hull of the feasible region* X' *defined by (2.3) earlier. This convex hull is defined by:*

$$[X'] = X'^{(s)}_R = \{x \in R^n \mid A^{(s)} x \le b^{(s)}, x \ge 0\} \tag{2.6}$$

and in addition,

$$A^{(s)} = \begin{bmatrix} A \\ . \\ a_1 \\ . \\ . \\ a_s \end{bmatrix} \quad \text{and} \quad b^{(s)} = \begin{bmatrix} b_1 \\ . \\ b_m \\ d_1 \\ . \\ d_s \end{bmatrix} \tag{2.7}$$

The two matrices are the original constraint matrix A and the right-hand side vector b , respectively, with s-additional constraints each corresponding to an efficient cut in the form $a_i x \le d_i$. *By an efficient cut, we mean that a cut which is not redundant.*

Now it can be observed, from the nature of problem (MOQLP) above, that a suitable scalarization technique for treating such problems is to use the ∈-constraint method. For this purpose, we consider the following integer nonlinear programming problem with a single-objective function as:

$$P_a(\varepsilon): \quad \max \quad f_a(x), \tag{2.8}$$

$$\text{subject to}$$

$$X(\varepsilon) = \left\{x \in R^n \mid f_r(x) \ge \varepsilon_r, r \in K - \{a\},\ x \in [X']\right\}$$

Where $a \in K = \{1, 2, \ldots, k\}$ *which can be taken arbitrary.*

It should be stated here that an efficient solution x for problem (CHMOIQP) can be found by solving the scalar problem* $P_a(\varepsilon)$ *and this can be done when the minimum allowable levels* $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{s-1}, \varepsilon_{s+1}, \ldots, \varepsilon_k)$ *for the (k-1) objectives* $(f_1, f_2, \ldots, f_{s-1}, f_{s+1}, \ldots, f_k)$ *are determined in the feasible region of solutions* $X(\varepsilon)$.

It is clear from [6] that a systematic variation of ε_i*'s will yield a set of efficient solutions. On the other hand, the resulting scalar problem* $P_a(\varepsilon)$ *can be solved easily at a certain parameter* $\varepsilon = \varepsilon^*$ *using the cutting plane technique. If* $x^* \in X(\varepsilon^*)$ *is a unique optimal integer solution of problem* $P_a(\varepsilon^*)$*, then x* becomes an efficient solution to problem (CHMOIQP) with probability* $\prod_{i=1}^{m} \alpha_i$.

3 A Parametric Study of Problem (CHMOIQP)

Now and before we go any further, we can rewrite problem $P_a(\varepsilon)$ in the following scalar relaxed subproblem which may occur in the cutting plane technique process as:

$$P_a^{'}(\varepsilon): \quad \max f_a(x), \quad \text{subject to} \quad x \in [X_s(\varepsilon)]. \tag{3.1}$$

Where

$$X_s(\varepsilon) = \left\{ \begin{aligned} x \in R^n \,\Big|\, & f_r(x) \geq \varepsilon_r, r \in K - \{a\}, \\ & g_i(x) = \sum_{j=1}^{n} a_{ij} x_j \leq C_i,\ i = 1, 2, ..., m, \\ & \sum_{j=1}^{n} a_{lj} x_j \leq d_l, l = 1,2,..., s, \\ & x_j \geq 0, (j = 1,2,..., n). \end{aligned} \right\}, \tag{3.2}$$

Where the s-additional constraints each corresponding to an efficient cut in the form $a_i x \leq d_i$. By an efficient cut, we mean that a cut which is not redundant of problem $P_a(\varepsilon)$ for obtaining its optimal integer solution x*.
In addition, it is supposed that:

$$C_i = E\{b_i\} + K_{\alpha_i} \sqrt{Var\{b_i\}}\ , (i = 1, 2,....m). \tag{3.3}$$

In what follows, definitions of some basic stability notions are given for the relaxed problem $P_a^{'}(\varepsilon)$ above. We shall be essentially concerned with three basic notions: the set of feasible parameters; the solvability set and the stability set of the first kind (SSK1). The qualitative and quantitative analysis of these notions have been introduced in details by ([4], [9]) for different classes of parametric optimization problems. Moreover, stability results for such problems have been derived.

Definition 3.1. *The set of feasible parameters of problem $P_s^{'}(\varepsilon)$, which is denoted by A, is defined by:*

$$A = \left\{ \varepsilon \in R^{k-1} \,\middle|\, X_s(\varepsilon) \neq \Phi \right\}.$$

Definition 3.2. *The solvability set of problem $P_s^{'}(\varepsilon)$, which is denoted by B, is defined by:*

$$B = \{\varepsilon \in A \mid \text{Problem } P_s(\varepsilon) \text{ has optimal integer solution}\}.$$

Definition 3.3. *Suppose that* $\varepsilon^* \in B$ *with a corresponding optimal integer solution x*, then the stability set of the first kind of problem* $P_s^{'}(\varepsilon)$ *corresponding to x*, which is denoted by S(x*), is defined by:*

$$S(x^*) = \{\varepsilon \in B \mid x^* \text{ remain optimal integer solution of problem } P_s^{'}(\varepsilon)\}.$$

4 Utilization of Kuhn-Tucker Necessary Optimality Conditions For $P_a^{'}(\varepsilon)$.

Now, given an optimal point x*, which may be found as described earlier in Section 2, the question is: For what values of the vector ε the Kuhn-Tucker necessary optimality conditions for the sub-problem $P_a^{'}(\varepsilon)$ are satisfied?
In the following, the Kuhn-Tucker necessary optimality conditions corresponding to problem $P_a^{'}(\varepsilon)$ will have the form:

$$\left.\begin{array}{lr}
\dfrac{\partial f_a(x)}{\partial x_j} + \sum\limits_{\substack{r=1\\ r\neq a}}^{k} \mu_r \dfrac{\partial f_r(x)}{\partial x_j} - \sum\limits_{i=1}^{m} \delta_i \dfrac{\partial g_i(x)}{\partial x_j} - \sum\limits_{l=1}^{s} u_l \dfrac{\partial t_l(x)}{\partial x_j} = 0, & (j=1,2,....,n) \\
f_r(x) \geq \varepsilon_r, & r \in K-\{a\}, \\
g_i(x) \leq C_i, & (i=1,2,....m), \\
t_l(x) \leq d_l, & (l=1,2,....s), \\
\mu_r[-f_r(x)+\varepsilon_r]=0, & r \in K-\{a\}, \\
\delta_i[g_i(x)-C_i]=0, & (i=1,2,....m), \\
u_l[t_l(x)-d_l]=0, & (l=1,2,...,s), \\
\mu_r \geq 0, & r \in K-\{a\}, \\
\delta_i \geq 0, & (i=1,2,....m), \\
u_l \geq 0, & l=\{1,2,...,s\}.
\end{array}\right\} \quad (4.1)$$

Where all the above relations of system (4.1) above are evaluated at the optimal integer solution x*. The variables μ_r, δ_i, u_j are the Langrangian multipliers.
The first and last three relations of the system (4.1) above represent a Polytope in $\mu\delta u$–space for which its vertices can be determined using any algorithm based upon the simplex method. According to whether any of the variables μ_r, $r\in K-\{a\}$, δ_i, (i=1,2,…m), u_l,(l=1,2,…,s) is zero or positive, then the set of parameters ε'a for which the Kuhn-Tucker necessary optimality conditions are utilized will be determined. This set is denoted by T(x*).

5 An Algorithm of Determination of the set T(x*)

In what follows, we propose an algorithm in series of steps to find the set of possible ε which will be denoted by T(x*). For the set T(x*), the point x* remains efficient for all values of the vector ε. Clearly,$T(x^*) \subseteq S(x^*)$.

Step 1. Determine the means E{b_i} and Var{b_i} (i =1, 2,…m).
Step 2. Convert the original set of constraints X of problem (CHMOIQP) into the equivalent set of constraints $X^{'}$.
Step 3. Formulate the deterministic multi-objective integer quadratic problem (MOIQP) corresponding to problem (CHMOIQP).
Step 4. (a) Use Balinski's algorithm to find all the vertices of the feasible region X′.
(b) Select one of the non-integer vertices $x^1 = \left(x_1^1, x_2^1, ..., x_n^1\right)$ of the solution space. In the tableau of this vertex, choose the row vector where the basic variable has the largest fractional value and construct its corresponding Gomory's fractional cut in the form $a_1 x \le c_1$.
(c) Add the first cut $a_1 x \le c_1$ to the original set of the constraints X. This will yield a new feasible region X^1.
(d) Repeat again the steps (a) → (c) until, at some step, r, the obtained vertices of the solution space all are integers.
(e) Eliminate (drop) all the redundant constraints of the applied cuts.
(f) Add all the constraints of applied s-efficient cuts to the original set of constraints X′ to get [X′].
Step 5. Formulate the equivalent linear fractional problem with the constraints [X′].
Step 6. Formulate the integer quadratic problem with a single-objective function $P_a(\varepsilon)$.
Step 7. Solve k-individual integer quadratic problem P_r, (r =1,2,…,k) where

$$P_r: \quad \max f_r(x), \quad (r=1,2,\ldots,k), \quad \text{subject to} \quad x \in [X^{'}],$$

to find the optimal integer solutions of the k-objectives.
Step 8. Construct the payoff table and determine n_r, M_r (the smallest and the largest numbers in the r^{th} column in the payoff table).
Step 9. Determine the ε_i's from the formula:

$$\varepsilon_r = n_r + \frac{t}{N-1}(M_r - n_r), r \in K - \{a\}$$

where t is the number of all partitions of the interval [n_r, M_r].
Step 10. Find the set $B = \left\{ \varepsilon \in R^{k-1} \middle| \; n_r \le \varepsilon_r \le M_r, r \in K - \{a\} \right\}$
Step 11. Choose $\varepsilon_r^* \in B$ and solve the problem $P_a(\varepsilon^*)$ to find its optimal integer solution x*.
Step 12. Determine the set T(x*) by utilizing the Kuhn-Tucker necessary optimality conditions (4.1) corresponding to problem $P_s^{'}(\varepsilon)$.
Step 13. If $T_1(x^*)$ is a one-point set, go to step 14. Otherwise, go to step 14.

Step 14. Define $T_1(x^*) = \{ \varepsilon \in R^{k-1} \mid \varepsilon_r^* - \Delta \le \varepsilon_r^* \le M_r, r \in K - \{a\} \}$, where Δ is any small Pre-specified positive real number.

Step 15. Determine $B - T_i(x^*)$. If $B - T_i(x^*) = \phi$, stop. Otherwise, go to step 16.

Step 16. Choose another $\varepsilon_r = \bar{\varepsilon}_r \in \Im - T_1(x^*)$ and go to step 11.

The above algorithm terminates when the range of B is fully exhausted.

6 An Illustrative Example

Here, we provide a numerical example to clarify the developed theory and the proposed algorithm. The problem under consideration is the following multi-objective integer quadratic programming problem involving random parameters in the right-hand side of the constraints (CHMOIQP).

(CHMOIQP): max $F(x) = [f_1(x), f_2(x)]$,

subject to

$$P\{x_1 + x_2 \le b_1\} \ge 0.90,$$
$$P\{-x_1 + 3x_2 \le b_2\} \ge 0.95,$$
$$P\{3x_1 + x_2 \le b_3\} \ge 0.90,$$

$x_1, x_2 \ge 0$ and integers.

where

$$f_1(x) = x_1^2 + x_2^2, \quad f_2(x) = x_1 + x_2 .$$

Suppose that b_i, ($i = 1, 2, 3$) are normally distributed random parameters with the following means and variances.

$E\{b_1\} = 1$, $E\{b_2\} = 3$, $E\{b_3\} = 9$,

$Var\{b_1\} = 25$, $Var\{b_2\} = 4$, $Var\{b_3\} = 4$.

From standard normal tables, we have:

$$K_{\alpha_1} = K_{\alpha_3} = K_{0.90} \cong 1.285, \; K_{\alpha_2} = K_{0.95} \cong 1.645$$

For the first constraint, the equivalent deterministic constraint is given by:

$$x_1 + x_2 \le C_1 = E\{b_1\} + K_{\alpha_1}\sqrt{Var\{b_1\}} = 1 + 1.285(5) = 7.425$$

For the second constraint:

$$-x_1 + 3x_2 \le C_2 = E\{b_2\} + K_{\alpha_2}\sqrt{Var\{b_2\}} = 3 + 1.645(2) = 6.29$$

For the third constraint:

$$3x_1 + x_2 \le C_3 = E\{b_3\} + K_{\alpha_3}\sqrt{Var\{b_3\}} = 9 + 1.285(2) = 11.57$$

Therefore, problem (CHMOIQP) can be understood as the corresponding deterministic bicriterion integer linear programming problem in the form:

(MOIQP): max [$f_1(x) = x_1^2 + x_2^2$, $f_2(x) = x_1 + x_2$],
subject to

$$x_1 + x_2 \le 7.425,$$
$$-x_1 + 3x_2 \le 6.29,$$
$$3x_1 + x_2 \le 11.57,$$
$$x_1, x_2 \ge 0 \text{ and integers.}$$

Now , (MOQLP) problem associated with (MOIQP) problem can be stated with the help of cutting-plane technique ([1], [7], [10]) together with Balinski algorithm [2] in the following form.

(MOQLP): max [$f_1(x) = x_1^2 + x_2^2$, $f_2(x) = x_1 + x_2$],
subject to

$$x_1 + x_2 \le 7.425,$$
$$-x_1 + 3x_2 \le 6.29,$$
$$3x_1 + x_2 \le 11.57,$$
$$x_1 \le 3,$$
$$x_2 \le 2,$$
$$x_1, x_2 \ge 0 .$$

Using the ε-constraint method [3], then problem above with a single-objective function becomes:

P_1 (ε): max $f_1(x) = x_1^2 + x_2^2$,
subject to

$$x_1 + x_2 \ge \varepsilon_2 ,$$
$$x_1 + x_2 \le 7.425,$$
$$-x_1 + 3x_2 \le 6.29,$$
$$3x_1 + x_2 \le 11.57$$
$$x_1, x_2 \ge 0 \text{ and integers.}$$

It can be shown easily that $2 \le \varepsilon_2 \le 7.425$.
Problem $P_1(\varepsilon)$ can be solved at $\varepsilon_2 = \varepsilon_2^* = 5$, $f_1(x^*) = 13$ and its optimal integer solution is found $(x_1^*, x_2^*) = (3, 2)$ with probability 0.7695.
Furthermore, problem $P_1(\varepsilon)$ can be rewritten in the following parametric form as:

$P_1'(\varepsilon)$: max $f_1(x) = x_1^2 + x_2^2$,
subject to

$$x_1 + x_2 \ge \varepsilon_2 ,$$
$$x_1 + x_2 \le 7.425,$$
$$-x_1 + 3x_2 \le 6.29,$$

$$\begin{aligned} 3x_1 + x_2 &\le 11.57, \\ x_1 \quad &\le 3, \\ x_2 &\le 2, \\ x_1, x_2 &\ge 0\ . \end{aligned}$$

Therefore, the Kuhn-Tucker necessary optimality conditions corresponding to problem $P_1'(\varepsilon)$ will take the following form:

$$\left.\begin{aligned} 2x_1 + \mu_1 - \delta_1 + \delta_2 - 3\delta_3 - u_1 &= 0, \\ 1 + \mu_1 - \delta_1 - 3\delta_2 - \delta_3 - u_2 &= 0, \\ x_1 + x_2 &\ge \varepsilon_2, \\ x_1 + x_2 &\le 7.425, \\ -x_1 + 3x_2 &\le 6.29, \\ 3x_1 + x_2 &\le 11.57, \\ x_1 &\le 3, \\ x_2 &\le 2, \\ \mu_1(-x_2 - x_2 + \varepsilon_2) &= 0, \\ \delta_1(x_1 + x_2 - 7.425) &= 0, \\ \delta_2(-x_1 + 3x_2 - 6.29) &= 0, \\ \delta_3(3x_2 + x_2 - 11.57) &= 0, \\ u_1(x_1 - 3) &= 0, \\ u_2(x_2 - 2) &= 0, \\ \mu_1, \delta_1, \delta_2, \delta_3, u_1, u_2 &\ge 0 \end{aligned}\right\} \quad (\#)$$

where all the above expressions of system (#) are evaluated at the optimal integer solution $(x_1^*, x_2^*) = (3, 2)$ with probability 0.7695 . In addition, it can be shown that

$\delta_1 = \delta_2 = \delta_3 = 0$, $u_1, u_2 > 0$, $\mu_1 \ge 0$

Therefore, the set $T_1(3, 2)$ is given by:

$T_1(3, 2) = \{\varepsilon \in R \mid 2 \le \varepsilon_2 \le 5\}$.

A systematic variation of $\varepsilon_2 \in R$ and $2.61 \le \varepsilon_2 \le 5$ will yield another stability set $T_2(3, 2)$, and so on.

7 Conclusions

The general purpose of this study was to investigate stability of the efficient solution for chance-constrained multi-objective integer quadratic programming problem. A parametric study has been carried out on the problem under consideration, where some basic stability notions have been defined and characterized for the formulated problem.

Many aspects and general questions remain to be studied and explored in the field of multi-objective integer optimization problems under randomness. This paper is an attempt to establish underlying results which hopefully will help others to answer some or all of these questions.

There are however several unsolved problems, in our opinion, to be studied in future. Some of these problems are:

(i) An algorithm is required for solving multi-objective integer quadratic programming problems involving random parameters in the left-hand side of the constraints,

(ii) An algorithm is needed for treating large-scale multi-objective integer quadratic nonlinear programming problems under randomness,

(iii) An algorithm should be handled for solving integer fractional goal programs involving random parameters.

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