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Weingarten and Linear Weingarten Canal Surfaces in Euclidean 3-Space

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Abstract

In this study, we investigated the (K,H),(K,K_{II}), (H,K_{II})-Weingarten and (K,H),(K,K_{II}),(H,K_{II}) and (K,H,K_{II})-linear Weingarten canal surfaces in IR^3.

Keywords: *Weingarten canal surfaces, linear Weingarten canal surfaces, Gauss curvature, mean curvature.*

1 Preliminaries

In 1863, Julius Weingarten was able to make a major step forward in the topic when he gave a class of surfaces isometric to a given surface of revolution. Surface for which there is a definite functional relation between the principal curvatures (which called curvature diagram) and also between the Gaussian and the mean curvatures is called Weingarten surface. The knowledge of first fundamental form I and second fundamental form II of a surface facilitates the analysis and the classification of surface shape. Especially recent years, the geometry of the second fundamental form II has become an important issue in terms of investigating intrinsic and extrinsic geometric properties of the surfaces. Very recent results concerning the curvature properties associated to II and other variational aspects can be found in [16]. One may associate to such a surface M geometrical objects measured by means of its second fundamental form, as second Gaussian curvature K_{II}, respectively. We are able to compute K_{II} of a surface by replacing the components of the first fundamental form E, F, G by the components of the second fundamental form e, f, g respectively in Brioschi formula which is given by Francesco Brioschi in the years of 1800's. Identification of the curvatures related to the second fundamental form of a surface opened a door to research the new classes of Weingarten surfaces. Since the middle of the last century, several geometers have studied Weingarten surfaces and linear Weingarten surfaces and obtained many interesting and valuable results [3, 4, 8, 9, 12, 13, 15]. For study of these surfaces, W. Kühnel and G.Stamou investigate ruled (X,Y)-Weingarten surface in Euclidean 3-space E^3 [12, 15]. Also,C.Baikoussis and Th. Koufogiorgos studied helicoidal (H,K_{II})-Weingarten surfaces [1]. F.Dillen and W. Kühnel and F.Dillen and W.Sodsiri gave a classification of ruled (X,Y)-Weingarten surface in Minkowski 3-space E_1^3, where X,Y$\in \{K, H, K_{II}\}$[3, 4, 5]. D. Koutroufiotis and Th.Koufogiorgos and T. Hasanis investigate closed ovaloid (X,Y)-linear Weingarten surface in E^3 [10, 11]. D. W. Yoon and D.E.Blair and Th.Koufogiorgos classified ruled (X,Y)-linear Weingarten surface in E^3[2, 20]. D.W. Yoon and J.S.Ro studied tubes in Euclidean 3-space which are (K,H),(K,K_{II}),(H,K_{II})-Weingarten and linear Weingarten tubes [14]. D. W. Yoon also studied the Weingarten and linear Weingarten types translation surfaces in Euclidean 3-space.

Surface theory has been a popular topic for many researchers in many aspects. Besides the using curves and surfaces, canal surfaces are the most popular in computer aided geometric design such that designing models of internal and external organs, preparing of terrain-infrastructures, constructing of blending surfaces, reconstructing of shape, robotic path planning, etc. (see, [6, 17, 18]).

In this study, we investigated the (K,H), (K,K_{II}), (H,K_{II})-Weingarten and (K,H), (K,K_{II}), (H,K_{II}) and $(K, H, K_{II})-$ linear Weingarten canal surfaces in IR^3 by using the definition of general canal surfaces. During the study, we faced a very large equations. It was not possible to give them all of course. So we had to make our processes via a computer time to time.

Let f and g be smooth functions on a surface M in Euclidean 3-space E^3.The Jacobi function $\Phi(f, g)$ formed with f, g is defined by

$$\Phi(f, g) = f_s g_t - f_t g_s$$

where $f_s = \frac{\partial f}{\partial s}$ and $f_t = \frac{\partial f}{\partial t}$. In particular,a surface satisfying the Jacobi equation Φ(K,H) $= 0$ with respect to the Gaussian curvature K and the mean curvature H on a surface M is called a Weingarten surface. Also, if a surface satisfies a linear equation with respect to K and H, that is, aK+bH=c $(a, b, c \in IR, (a, b, c) \neq (0, 0, 0))$, then it is said to be a linear Weingarten surface [14].

When the constant b=0, a linear Weingarten surface M reduces to a surface with constant Gaussian curvature. When the constant a=0, a linear Weingarten surface M reduces to a surface with constant mean curvature. In such a sense, the linear Weingarten surfaces can be regarded as a natural generalization of surfaces with constant Gaussian curvature or with constant mean curvature [14].

If the second fundamental form of a surface M in E^3 is non-degenerate, then it is regarded as a new pseudo-Riemannian metric. Therefore, the Gaussian curvature K_{II} of non-degenerate second fundamental form can be defined formally on the Riemannian or pseudo-Riemannian manifold (M,II).We call the curvature K_{II} the second Gaussian curvature on M [14].

Following the Jacobi equation and the linear equation with respect to the Gaussian curvature K, the mean curvature H and the second Gaussian curvature K_{II} an interesting geometric question is raised. Classify all surfaces in Euclidean 3-space satisfying the conditions

$$\Phi(X,Y)=0$$

$$aX+bY=c$$

where $(X,Y)\in\{K,H,K_{II}\}$, X≠Y and $(a,b,c)\neq(0,0,0)$. Let M be a surface immersed in Euclidean 3-space, the first fundamental form of the surface M is defined by

$$I=Edu^2+2Fdudv+Gdv^2$$

where $E=<M_s,M_s>$, $F=<M_s,M_t>$, $G=<M_t,M_t>$ are the coefficients of I. A surface is called degenerate if it has the degenerate first fundamental form. The second fundamental form of M is given by

$$II=edu^2+2fdudv+gdv^2$$

where $e=\langle M_{ss},n\rangle$, $f=\langle M_{st},n\rangle$, $g=\langle M_{tt},n\rangle$ and n is the unit normal of M. The Gaussian curvature K and the mean curvature H are given by, respectively

$$K=\frac{eg-f^2}{EG-F^2}, \tag{1}$$

$$H=\frac{Eg-2Ff+Ge}{2(EG-F^2)}. \tag{2}$$

A regular surface is flat if and only if its Gaussian curvature vanishes identically. A minimal surface in IR^3 is a regular surface for which mean curvature vanishes identically [7].

Furthermore, the second Gaussian curvature K_{II} of a surface is defined by

$$K_{II} \quad = \quad \frac{1}{(|eg|-f^2)^2}\{p-q\}. \tag{3}$$

where

$$p = \begin{vmatrix} -\frac{1}{2}e_{tt} + f_{st} - \frac{1}{2}g_{ss} & \frac{1}{2}e_s & f_s - \frac{1}{2}e_t \\ f_t - \frac{1}{2}g_s & e & f \\ \frac{1}{2}g_t & f & g \end{vmatrix}$$

and

$$q = \begin{vmatrix} 0 & \frac{1}{2}e_t & \frac{1}{2}g_s \\ \frac{1}{2}e_t & e & f \\ \frac{1}{2}g_s & f & g \end{vmatrix}$$

A surface is called II-flat if the second Gaussian curvature vanishes identically [19]. Having in mind the usual technique for computing the second mean curvature by using the normal variation of the area functional one gets

$$H_{II} = H - \frac{1}{2\sqrt{|\det(II)|}} \sum_{i,j} \frac{\partial}{\partial u^i} \left(\sqrt{|\det(II)|} L^{ij} \frac{\partial}{\partial u^j} \left(\ln \sqrt{|K|} \right) \right) \tag{4}$$

where u^i and u^j stand for "s" and "t", respectively, and $(L^{ij}) = (L_{ij})^{-1}$, where L_{ij} are the coefficients of second fundamental forms[16]. A surface is called II-minimal if the second mean curvature vanishes identically[19].

A canal surface is an envelope of a 1-parameter family of surface. The envelope of a 1-parameter family $s \longrightarrow S^2(s)$ of spheres in $\mathbb{R}^3$ is called a *canal surface*[7]. The curve formed by the centers of the spheres is called *center curve* of the canal surface. The radius of canal surface is the function r such that $r(s)$ is the radius of the sphere $S^2(s)$. Suppose that the center curve of a canal surface is a unit speed curve $\alpha : I \to \mathbb{R}^3$. Then the canal surface can be parametrized by the formula

$$C(s,t) = \alpha(s) - R(s)T - Q(s)\cos(t)N + Q(s)\sin(t)B \tag{5}$$

where

$$R(s) = r(s)r'(s)Q(s) = \pm r(s)\sqrt{1 - r'(s)^2}. \tag{6}$$

All the tubes and the surfaces of revolution are subclass of the canal surface.

Theorem 1.1 *The center curve of a canal surface M is a straight line if and only if M is a surface of revolution for which no normal line to the surface is parallel o the axis of revolution [7].*

Theorem 1.2 *The following conditions are equivalent for a canal surface M: i. M is a tube parametrized by (5); ii. the radius of M is constant; iii. the radius vector of each sphere in family that defines the canal surface M meets the center curve orthogonally [7].*

Coefficients of first and second fundamental forms of canal surface are

$$E(s,t) = Q^2\kappa^2\cos^2(t) + p_1\kappa\cos(t) + 2QR\kappa\tau\sin(t) + p_2 \quad (7)$$

$$F(s,t) = -Q(R\kappa\sin(t) + Q\tau) \quad (8)$$

$$G(s,t) = Q^2 \quad (9)$$

and

$$e(s,t) = \frac{-1}{r(s)}\{E - Q\kappa\cos(t) - p_5\} \quad (10)$$

$$f(s,t) = \frac{-1}{r(s)}F(s,t) \quad (11)$$

$$g(s,t) = \frac{-1}{r(s)}G(s,t) \quad (12)$$

Let us take $\psi(s,t) = \det I$ and $\phi(s,t) = \det II$. Thus, we have

$$\phi(s,t) = \frac{1}{r^2}\left\{\psi(s,t) - Q^3\kappa\cos(t) - Q^2p_5\right\} \quad (13)$$

$$\psi(s,t) = Q^2\left\{\begin{array}{c}\kappa^2(R^2+Q^2)\cos^2(t) + \kappa p_1\cos(t) \\ +1 - 2R' + (R')^2 + (Q')^2\end{array}\right\}. \quad (14)$$

and

$$p_1 = 2(Q - QR' + Q'R) \quad (15)$$

$$p_2 = Q^2\tau^2 + R^2\kappa^2 + (R')^2 + (Q')^2 - 2R' + 1 \quad (16)$$

$$p_3 = p_1 - Q \quad (17)$$

$$p_4 = p_2 - p_5 \quad (18)$$

$$p_5 = (R')^2 + (Q')^2 - 2R' + 1 + RR'' + QQ'' \quad (19)$$

If $Q(s) = 0$, then the first and the second fundamental forms are degenerate. So the canal surface is degenerate surface and the radius is $r(s) = \pm s + c$. Furthermore, in the case $\kappa(s) = 0$ and $1 - 2R' + (R')^2 + (Q')^2 = 0$, the radius is

$$r(s) = \sqrt{s^2 - 2c_1s + 2c_2}.$$

Let the center curve be $\alpha(s) = (s,0,0)$. Then $T = e_1$, $N = e_2$ and $B = e_3$. Hence, $R(s) = s - c_1$ and $C(s,t)$ is the curve in the plane $x = c_1$. The conditions that $r(s) \neq \pm s + c$ and $\left(\kappa(s) = 0, r(s) \neq \sqrt{s^2 - 2c_1s + 2c_2}\right)$ are the necessary conditions to define a non-degenerate canal surface with the equation (6). At this point, we can write the following theorem.

Theorem 1.3 *Let M be a canal surface with the center curve $\alpha(s)$ and the radius $r(s)$. If the center curve is a line then M is a regular surface in IR^3 iff the radius is $r(s) \neq \pm s + c$ and $r(s) \neq \sqrt{s^2 - 2c_1s + 2c_2}$.*

Additionally, if $\phi(s,t) = 0$ then M has degenerate second fundamental form. A canal surface has degenerate second fundamental form if canal surface is a surface of revolution with the radious $r(s) = \sqrt{s^2 - 2c_1 s + 2c_2}$ or $r(s) = c_1 s + c_2$. From (1), (2) and (3), we obtained the Gauss curvature, mean curvature such that

$$K(s,t) = \frac{-1}{\psi(s,t)r^2}\left\{Q^3\kappa\cos(t) + Q^2 p_5 - \psi(s,t)\right\} \tag{20}$$

$$H(s,t) = \frac{1}{2\psi(s,t)r^2}\left\{Q^3\kappa\cos(t) + Q^2 p_5 - 2\psi(s,t)\right\}. \tag{21}$$

2 Weingarten Type Canal Surfaces

Let M be a canal surface with the center curve $\alpha(s)$ and the radius $r(s)$. The existence of a Weingarten relation Φ(H,K) = 0 means that curvatures H and K are functionally related, and since H and K are differentiable functions depending on s and t, this implies the Jacobian condition Φ(H,K)=0. More precisely the following condition

$$H_t K_s - H_s K_t = 0 \tag{22}$$

needs to be satisfied. By using equations (20) and (21) we get

$$H_t K_s - H_s K_t = \frac{1}{2\psi^3 r^4}\sum_{i=0}^{2} h_i \cos^i(t) \tag{23}$$

where

$$\begin{aligned}
h_2 &= -Q^6\kappa^2\psi_t r' \\
h_1 &= -Q^2\left\{\begin{array}{c} Q^4\kappa^2\psi r'\sin(t) + 3Q\kappa\psi^2\psi_t r' - Q\kappa'\psi^2\psi_t r - 3Q'\kappa\psi^2\psi_t r \\ +Q\kappa\psi\theta_t r' - 2Q\kappa\psi_t\theta r' \end{array}\right\} \\
h_0 &= \psi\theta_t\left\{r\psi\psi_s - r'(\psi^2-\theta)\right\} + \psi_t\left\{3\theta r'(\psi^2 - \frac{1}{3}\theta) - r\psi^2\theta_s\right\} \\
&\quad +Q^3\psi\kappa\left\{r\psi\psi_s - r'(\psi^2-\theta)\right\}\sin(t).
\end{aligned}$$

and $\theta = \psi - Q^2 p_5$. The Jacobian condition requires $h_0 = h_1 = h_2 = 0$. From $h_2 = 0$, the cases $\kappa = 0, \psi_t = 0, r' = 0$ are possible. If $\kappa = 0$ then, $\psi = Q^2 p_2$ and $h_0 = h_1 = 0$ satisfies. If $\psi_t = 0$ then, from (14) $\kappa = 0$. If $r = c$ then, from $h_1 = 0$ we have

$$c^3\psi_t\kappa' = 0$$

In this case, if $\psi_t = 0$ then, from (14) $\kappa = 0$ and so (20) satisfy. If $\kappa = a$ =constant then, $h_1 = h_2 = 0$ and $h_0 = 0$ is

$$\psi_s(ac^3 \sin(t) + \theta_t) - \psi_t\theta_s = 0.$$

Since the relation $ac^3 \sin(t) + \theta_t \neq 0$ and $\psi_t \neq 0$ then, $\psi_s = 0$ and $\psi_t\theta_s = 0$. Also from (17), $p_1 = 2c$, $p_5 = 1$ and $p_2 = c^2\tau^2 + 1$. Thus $\psi_s = \theta_s = 0$ satisfy. Hence we proved the following theorem.

Theorem 2.1 *Let M be a regular canal surface then, M is a (H,K)-Weingarten canal surface if M is one of the surfaces that surface of revolution, cyclinder and tubular surface whose centered curve with non-zero constant curvature.*

Thus, there are three cases for (5) such as $(r' \neq 0, \kappa = 0)$, $(r' = 0, \kappa \neq 0)$ and $(r' = 0, \kappa = 0)$, If $\kappa = 0$ then, all of the coefficients h_i in (23) are zero so the condition (22) is satisfy. If $r' = 0$ and $\kappa \neq 0$ then $r(s) = c \neq 0$ and (5) turns to a tubular suface such that

$$C(s,t) = \alpha(s) \mp c\cos(t)\,N \pm c\sin(t)\,B. \tag{24}$$

If $\kappa = 0$ then let assume that the center curve is the $x-$axis (5) turns to a surface of revolution and a cylinder such that

$$C(s,t) = \left(s - r(s)r'(s), \mp r(s)\sqrt{1 - r'(s)^2}\cos(t), \pm r(s)\sqrt{1 - r'(s)^2}\sin(t)\right) \tag{25}$$

and

$$C(s,t) = (s, \mp c\cos(t), \pm c\sin(t)) \tag{26}$$

respectively.

From (3), we can write the term p as

$$p = \left(-\frac{1}{2}e_{tt} + f_{st} - \frac{1}{2}g_{ss}\right)\phi + \left(f_t - \frac{1}{2}g_s\right)\left\{\left(f_s - \frac{1}{2}e_t\right)f - \frac{1}{2}e_s g\right\} \tag{27}$$

by taking $g_t = 0$. From (21), (27) and with the aid of prog 2, the Jacobi function $\Phi(\mathrm{H},\mathrm{K}_{II})$ is obtained a polynomial expressions in $\cos(t)$ such that

$$H_t\,(K_{II})_s - H_s\,(K_{II})_t = \frac{1}{denom}\sum_{i=0}^{6} g_i \cos^i(t). \tag{28}$$

For n=6 in Prog.2, g_6 is

$$g_6 = -Q^{13}\kappa^5 r^4 \psi_t \{4Q\kappa r' - 3Q\kappa' r - 5Q'\kappa r\}.$$

The Jacobian condition $\Phi(\mathrm{H},\mathrm{K}_{II})=0$ requires $g_0 = g_1 = ...g_6 = 0$. In the case $\kappa = 0$, $g_1 = ...g_6 = 0$ satisfies. By using prog.1 for $\kappa = \tau = 0$, it is easy to see that $g_0 = 0$ satisfy. If $\psi_t = 0$ then, from (8) we obtain $\kappa = 0$ so Φ(H,K$_{II}$)=0 satisfy also. If $4Q\kappa r' - 3Q\kappa' r - 5Q'\kappa r = 0$ then, from (6) and (6), we obtain

$$5\kappa r r' r'' + ((r')^2 - 1)(3r\kappa' + r'\kappa) = 0. \tag{29}$$

It may be hard to solve (29), but ofcourse we consider the special solutions of (29). If r is constant then from (29) κ is non-zero constant. If $r = c_1 s + c_2$ then, (29) turn to

$$((c_1)^2 - 1)(3r\kappa' + c_1\kappa) = 0$$

and the solutions are $c_1 = \pm 1$(M is not regular) or

$$\kappa = \frac{c_3}{r^{1/3}}.$$

If $\kappa = c_1$ is non-zero constant in (29) then, (29) turn to

$$rr'(5rr'' + (r')^2 - 1) = 0$$

and the real solution is r =constant$\neq \pm 1$. Thus we can give the following theorem.

Theorem 2.2 *Let M be a regular (H,K$_{II}$)-Weingarten canal surface then followings are ture for M.*

i.M is the surface of revolution,

ii.M is a canal surface with $r = c_1 s + c_2$,($c_1 \neq \pm 1$) and with the centered curve whose curvature is $\kappa = \frac{c_3}{r^{1/3}}$,

iii.M is a tubular surface whose centered curve with non-zero constant curvature.

Jacobi function Φ(K,K$_{II}$) is obtained a polynomial expressions in $\cos(t)$ by using (20) and (27) as follows.

$$K_t\,(K_{II})_s - K_s\,(K_{II})_t = \frac{1}{denom}\sum_{i=0}^{6} f_i \cos^i(t) \tag{30}$$

Jacobian condition Φ(K,K$_{II}$)=0 requires $f_0 = f_1 = ...f_6 = 0$.For $i = 6$ in Prog.3, f_6 is the same as g_6 in (28).

$$f_6 = Q^{13}\kappa^5 r^4 \psi_t \left\{4Q\kappa r' - 3Q\kappa' r - 5Q'\kappa r\right\}.$$

If $\kappa = 0$ then, Φ(K,K$_{II}$)=0 satisfy. If $\psi_t = 0$ then, from (14) $\kappa = 0$ and so Φ(K,K$_{II}$)=0 satisfy. If $4Q\kappa r' - 3Q\kappa' r - 5Q'\kappa r = 0$ then, we obtain the same differential equation (29) and we consider the special solution again. If r is constant then from (29) κ is non-zero constant. In this case, using Prog.1 and Prog.3, we obtain $f_6 = f_5 = f_4 = 0$ and $f_3 = -3\kappa^5 r^2 \tau' \sin^2(t)$, $f_2 = -6\kappa^4 r\tau' \sin^2(t)$, $f_1 = \kappa^3(4\kappa^2 r^2 - 1)\tau' \sin^2(t)$, $f_0 = 2\kappa^4 r\tau' \sin^2(t)$. The Jacobien condition reguires that τ is constant. If $r = c_1 s + c_2$ then, $\kappa = \frac{c_3}{r^{1/3}}$ and also, Φ(K,K$_{II}$)=0 satisfy. Thus, we can write the following theorem.

Theorem 2.3 *Let M be a regular (K, K_{II})-Weingarten canal surface then followings are ture for M.*

i.M is the surface of revolution,

ii.M is a canal surface with $r = c_1 s + c_2$,$(c_1 \neq \pm 1)$ and with the centered curve whose curvature is $\kappa = \frac{c_3}{r^{1/3}}$,

iii.M is a tubular surface whose centered curve is cylindirical helix.

3 Linear Weingarten Type Canal Surfaces

Let M be a canal surface with the center curve $\alpha(s)$ and the radius $r(s)$ then M is called (K, H) −linear Weingarten surface if Gaussian and the mean curvatures of M satisfies a linear equation with the constants a, b and d such that

$$aK + bH = d.$$

By using equations (20) and (21), we get the relation between K(s, t) and H(s, t) such as

$$H(s,t) + \frac{1}{2}K(s,t) = -\frac{1}{2r^2}. \tag{31}$$

Thus we have the following theorem.

Theorem 3.1 *Let M be a regular canal surface then M is $(K, H)-$ linear Weingarten surface if and only if M is a tubular surface.*

From (4),we can write

$$denom(\delta)\sqrt{\mu_1\phi}H_{II} - denom(\delta)\sqrt{\mu_1\phi}H = \frac{numer(\delta)}{2} \tag{32}$$

where $\delta = \delta_1 + \delta_2$,

$$\begin{aligned}
\delta_1 &= \frac{\partial}{\partial s}\left(\sqrt{\mu_1\phi}L^{11}\frac{\partial}{\partial s}\left(\ln\sqrt{\mu_2 K}\right) + \sqrt{\mu_1\phi}L^{12}\frac{\partial}{\partial t}\left(\ln\sqrt{\mu_2 K}\right)\right)\\
\delta_2 &= \frac{\partial}{\partial t}\left(\sqrt{\mu_1\phi}L^{21}\frac{\partial}{\partial s}\left(\ln\sqrt{\mu_2 K}\right) + \sqrt{\mu_1\phi}L^{22}\frac{\partial}{\partial t}\left(\ln\sqrt{\mu_2 K}\right)\right),
\end{aligned}$$

$$\mu_1 = \begin{cases} 1 & ; if\phi > 0 \\ -1 & ; if\phi < 0 \end{cases} \quad and\mu_2 = \begin{cases} 1 & ; if K > 0 \\ -1 & ; if K < 0 \end{cases}$$

and with the aid of Prog.4

$$denom(\delta) = 4\phi^2\psi^2 r^2\sqrt{\mu_1\phi}.$$

Thus, (32) turn to

$$4\phi^3\psi^2 r^2 H_{II} - 4\phi^3\psi^2 r^2 H = \frac{numer(\delta)}{2} \tag{33}$$

In this case, from (33), if $\phi^3\psi^2 r^2$ and numer(δ) are constant then, we can say that there is a linear relation between H_{II} and H. By using (13), (14) and Prog.5

$$(r^2\phi^3\psi^2)_t = \frac{1}{r^4}\sin(x)\sum_{i=0}^{9} m_i \cos^i(x) \tag{34}$$

for $i = 9$

$$m_9 = -10Q^{10}\kappa^{10}(Q^2 + R^2)^5$$

from $m_9 = 0$ then $\kappa = 0$ and also all of m_i are zero. In the case of $\kappa = 0$, $r^2\phi^3\psi^2$ is

$$r^2\phi^3\psi^2 = \frac{1}{r^4}Q^{10}(p_2 - p_5)^3(p_2)^2. \tag{35}$$

By using prog.5 and 6, the real non-zero solutions of $(r^2\phi^3\psi^2)_s = 0$ are $r = \pm\sqrt{s^2 - 2c_1 s + 2c_2}$ and $r = c_1 s + c_2$. For first r, M is degenerate, and for the second r, $\phi = 0$. Thus we have the following theorem.

Theorem 3.2 *Let M be a regular canal surface then, there is no (H,H_{II})-linear Weingarten surface in IR^3.*

From (1),

$$\phi(s,t) = \psi(s,t)\,K(s,t),$$

and by using (13), (20) and (21), we get

$$\phi^2 K_{II} - A\psi K = (B - q) \tag{36}$$

where

$$A = \left(-\frac{1}{2}e_{tt} + f_{st} - \frac{1}{2}g_{ss}\right)$$

and

$$B = \left(f_t - \frac{1}{2}g_s\right)\left\{\left(f_s - \frac{1}{2}e_t\right)f - \frac{1}{2}e_s g\right\}.$$

If ϕ, $r^4 A\psi$ and $r^4(B - q)$ are non-zero constans then, we called M is (K_{II},K)-linear Weingarten canal surface.

$$\frac{\partial\phi}{\partial t} = 0$$

$$\frac{\kappa Q^2}{r^4}\left\{-2\kappa\left(R^2 + Q^2\right)\cos(t) - (p_1 - Q)\right\}\sin(t) = 0$$

Thus, $\kappa = 0$ and for $\kappa = 0$,

$$\phi = \frac{-Q^2}{r^4}\left\{RR'' + QQ''\right\}$$

and by using (1) and (2), we obtain

$$\frac{-Q^2}{r^4}\{RR''+QQ''\}=rr''((r')^2+rr''-1)$$

since $\frac{\partial\phi}{\partial s}=0$ then, $\frac{-Q^2}{r^4}\{RR''+QQ''\}=c=$constant so we can write

$$rr''((r')^2+rr''-1)=c$$

There are the only three real non-zero solution of last equation for $c=0$ such that, $r=\pm\sqrt{s^2-2c_1s+2c_2}$ and $r=c_1s+c_2$ but in the case of $r=\pm\sqrt{s^2-2c_1s+2c_2}$ and $r=c_1s+c_2$, M have degenerate first and second fundamental forms. Thus, we can give the following theorem.

Theorem 3.3 *Let M be a regular canal surface then, there is no (K,K_{II})-linear Weingarten surface in IR^3.*

By substituting H in the equation (31) into (36), we get

$$8\phi^3\psi^2r^2H_{II}+4\phi^3\psi^2r^2K=numer(\delta)-\frac{4\phi^3\psi^2r^2}{r^2} \tag{37}$$

and also, we found before that $\phi=0$ when $\phi^3\psi^2r^2$ is a constant then, we have the following theorem.

Theorem 3.4 *Let M be a regular canal surface then, there is no (K,H_{II})-linear Weingarten surface in IR^3.*

Similarly, by substituting K in the equation (31) into (36), we get

$$\phi^2r^2K_{II}+2r^2A\psi H=r^2(B-q)-A\psi. \tag{38}$$

Also, we found before that M have degenerate first and second fundamental forms when $\{\psi(s,t)-Q^3\kappa\cos(t)-Q^2p_5\}$ is a constant then, we have the following theorem.

Theorem 3.5 *Let M be a regular canal surface then, there is no (H,K_{II})-linear Weingarten surface in IR^3.*

We can find the relation between K_{II} and H_{II} by using (36) and (37) as follow

$$8\phi^3\psi^2r^2H_{II}+\frac{4\phi^3\psi\{\phi r^2\}^2}{r^2A}K_{II}=numer(\delta)+\Gamma \tag{39}$$

where $\Gamma=\frac{4\phi^3\psi(r^2(B-q)-A\psi)}{A}$. Since $\phi=0$ when $\phi^3\psi^2r^2$ is a constant then, we give the following theorem.

Theorem 3.6 *Let M be a regular canal surface then, there is no (K_{II},H_{II})-linear Weingarten surface in IR^3.*

It easy to obtain the relations of trible of $\{H, K, H_{II}, K_{II}\}$. From (31) and (33),

$$2(1-4\gamma_1)H + K + 8\gamma_1 H_{II} = \frac{r^2 numer(\delta) - 1}{r^2}, \quad (40)$$

from (31) and (33),

$$2H + (1-2r^4 A\psi)K + 2(\gamma_2)^2 K_{II} = \frac{2r^6(B-q) - 1}{r^2}, \quad (41)$$

from (36) and (37),

$$\psi r^2(4\phi^3\psi - r^2 A)K + 8\gamma_1 H_{II} + (\gamma_2)^2 K_{II} = numer(\delta) + r^4(B-q) - 4\phi^3\psi^2, \quad (42)$$

from (38) and (39),

$$2r^4 A\psi H + 8\gamma_1 H_{II} + \frac{(4\phi^3\psi + r^2 A)(\gamma_2)^2}{r^2 A} K_{II} = numer(\delta) + \gamma_3 \quad (43)$$

where

$$\begin{aligned} \gamma_1 &= \phi^3\psi^2 r^2 \\ \gamma_2 &= \phi r^2 \\ \gamma_3 &= \frac{(4\phi^3\psi + Ar^2)(r^2(B-q) - A\psi)}{A}. \end{aligned}$$

In (40), (41), (42) and (43), the conditions γ_1 and γ_2 are constants requires the first and the second fundamental forms are degenerate. Thus, we have the following theorem.

Theorem 3.7 *Let M be a regular canal surface then, there is no (H,K,H_{II}), (H,K,K_{II}), (K,K_{II},H_{II}) and (H,K_{II},H_{II})-linear Weingarten surfaces in IR^3.*

Finally, from (40) and (41) we get

$$4(1-2\gamma_1))H + 2(1-r^4 A\psi)K + 8\gamma_1 H_{II} + 2(\gamma_2)^2 K_{II} = \frac{r^2 numer(\delta) + 2r^6(B-q) - 2}{r^2}$$

and the condition γ_1 is constant requires the second fundamental form is degenerate. Thus, we have the following theorem.

Theorem 3.8 *Let M be a regular canal surface then, there is no (H,K,H_{II},K_{II})-linear Weingarten surfaces in IR^3.*

Prog.1

```
R(s):=r(s)*diff(r(s),s):
Q(s):=r(s)*((1-diff(r(s),s)^2)^(1/2)):
G(s):=(Q(s))^2:
p1(s):=2*Q(s)+2*R(s)*diff(Q(s),s)-2*Q(s)*diff(R(s),s):
p2(s):=(Q(s)^2)*(tau(s)^2)+(R(s)^2)*(kappa(s)^2)+diff(R(s),s)^2+diff(Q(s),s)^2-
2*diff(R(s),s)+1;
p3(s):=p1(s)-Q(s):
p5(s):=diff(R(s),s)^2+diff(Q(s),s)^2-2*diff(R(s),s)+1+R(s)*diff(diff(R(s),s),s)
+Q(s)*diff(diff(Q(s),s),s);
p4(s):=p2(s)-p5(s):
E(s,t):=(Q(s)^2)*(kappa(s)^2)*(cos(t))^2+p1(s)*kappa(s)*cos(t)
+2*Q(s)*R(s)*kappa(s)*tau(s)*sin(t)+p2(s);
F(s,t):=-Q(s)*R(s)*kappa(s)*sin(t)-G(s)*tau(s):
psi(s,t):=E(s,t)*G(s)-(F(s,t))^2:
theta(s,t):=-(Q(s)^2)*p5(s)+psi(s,t):
```

Prog.2

```
G(s):=(Q(s))^2;
e:=(-1/r(s))*(E(s,t)-Q(s)*kappa(s)*cos(t)-p5(s)):
f:=(-1/r(s))*F(s,t):
g:=(-1/r(s))*G(s):
es:=diff(e,s):
fs:=diff(f,s):
gs:=diff(g,s):
ess:=diff(diff(e,s),s):
fss:=diff(diff(f,s),s):
gss:=diff(diff(g,s),s):
et:=diff(e,t):
ft:=diff(f,t):
gt:=diff(g,t):
ett:=diff(diff(e,t),t):
ftt:=diff(diff(f,t),t):
gtt:=diff(diff(g,t),t):
est:=diff(diff(e,s),t):
fst:=diff(diff(f,s),t):
gst:=diff(diff(g,s),t):
ets:=diff(diff(e,t),s):
fts:=diff(diff(f,t),s):
gts:=diff(diff(g,t),s):
phi(s,t):=(1/(r(s))^2)*(psi(s,t)-(Q(s)^3)*kappa(s)*cos(t)
-(Q(s)^2)*p5(s)):
```

```
V1:=Matrix([[phi(s,t)*((-ett/2)+fst-(gss/2)),0],[0,phi(s,t)*((-ett/2)+fst-(gss/2))]])
+Matrix([[ft-(gs/2),((fs-(et/2))*e-(f*es/2))],[(gt/2),(fs-(et/2))*f-(g*es/2)]]):
V2:=Matrix([[0,et/2,gs/2],[et/2,e,f],[gs/2,f,g]]):
v1:=LinearAlgebra:-Determinant(V1):
v2:=LinearAlgebra:-Determinant(V2):
K2:=simplify((v1-v2)/(phi(s,t))):
H:=-1/2*(-Q(s)^3*kappa(s)*cos(t)+theta(s,t)+psi(s,t)^2)/psi(s,t)/r(s):
simplify(coeff(numer(subs(sin(t)=B,subs(cos(t)=A,simplify(diff(H,t)*diff(K2,s)
-diff(H,s)*diff(K2,t)))))),A,n),'size');
```

Prog.3

```
G(s):=(Q(s))^2;
e:=(-1/r(s))*(E(s,t)-Q(s)*kappa(s)*cos(t)-p5(s)):
f:=(-1/r(s))*F(s,t):
g:=(-1/r(s))*G(s):
es:=diff(e,s):
fs:=diff(f,s):
gs:=diff(g,s):
ess:=diff(diff(e,s),s):
fss:=diff(diff(f,s),s):
gss:=diff(diff(g,s),s):
et:=diff(e,t):
ft:=diff(f,t):
gt:=diff(g,t):
ett:=diff(diff(e,t),t):
ftt:=diff(diff(f,t),t):
gtt:=diff(diff(g,t),t):
est:=diff(diff(e,s),t):
fst:=diff(diff(f,s),t):
gst:=diff(diff(g,s),t):
ets:=diff(diff(e,t),s):
fts:=diff(diff(f,t),s):
gts:=diff(diff(g,t),s):
phi(s,t):=(1/(r(s))^2)*(psi(s,t)-(Q(s)^3)*kappa(s)*cos(t)
-(Q(s)^2)*p5(s)):
V1 := Matrix([[phi(s,t)*((-ett/2)+fst-(gss/2)),0],[0,phi(s,t)*((-ett/2)+fst-(gss/2))]])
+Matrix([[ft-(gs/2),((fs-(et/2))*e-(f*es/2))],[(gt/2),(fs-(et/2))*f-(g*es/2)]]):
V2 := Matrix([[0,et/2,gs/2],[et/2,e,f],[gs/2,f,g]]):
v1 := LinearAlgebra:-Determinant(V1):
v2 := LinearAlgebra:-Determinant(V2):
K2:=simplify((v1-v2)/(phi(s,t))):
K:=1/psi(s,t)*(-Q(s)^3*kappa(s)*cos(t)+theta(s,t))/r(s)^2:
```

```
simplify(coeff(numer(subs(sin(t)=B,subs(cos(t)=A,simplify(diff(K,t)*diff(K2,s)
-diff(K,s)*diff(K2,t))))),A,i),'size');
```

Prog.4

```
G(s):=(Q(s))^2:
e:=(-1/r(s))*(E(s,t)-Q(s)*kappa(s)*cos(t)-p5(s)):
f:=(-1/r(s))*F(s,t):
g:=(-1/r(s))*G(s):
K:=phi(s,t)/psi(s,t);
L11:=g/(phi(s,t)):
L12:=-f/(phi(s,t)):
L21:=-f/(phi(s,t)):
L22:=e/(phi(s,t)):
delta1:=simplify(diff(((mu1*(phi(s,t)))^(1/2))*L11*diff(ln((mu2*K)^(1/2)),s)
+((mu1*(phi(s,t)))^(1/2))*L12*diff(ln((mu2*K)^(1/2)),t),s)):
delta2:=simplify(diff(((mu1*(phi(s,t)))^(1/2))*L21*diff(ln((mu2*K)^(1/2)),s)
+((mu1*(phi(s,t)))^(1/2))*L22*diff(ln((mu2*K)^(1/2)),t),t)):
denom(delta1+delta2);
```

Prog.5

```
E(s,t):=(Q(s)^2)*(kappa(s)^2)*(cos(t))^2+p1(s)*kappa(s)*cos(t)
+2*Q(s)*R(s)*kappa(s)*tau(s)*sin(t)+p2(s):
F(s,t):=-Q(s)*R(s)*kappa(s)*sin(t)-G(s)*tau(s):
G(s):=(Q(s))^2:
psi(s,t):=E(s,t)*G(s)-F(s,t)^2:
phi(s,t):=simplify((1/(r(s))^2)*(psi(s,t)-(Q(s)^3)*kappa(s)*cos(t)
-(Q(s)^2)*p5(s))):
simplify(coeff(numer(subs(sin(t)=B,subs(cos(t)=A,
diff(expand(simplify(phi(s,t)^3*psi(s,t)^2*r(s)^2)),t)))),A,9),'size');
```

Prog.6

```
dsolve(diff(expand(simplify(phi(s,t)^3*psi(s,t)^2*r(s)^2)),s), { r(s) } );
```

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