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Lacunary Spline Function for Solving Second Order Boundary Value Problems

Karwan H.F. Jwamer[1] and Najim A. Ismael[2]

[1]Mathematical Department, Faculty of Science
University of Sulaimani, Kurdistan Region, Sulaimani, Iraq
E-mail: jwameri1973@gmail.com
[2]Mathematical Department, College of Education
University of Garmian, Kurdistan Region, Kalar, Iraq
E-mail: najimabdallah51@yahoo.com

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Abstract

This article is dedicated to constructing of spline function of seven-degree lacunary spline function of the type (0, 1, 3, 6) for solving a two-point boundary value problem of type (B.V.P):

$y''(x)+f_1(x)y'(x)+f_2(x)y(x)=r(x),\ y(x_0)=y_0,\ y(x_n)=y_n \ldots \quad (1.1)$

where $f_1(x)$, $f_2(x)$ and $r(x)$ are continuous functions of x. Existence and uniqueness of the spline function discussed. Convergence and error bounds are investigated. Numerical illustrations have given, with the example for calculating absolute error between spline functions and exact solution with their derivatives.

Keywords: *Spline functions, boundary value problem, convergence and error bounds.*

1 Introduction

There are numerous studies on spline interpolation. In [8] the existence, uniqueness and error bound of approximation spline interpolation for solving the second order initial value problem (I.V.P) are studied. In [5] spline solution and asymptotic behaviors of eigen values and eigen functions for some B.V.P. is presented. Numerical solution of B.V.P. and I.V.P by using spline function studied by Haq [2]. Approximation solution of second order I.V.Ps by spline function of degree seven is obtained by [3], and solution of the second order I.V.P by eight-degree spline studied in [4]. A computer based numerical method for singular B.V.Ps. is discussed in [1]. Two B.V.Ps solved by piecewise cubic interpolation in [7].

In this paper, we construct and use seven-degree lacunary spline function of type (0, 1, 3, 6) to solve a two-point B.V.Ps for linear differential equation (1.1). Also, existence and uniqueness of spline function of seven-degree have discussed, reduction of linear B.V.Ps to I.V.Ps have studied. An illustration example used to show applicability and efficiency of our construction.

2 Solution of Two Point Boundary Value Problem (B.V.P) by Seven-Degree Spline Function and Superposition Methods

To solve a two-point boundary value problem for linear ordinary differential equation of the type:

$$y''(x) + f_1(x)y'(x) + f_2(x)y(x) = r(x)\,, y(x_0) = y_0\,, y(x_n) = y_n, \quad (2.1)$$

Where $f_1(x), f_2(x)$ *and* $r(x)$ are continuous functions of x by using seven-degree lacunary spline function of the type (0, 1, 3, 6). We go ahead with the following section:

2.1 Construction of the Spline

Consider an interval [0, 1] which is subdivided by a mesh of (n+1) points$\{x_i\}$into n equal parts defined by $x_i = x_0 + ih$, where $h = \frac{1}{n}$for $i = 0,1,2, \dots, n$, $0 = x_0 < x_1 < \cdots < x_n = 1$, and h is the length of each subintervals. Let $y = f(x)$ be a smooth function defined on [0, 1], and $y = f(x) \in C^{n-1}([0,1] \times R^2), n \geq 2$.

We define a seven-degree spline interpolation $S_i(x)$ for one dimensional on the interval $[x_i\,, x_{i+1}]$ for i= 1, 2,…, n-1 as:

$$S_i(x) = y(x_i) + (x - x_i)y'(x_i) + (x - x_i)^2 a_{i,2} + \frac{(x - x_i)^3}{6} y'''(x_i) + (x - x_i)^4 a_{i,4} + (x - x_i)^5 a_{i,5} + \frac{(x - x_i)^6}{720} y^{(6)} + (x - x_i)^7 a_{i,7}. \qquad (2.2)$$

where $a_{i,j}$, $i = 1,2, \dots, n - 1$, $j = 2,4,5,7$ are unknowns to be determined.

2.2 Existence and Uniqueness of the Spline Function

In this section, we present the existence and uniqueness of spline function of our defined lacunary spline function of seven-degree (2.2) which is used to solve the given boundary value problem (2.1). We also give the conditions that guarantee the existence and uniqueness of the given lacunary spline function by the following theorem:

Theorem 2.1: *Let $y(x_i), y'(x_i), y'''(x_i) and y^{(6)}(x_i)$ for $i = 0,1, \dots, n$ be a given real numbers. Then there exists a unique lacunary spline function of degree seven as given in the equation (2.2) such that:*

$$\left.\begin{array}{r} S(x_i) = y(x_i) \\ S^{(r)}(x_i) = y^{(r)}(x_i) \end{array}\right\} for\ r = 1,3,6\ and\ \ i = 0,1,2, \dots, n \quad \dots (2.3)$$

The spline function $S(x)$ is defined as $S(x) = S_i(x) where\ x \in [x_i, x_{i+1}]$ i= $1,2, \dots, n - 1$, and the coefficients of this polynomial are to be determined by the following conditions:

$$\left.\begin{array}{r} S_i(x_{i+1}) = S_{i+1}(x_{i+1}) = y(x_{i+1}) \\ S^{(r)}{}_i(x_{i+1}) = S^{(r)}{}_{i+1}(x_{i+1}) = y^{(r)}(x_{i+1}) \end{array}\right\} for\ r = 1,3,6 \quad \dots (2.4)$$

Proof: To find uniquely the coefficients $a_{i,j}$, $j = 2,4,5,7$ of $S_i(x)$ for $i = 1,2, \dots, n - 1$ which is defined by (2.2).

We start with the first, third and sixth derivatives of $S_i(x)$ withthe condition (2.4) we get the following equations:

$$y_{i+1}(x) = y(x_i) + hy'(x_i) + h^2 a_{i,2} + \frac{h^3}{6} y'''(x_i) + h^4 a_{i,4} + h^5 a_{i,5} + \frac{h^6}{720} y^{(6)} + h^7 a_{i,7},$$

Then

$$h^2 a_{i,2} + h^4 a_{i,4} + h^5 a_{i,5} + h^7 a_{i,7} = y_{i+1} - y_i - hy_i' - \frac{h^3}{6} y_i'{}'' - \frac{h^6}{720} y_i^{(6)} \qquad (2.5)$$

$$2h^2 a_{i,2} + 4h^3 a_{i,4} + 5h^4 a_{i,5} + 7\, h^6 a_{i,7} = y'_{i+1} - y'_i - \frac{h^2}{2} y_i''' - \frac{h^5}{120} y_i^{(6)} \qquad (2.6)$$

$$24\,h\,a_{i,4} + 60\,h^2 a_{i,5} + 210\,h^4 a_{i,7} = y'''_{i+1} - y'''_i - \frac{h^3}{6} y_i^{(6)}, \tag{2.7}$$

$$5040 h a_{i,7} = y_{i+1}^{(6)} - y_i^{(6)}. \tag{2.8}$$

From equations (2.2) and (2.4), then by taking first, third, and sixth derivatives of $S_i(x)$ we get

$$S_i(x) = y(x_i) + hy'(x_i) + h^2 a_{i,2} + \frac{h^3}{6} y'''(x_i) + h^4 a_{i,4} + h^5 a_{i,5} + \frac{h^6}{720} y_i^{(6)} + h^7 a_{i,7},$$

$$h^2 a_{i,2} + h^4 a_{i,4} + h^5 a_{i,5} + h^7 a_{i,7} = y_{i+1} - y_i - hy'_i - \frac{h^3}{6} y'''_i - \frac{h^6}{720} y_i^{(6)},$$

$$2h a_{i,2} + 4h^3 a_{i,4} + 5h^4 a_{i,5} + 7\,h^6 a_{i,7} = y'_{i+1} - y'_i - \frac{h^2}{2} y'''_i - \frac{h^5}{120} y_i^{(6)}$$

$$2a_{i,2} + 12h^2 a_{i,4} + 20h^3 a_{i,5} + 42h^5 a_{i,7} = y''_{i+1} - hy'''_i - \frac{h^4}{24} y_i^{(6)}$$

$$24\,h\,a_{i,4} + 60\,h^2 a_{i,5} + 210\,h^4 a_{i,7} = y'''_{i+1} - y'''_i - \frac{h^3}{6} y_i^{(6)}$$

$$24\; a_{i,4} + 120\,h\,a_{i,5} + 840\,h^3 a_{i,7} = y_{i+1}^{(4)} - \frac{h^2}{2} y_i^{(6)}$$

$$120\,a_{i,5} + 2520\,h^2 a_{i,7} = y_{i+1}^{(5)} - hy_i^{(6)}$$

$$5040 h a_{i,7} = y_{i+1}^{(6)} - y_i^{(6)}.$$

From (2.5) – (2.8) we get the following equations:

$$a_{i,2} + h^2 a_{i,4} + h^3 a_{i,5} + h^5 a_{i,7} = \frac{1}{h^2}[\,y_{i+1} - y_i - hy'_i - \frac{h^3}{6} y'''_i - \frac{h^6}{720} y_i^{(6)}] = \alpha_1$$

$$2a_{i,2} + 4h^2 a_{i,4} + 5h^3 a_{i,5} + 7\,h^5 a_{i,7} = \frac{1}{h} y'_{i+1} - y'_i - \frac{h^2}{2} y'''_i - \frac{h^5}{120} y_i^{(6)}] = \alpha_2$$

$$4a_{i,4} + 10h a_{i,5} + 35\,h^3 a_{i,7} = \frac{1}{6h}[\,y'''_{i+1} - y'''_i - \frac{h^3}{6} y_i^{(6)}] = \alpha_3$$

Hence,

$a_{i,7} = \frac{1}{5040h}\left[y_{i+1}^{(6)} - y_i^{(6)}\right] = \alpha_4$, then

$$a_{i,2} + h^2 a_{i,4} + h^3 a_{i,5} + h^5 a_{i,7} = \alpha_1$$
$$2a_{i,2} + 4h^2 a_{i,4} + 5h^3 a_{i,5} + 7h^5 a_{i,7} = \alpha_2$$
$$4a_{i,4} + 10h a_{i,5} + 35h^3 a_{i,7} = \alpha_3$$
$$a_{i,7} = \alpha_4.$$

From the last four equations which form a 4×4 linear system and the coefficient matrix of the system in the unknown $a_{i,j}$, $j = 2,4,5,7$ and $i = 1,2,\dots,n-1$ is non - singular matrix and hence the coefficient $a_{i,j}$ are uniquely determined. By solving the resulting system, we can specify uniquely the equations of the coefficients $a_{i,j}$ as follows:

Since $|A| \neq 0$ (A is non $-$ singular matrix), $|A| = 8h^3$

$$a_{i,2} = \frac{5}{2}\frac{1}{h^2}[y_{i+1} - y_i] - \frac{1}{4h}[3y'_{i+1} + 7y'_i] + \frac{h}{48}[y'''_{i+1} - 3y'''_i] - \frac{1}{40320}h^4\left[13y^{(6)}_{i+1} + 15y^{(6)}_i\right], \quad (2.9)$$

$$a_{i,4} = -\frac{5}{2}\frac{1}{h^4}[y_{i+1} - y_i] + \frac{5}{4h^3}[y'_{i+1} + y'_i] - \frac{1}{48h}[3y'''_{i+1} + 7y'''_i] + \frac{1}{8064}h^2 \left[11y^{(6)}_{i+1} + 17y^{(6)}_i\right], \quad (2.10)$$

$$a_{i,5} = \frac{1}{h^5}[y_{i+1} - y_i] - \frac{1}{2h^4}[y'_{i+1} + y'_i] + \frac{1}{24h^2}[y'''_{i+1} + y'''_i] - \frac{1}{20160}h\left[25y^{(6)}_{i+1} + 59y^{(6)}_i\right], \quad (2.11)$$

$$a_{i,7} = \frac{1}{5040h}\left[y^{(6)}_{i+1} - y^{(6)}_i\right], \quad (2.12)$$

for i= 1,2,…, n-1.

Consequently, we can determine uniquely the coefficient $a_{i,2}$, $a_{i,4}$, $a_{i,5}$ and, $a_{i,7}$ for i= 1, 2,…, n-1 from the equations (2.9), (2.10), (2.11) and (2.12) successively; and hence the proof has completed .

2.3 Reduction of Linear (B.V.Ps) to (I.V.Ps)

The method of superposition is based on transforming linear ordinary differential equations from boundary value problems to initial value problems. For linear ordinary differential equations, it is possible to reduce the boundary value problem to two or more initial value problems. By using one of the known methods; Taylor's method; shooting method or Runge–Kutta method, the initial value problems can be solved. Combining these results then gives the solution of the original boundary value problem. Here, we use Taylor's approximations method to the initial value problems.

We explain the technique of the method as shown below.

Consider a second order linear ordinary differential equation:

$$y''(x) + f_1(x)y'(x) + f_2(x)y(x) = r(x). \quad (2.13)$$

Subject to the boundary condition

$$y(x_0) = y_0 \ , \ y(x_n) = y_n, \quad (2.14)$$

where $f_1(x), f_2(x)$ *and* $r(x)$ are continuous functions of x and the continuity of $f_1(x), f_2(x)$ *and* $r(x)$ assures the existence and uniqueness of the solution of equation (2.13).

To transform equation (2.13) and (2.14) into an initial value problem, we assume:

$$y(x) = y_1(x) + \mu\, y_2(x), \quad (2.15)$$

where μ is a constant to be determined.

By substituting $y(x)$ from equation (2.15) in equation (2.13), we get

$$(y_1''(x) + f_1(x)y_1'(x) + f_2(x)y_1(x) - r(x)) + \mu\,(y_2''(x) + f_1(x)y_2'(x) + f_2(x)y_2(x)) = 0. \quad (2.16)$$

Then from (2.16), we get

$$y_1''(x) + f_1(x)y_1'(x) + f_2(x)y_1(x) = r(x) \quad (2.17)$$

And

$$y_2''(x) + f_1(x)y_2'(x) + f_2(x)y_2(x) = 0. \quad (2.18)$$

The first boundary condition in equation (2.14) is next transformed to

$y(x_0) = y_1(x_0) + \mu\, y_2(x_0)$ or $y_0 = y_1(x_0) + \mu\, y_2(x_0)$.

From which

$$y_1(x_0) = y_0, \quad (2.19)$$

And

$$y_2(x_0) = 0\,. \quad (2.20)$$

Differentiating equation (2.15) and setting x equal to x_0, we get:

$$y'(x_0) = y'_1(x_0) + \mu\, y'_2(x_0). \tag{2.21}$$

If the two unknown boundary conditions are set equal to

$$y'_1(x_0) = 0, \tag{2.22}$$

And

$$y'_2(x_0) = 1. \tag{2.23}$$

Then the equation (2.21) gives

$$y'(x_0) = \mu. \tag{2.24}$$

Thus, the unknown constant μ is identified as the missing initial slope.

Finally, the boundary condition at the second point is transformed to

$$y(x_n) = y_1(x_n) + \mu\, y_2(x_n) \text{ or } y_n = y_1(x_n) + \mu\, y_2(x_n).$$

So

$$\mu = \frac{y_n - y_1(x_n)}{y_2(x_n)}. \tag{2.25}$$

Thus, the solution of equation (2.13) consists of the following steps:

1- Solving equation (2.17) with initial condition (2.19) and (2.22) from $x = x_0$ to $x = x_n$. Then $y_1(x_n)$ is obtained.
2- Solving equation (2.18) with initial condition (2.20) and (2.23) from $x = x_0$ to $x = x_n$. Then $y_2(x_n)$ is obtained.
3- We find the missing initial slop μ by equation (2.25).
4- We solve the original differential equation (2.13) from the equation (2.15).

2.4 Convergence and Error Bounds

In this section, we show a criterion of convergence of the given initial value problems (2.17) and (2.18), and the solution of the given boundary value problems (1.1) and the convergence of non- interpolating points of the second and fourth derivatives of the given Lacunary spline functions.

Theorem 2.2: *Let $y_1(x)$ and $y_2(x)$ be the solutions of the initial value problems (2.17) and (2.18) respectively which are found by Taylor's series expansion formula where $y_1(x)$ and $y_2(x) \in C^7[0,1]$ and $y^{(r)}{}_1(x_i)$ and $y^{(r)}{}_2(x_i)$ are given for r=0,1,6 and i= 0,1,…,n. Let $S_i(x)$ be a unique spline function of degree seven which is defined by equation (1.2), then for*

$x \in [x_i , x_{i+1}], i = 1,2, \dots , n - 1.$

$$\|S_i(x) - y_1(x)\|_\infty \le \frac{19}{6720} w_7(f;h)$$

And

$$\|S_i(x) - y_2(x)\|_\infty \le \frac{19}{6720} w_7(f;h)$$

Where$w_7(f;h)$ is the module of continuity which is defined by:

$$w_7(f;h) = Max\,\{ \quad |y^{(7)}(s) - y^{(7)}(t)|: |s-t| \le h\, \forall\, s,t \in [0,1]\}$$

And$||f(x)||_\infty = Max\,\{ \quad |f(x)|:\ \forall\, x \in [0,1]\}$

Proof: Let $x \in [x_i , x_{i+1}]$, where i=1,2,…,n-1.

To find $\|S_i(x) - y_1(x)\|_\infty$, from Taylor's series expansion about $x = x_i$for $y_{i+1}^{(r)}$, r=0, 1, 3,6of the equations (2.2.9) – (2.2.12), we get

$$a_{i,2} = \frac{5}{2}\frac{1}{h^2}[\, y_{i+1} - y_i] - \frac{1}{4h}[3y'_{i+1} + 7y'_{i}] + \frac{h}{48}[y'''_{i+1} - 3y'''_{i}] - \frac{1}{40320}h^4\left[13y_{i+1}^{(6)} + \quad 15y_i^{(6)}\right]$$

$$\frac{1}{2}y''_i + \frac{h^5}{2016}y^{(7)}(\varphi_0) - \frac{h^5}{960}y^{(7)}(\varphi_1) + \frac{h^5}{1152}y^{(7)}(\varphi_2)] - \frac{13}{40320}h^5\left[y^{(7)}(\varphi_3)\right], \qquad (2.26)$$

Where $x_i < \varphi_0 < \varphi_1 < \varphi_2 < \varphi_3 < x_{i+1}$, i=1,2,...,n-1.

$$a_{i,4} = -\frac{5}{2h^4}[\, y_{i+1} - y_i] + \frac{5}{4h^3}[y'_{i+1} + y'_{i}] - \frac{1}{48h}[\,3y'''_{i+1} + 7y'''_{i}] + \frac{1}{8064}h^2\left[11y_{i+1}^{(6)} + 17y_i^{(6)}\right]$$

Hence by simplification:

$$a_{i,4} = \frac{1}{24}y_i^{(4)} - \frac{h^3}{2016}y^{(7)}(\alpha_0) + \frac{h^3}{576}y^{(7)}(\alpha_1) - \frac{h^3}{384}y^{(7)}(\alpha_2)] + \frac{11}{8064}h^3y^{(7)}(\alpha_3), \qquad (2.27)$$

where $x_i < \alpha_0 < \alpha_1 < \alpha_2 < \alpha_3 < x_{i+1}$, i = 1, 2,..., n-1.

$$a_{i,5} = \frac{1}{h^5}[\, y_{i+1} - y_i] - \frac{1}{2h^4}[y'_{i+1} + y'_i] + \frac{1}{24\, h^2}[y'''_{i+1} + y'''_i] - \frac{h}{20160}\left[25y^{(6)}_{i+1} + 59y^{(6)}_i\right]$$

Also by simplification:

$$a_{i,5} = \frac{1}{120}y_i^{(5)} + \frac{h^2}{5040}y^7(\beta_0) - \frac{h^2}{1440}y^7(\beta_1) + \frac{h^2}{576}y^7(\beta_2)] - \frac{5}{4032}h^2[y^7(\beta_3)]\,, \qquad (2.28)$$

where $x_i < \beta_0 < \beta_1 < \beta_2 < \beta_3 < x_{i+1}$i=1,2,...,n-1

$$a_{i,7} = \frac{1}{5040h}\left[y^{(6)}_{i+1} - y^{(6)}_i\right] = \frac{1}{5040}y_i^{(7)}(\theta_0), \qquad (2.29)$$

where $x_i < \theta_0 < x_{i+1}$

And also from Taylor's series expansion about $x = x_i$ for y= y_1, $y = y_1 \in C^7[0,1]$ is of the form:

$$y_1 = y_1(x_i) + (x - x_i)y_1'(x_i) + \frac{(x-x_i)^2}{2}y_1''(x_i) + \frac{(x-x_i)^3}{6}y_1'''(x_i) + \frac{(x-x_i)^4}{24}y_1^{(4)}(x_i) + \frac{(x-x_i)^5}{120}y_1^{(5)}(x_i) + \frac{(x-x_i)^6}{720}y_1^{(6)}(x_i) + \frac{(x-x_i)^7}{5040}y_1^{(7)}(\delta)$$

where $x_i < \delta < x_{i+1}$.

By substituting equations $(2.26) - (2.29)$ into the equation (1.2) and then taken the infinite norm for $S_i(x) - y_1(x)$, we get:

$$||S_i(x) - y_1(x)||_\infty \le \frac{11h^7}{8064}|y^{(7)}(\varphi_i) - y^{(7)}(\varphi_j)| + \frac{25h^7}{8064}|y^{(7)}(\alpha_k) - y^{(7)}(\alpha_l)\Big| + \frac{13h^7}{6720}|y^{(7)}(\beta_m) - y^{(7)}(\beta_n)\Big| + \frac{h^7}{5040}|y^{(7)}(\theta_s) - y^{(7)}(\theta_t)|$$

for all $x \in [x_i, x_{i+1}]$ where $x_i < \varphi_i, \varphi_j, \alpha_k, \alpha_l, \beta_m, \beta_n, \theta_s, \theta_t < x_{i+1}$.

Hence

$$||S_i(x) - y_1(x)||_\infty \le \frac{19}{6720}W_7(f;h)\,,$$

and the proof has completed with respect to y_1.

Similarly, we will prove that

$$||S_i(x) - y_1(x)||_\infty \leq \frac{19}{6720} W_7(f;h)$$

Theorem 2.3: *Let $y(x)$ be a solution to the boundary value problem (2.13) defined by equation (2.15) and $S_i(x)$ be a unique spline function of seven-degree which is defined by equation (1.2) and it is a solution of the boundary value problem(2.13), then*

$||S_i(x) - y(x)||_\infty \leq \frac{1}{1-|\mu|}\frac{19\,h^7}{6720} W_7(f;h)$, $|\mu| \neq 1$, where μ is a constant, and it is determined by equation (2.2.13).

Proof: From equation (2.15), we have; $y(x) = y_1(x) + \mu\, y_2(x)$, where x is any point in [0, 1], then

$$\begin{aligned} ||S_i(x) - y(x)||_\infty &= ||S_i(x) - (y_1(x) + \mu\, y_2(x))||_\infty \\ &= ||S_i(x) - y_1(x) - \mu\, y_2(x)||_\infty \\ &= ||S_i(x) - y_1(x) + (-\mu)\, y_2(x))||_\infty \\ &\leq ||S_i(x) - y_1(x)||_\infty + ||(-\mu)\, y_2(x))||_\infty \end{aligned}$$

But

$||S_i(x) - y_1(x)||_\infty \leq \frac{19\,h^7}{6720} W_7(f;h)$, Theorem 2.2 (2.30)

So $||S_i(x) - y(x)||_\infty \leq \frac{19\,h^7}{6720} W_7(f;h) + |-\mu|\, ||\,y_2(x)||_\infty$

$$\leq \frac{19\,h^7}{6720} W_7(f;h) + |\mu|\, ||\,y_2(x)||_\infty$$

And since $S_i(x)$ *and* $y(x)$ are solutions for the equation (2.13), therefore

$S''_i(x) + f_1(x)S'_i(x) + f_2(x)S_i(x) = r(x)$, and

$$y''(x) + f_1(x)y'(x) + f_2(x)y(x) = r(x)$$

Therefore

$$(S_i(x) - y(x))'' + f_1(x)(S_i(x) - y(x))' + f_2(x)(S_i(x) - y(x)) = 0. \quad (2.31)$$

So

$S_i(x) - y(x)$ is also a solution of equation (2.18), but $y_2(x)$ is also a solution of equation (2.18), so

$$y_2''(x) + f_1(x)y'_2(x) + f_2(x)y_2(x) = 0. \quad (2.32)$$

From the final two equations (2.31) and (23.2) we get

$$S_i(x) - y(x) = y_2(x).$$

Hence

$$||S_i(x) - y(x)||_\infty \le \frac{19\, h^7}{6720} W_7(f;h) + |\mu|\, ||\, y_2(x)||_\infty$$
$$\le \frac{19\, h^7}{6720} |\mu|\, ||S_i(x) - y(x)||_\infty .$$

So

$$||S_i(x) - y(x)||_\infty (1 - |\mu|) \le \frac{19\, h^7}{6720} W_7(f;h).$$

Hence,

$||S_i(x) - y(x)||_\infty \le \frac{19\, h^7}{(1-|\mu|)\, 6720} W_7(f;h)$, hence the proof has been achieved.

Theorem 2.4: *Let $y_1(x), y_2(x)$ be the solutions of the initial value problems(2.17) and(2.18) respectively which are found by Taylor's series expansion formula where $y_1(x), y_2(x) \in C^7[0,1]$, and $y_1^{(r)}(x_i)$, $y_2^{(r)}(x_i)$ are given for r=0,1,3,6 and i=0,1,…,n. Let $S_i(x)$ be a unique spline function of degree seven which is defined by equation (2.1.2), then for $x \in [x_i, x_{i+1}]$, $i = 1,2, \dots, n-1$.*

$$||S_i''(x) - y_1''(x)||_\infty \le \frac{1753\, h^5}{20160} W_7(f;h)$$

$$||S_i'''(x) - y_1'''(x)||_\infty \le \frac{h^4}{5} W_7(f;h)$$

$$||S_i^{(4)}(x) - y_1^{(4)}(x)||_\infty \le \frac{h^3}{3} W_7(f;h)$$

$$||S_i^{(5)}(x) - y_1^{(5)}(x)||_\infty \le \frac{3h^2}{5} W_7(f;h)$$

$$||S_i^{(6)}(x) - y_1^{(6)}(x)||_\infty \le h\, W_7(f;h)$$

$$||S_i''(x) - y_2''(x)||_\infty \le \frac{1753\, h^5}{20160} W_7(f;h)$$

$$||S_i'''(x) - y_2'''(x)||_\infty \le \frac{h^4}{5} W_7(f;h)$$

$$||S_i^{(4)}(x) - y_2^{(4)}(x)||_\infty \leq \frac{h^3}{3} W_7(f;h)$$

$$||S_i^{(5)}(x) - y_2^{(5)}(x)||_\infty \leq \frac{3h^2}{5} W_7(f;h)$$

$$||S_i^{(6)}(x) - y_2^{(6)}(x)||_\infty \leq h\, W_7(f;h)$$

where $W_7(f,h)$ is the module of continuity which defined by

$W_7(f;h) = \max\{ \ |y^{(7)}(s) - y^{(7)}(t)| : |s-t| \leq h \, ; s,t \in [0,1]\}$ and

$||f(x)||_\infty = \max\{ \ |f(x)| : \forall x \in [0,1]\}$

Proof: Let x be any point in the interval $[x_i, x_{i+1}]$ where i = 1, 2,…, n-1.

To find $||S_i''(x) - y_1''(x)||_\infty$

Since

$$S_i(x) = y(x_i) + (x-x_i)y'(x_i) + (x-x_i)^2 a_{i,2} + \frac{(x-x_i)^3}{6} y_1'''(x_i) + (x-x_i)^4 a_{i,4} + (x-x_i)^5 a_{i,5} + \frac{(x-x_i)^6}{720} y^{(6)}(x_i) + (x-x_i)^7 a_{i,7}$$

$$S_i'(x) = y'^{(x_i)} + 2(x-x_i)a_{i,2} + \frac{(x-x_i)^2}{2} y_1'''(x_i) + 4(x-x_i)^3 a_{i,4} + 5(x-x_i)^4 a_{i,5} + \frac{(x-x_i)^5}{120} y_1^{(6)}(x_i) + 7(x-x_i)^6 a_{i,7}$$

$$S_i''(x) = 2a_{i,2} + (x-x_i)y_1'''(x_i) + 12(x-x_i)^2 a_{i,4} + 20(x-x_i)^3 a_{i,5} + \frac{(x-x_i)^4}{24} y_1^{(6)}(x_i) + 42(x-x_i)^5 a_{i,7}$$

Using Taylor's series expansion about $x = x_i \, for \; y_{i+1}^{(r)}, r = 0,1,3,6$ of the equations (2.26) – (2.29), and substituting these equations in $S_i^{(r)}(x), r = 2,3,4,5,6$. We get

$$S_i''(x) = 2\left\{\frac{1}{2} y_1''(x_i) + \frac{h^5}{2016} y_1^{(7)}(\varphi_0) - \frac{h^5}{960} y_1^{(7)}(\varphi_1) + \frac{h^5}{1152} y_1^{(7)}(\varphi_2) - \frac{13\, h^5}{40320} y_1^{(7)}(\varphi_3)\right\} + (x-x_i)y'''(x_i) + 12(x-x_i)^2$$

$$+\{\frac{1}{24}y_1^{(4)}(x_i)-\frac{h^3}{2016}y_1^{(7)}(\alpha_0)+\frac{h^3}{576}y_1^{(7)}(\alpha_1)-\frac{h^3}{384}y_1^{(7)}(\alpha_2)+\frac{11h^3}{8064}y_1^{(7)}(\alpha_3)\}+20(x-x_i)^3\{\frac{1}{120}y_1^{(5)}(x_i)+\frac{h^2}{5040}y_1^{(7)}(\beta_0)-\frac{h^2}{1440}y_1^{(7)}(\beta_1)+\frac{h^2}{576}y_1^{(7)}(\beta_2)-\frac{5h^2}{4032}y_i^{(7)}(\beta_3)\}+\frac{(x-x_i)^4}{24}y_1^{(6)}(x_i)+42(x-x_i)^5\{\frac{1}{5040}y_1^{(7)}(\theta_0)\}$$

where $x_i < \varphi_k, \alpha_k,\ \beta_k, \theta_0 < x_{i+1}\ for\ k = 0,1,2,3.$

Now by Taylor's series expansion on $y_1''(x_i)$ about $x = x_1$, we get

$$||S_i''(x)-y_1''(x_i)||_\infty \le \frac{(2)11\,h^5}{8064}W_7(f;h)+\frac{(12)25\,h^5}{8064}W_7(f;h)+\frac{(20)39\,h^5}{20160}W_7(f;h)+\frac{h^5}{120}W_7(f;h)$$

$$\le \frac{(322)h^5}{8064}W_7(f;h)+\frac{948\,h^5}{20160}W_7(f;h)\ \le \frac{1753\,h^5}{20160}W_7(f;h)$$

Now to find $||S_i^{(3)}(x)-\ y_1^{(3)}(x_i)||_\infty$

Since

$$S_i'''(x)=y_1'''(x_i)+\ 24(x-x_i)a_{i,4}+60(x-x_i)^2a_{i,5}+\frac{(x-x_i)^3}{6}y_1^{(6)}(x_i)+\ 210(x-x_i)^4a_{i,7}$$

Now

$$||S_i'''(x)-\ y_1'''(x_i)||_\infty = ||\ y_1'''(x_i)+24(x-x_i)\{\frac{1}{24}y_1^{(4)}(x_i)-\frac{h^3}{2016}y_1^{(7)}(\alpha_0)+\frac{h^3}{576}y_1^{(7)}(\alpha_1)-\frac{h^3}{384}y_1^{(7)}(\alpha_2)+\frac{11h^3}{8064}y_1^{(7)}(\alpha_3)\}+\ 60(x-x_i)^2\{\frac{1}{120}y_1^{(5)}(x_i)+\frac{h^2}{5040}y_1^{(7)}(\beta_0)-\frac{h^2}{1440}y_1^{(7)}(\beta_1)+\frac{h^2}{576}y_1^{(7)}(\beta_2)-\frac{5}{4032}y_1^{(7)}(\beta_3)\}+\frac{(x-x_i)^3}{6}y_1^{(6)}(x_i)+210\,(x-x_i)^4\left\{\frac{1}{5040}y_1^{(7)}(\theta_0)\right\}-\ y_1'''(x_i)||_\infty$$

Also by using Taylor's series expansion on $y_1'''(x_i)\,about\ x = x_i$ we deduce $||S_i'''(x) - y_1'''(x_i)||_\infty \le \frac{600\,h^4}{8064} W_7(f;h) + \frac{2340\,h^4}{20160} W_7(f;h) + \frac{h^4}{24} W_7(f;h)$

$$= \frac{13\,h^4}{56} W_7(f;h) \le \frac{h^4}{5} W_7(f;h)$$

To find $||S_i^{(4)}(x) - y_1^{(4)}(x)||_\infty$

Since

$$S_i^{(4)}(x) = 24a_{i,4} + 120(x-x_i)a_{i,5} + \frac{(x-x_i)^2}{2} y_1^{(6)}(x_i) + 840(x-x_i)^3 a_{i,7}$$

So

$$\begin{aligned} ||S_i^{(4)}(x) - y_1^{(4)}(x)||_\infty \\ &= ||\,24\,\{\frac{1}{24} y_1^{(4)}(x_i) - \frac{h^3}{2016} y_1^{(7)}(\alpha_0) + \frac{h^3}{576} y_1^{(7)}(\alpha_1) \\ &- \frac{h^3}{384}\, y_1^{(7)}(\alpha_2) + \frac{11h^3}{8064} y_1^{(7)}(\alpha_3)\,\} \\ &+ \quad 120(x-x_i)\,\{\frac{1}{120} y_1^{(5)}(x_i) + \frac{h^2}{5040} y_1^{(7)}(\beta_0) \\ &- \frac{h^2}{1440} y_1^{(7)}(\beta_1) + \frac{h^2}{576} y_1^{(7)}(\beta_2) - \frac{5h^2}{4032} y_1^{(7)}(\beta_3)\,\} \\ &+ \frac{(x-x_i)^2}{2} y_1^{(6)}(x_i) + 840\,(x-x_i)^3 \left\{\frac{1}{5040} y_1^{(7)}(\theta_0)\right\} \\ &- y_1^{(4)}(x_i)||_\infty \end{aligned}$$

Also by using Taylor's series expansion on $y_1^{(4)}(x_i)\,about\ x = x_i$ we deduce:

$$||S_i^{(4)}(x) - y_1^{(4)}(x_i)||_\infty \le \frac{600\,h^3}{8064} W_7(f;h) + \frac{4680\,h^3}{20160} W_7(f;h) + \frac{h^3}{6} W_7(f;h)$$

$$= \frac{19080\,h^3}{40320} W_7(f;h) \le \frac{h^4}{3} W_7(f;h)$$

To find $||S_i^{(5)}(x) - y_1^{(5)}(x)||_\infty$

Since

$$S_i^{(5)}(x) = 120a_{i,5} + (x-x_i)y_1^{(6)}(x_i) + 2520(x-x_i)^2 a_{i,7}$$

Now

$$||S_i^{(5)}(x) - y_1^{(5)}(x_i)||_\infty$$
$$= ||\ 120\,\{\frac{1}{120}y_1^{(5)}(x_i) + \frac{h^2}{5040}y_1^{(7)}(\beta_0) - \frac{h^2}{1440}y_1^{(7)}(\beta_1) + \frac{h^2}{576}y_1^{(7)}(\beta_2) - \frac{5h^2}{4032}y_1^{(7)}(\beta_3)\,\} + (x-x_i)y_1^{(6)}(x_i) + 2520\,(x - x_i)^2\left\{\frac{1}{5040}y_1^{(7)}(\theta_0)\right\} - y_1^{(5)}(x_i)||_\infty$$

Also by using Taylor's series expansion on$y_1^{(5)}(x_i)$ $about\ \ x = x_i$ we deduce:

$$||S_i^{(5)}(x) - y_1^{(5)}(x_i)||_\infty \le \frac{(120)39\,h^2}{20160}W_7(f;h) + \frac{h^2}{2}W_7(f;h)$$

$$= (\frac{39}{168} + \frac{1}{2})h^2\,W_7(f;h) = \frac{123\,h^2}{168}W_7(f;h) \le \frac{3h^2}{5}W_7(f;h)$$

To find $||S_i^{(6)}(x) - y_1^{(6)}(x_i)||_\infty$

Since $S_i^{(6)}(x) = y_1^{(6)}(x_i) + 5040(x-x_i)a_{i,7}$. And by using Taylor's series expansion on $y_1^{(6)}(x_i)$ $about\ x = x_i$ we deduce:

$$||S_i^{(6)}(x) - y_1^{(6)}(x_i)||_\infty$$
$$= ||y_1^{(6)}(x_i) + 5040\,(x-x_i)\left\{\frac{1}{5040}y_1^{(7)}(\theta_0)\right\} - y_1^{(6)}(x)||_\infty$$
$$\le h\,W_7(f;h)$$

Hence the proof completed with respect to $y_1(x)$.

Similarly the proof is obtained for $y_2(x)$.

Theorem 2.5: *Let $y(x) \in C^7[0,1]$ be a solution of the boundary value problem (2.13) defined by equations (2.15) and $S_i(x)$ be a unique spline function of seven-degree that is defined by equation (1.2) and is a solution of the problem (2.13), then $x \in [x_i, x_{i+1}],\ i = 1,2,\dots,n-1$.*

$$||S_i''(x) - y''(x)||_\infty \le \frac{1}{1-|\mu|}\frac{1753\,h^5}{20160}W_7(f;h)\ ,||S_i'''(x) - y'''(x)||_\infty \le \frac{1}{1-|\mu|}\frac{h^4}{5}W_7(f;h)$$

$$||S_i^{(4)}(x) - y^{(4)}(x)||_\infty \leq \frac{1}{1-|\mu|}\frac{h^3}{3}W_7(f;h)\ , \ \ ||S_i^{(5)}(x) - y^{(5)}(x)||_\infty \leq \frac{1}{1-|\mu|}\frac{3h^2}{5}W_7(f;h)$$

$$||S_i^{(6)}(x) - y^{(6)}(x)||_\infty \leq \frac{1}{1-|\mu|}h\, W_7(f;h),$$

where $|\mu| \neq 1$ is a constant and it is determined by equation (2.15).

Proof: Let $x \in [x_i, x_{i+1}],\ i = 1, 2, \dots, n-1$. To find $||S_i''(x) - y''(x)||_\infty$.

Since $y(x) = y_1(x) + \mu\, y_2(x)$, then differentiating this equation twice with respect to x, we get:

$$y''(x) = y_1''(x) + \mu\, y_2''(x), \qquad \text{and then}$$

$$||S_i''(x) - y''(x)||_\infty = ||S_i''(x) - (y_1''(x) + \mu\, y_2''(x))||_\infty$$

$$= ||S_i''(x) - y_1''(x) + (-\mu)\, y_2''(x))||_\infty$$

$$\leq ||S_i''(x) - y_1''(x)||_\infty + ||(-\mu)\, y_2''(x)||_\infty$$

$$\leq \frac{1753\, h^5}{20160}W_7(f;h) + |-\mu|\, ||\, y_2''(x)||_\infty$$

Hence $||S_i''(x) - y''(x)||_\infty \leq \frac{1753\, ;}{20160}W_7(f;h) + |\mu|\, ||\, y_2''(x)||_\infty$

Since $S_i(x)\ and\ y(x)$ are solutions of the equation (2.13), therefore

$S''_i(x) + f_1(x)S_i'(x) + f_2(x)S_i(x) = r(x)$, and

$$y''(x) + f_1(x)y'(x) + f_2(x)y(x) = r(x)$$

From which

$(S_i(x) - y(x))'' + f_1(x)(S_i(x) - y(x))' + f_2(x)(S_i(x) - y(x)) = 0.$(2.32)

This means that$S_i(x) - y(x)$ is a solution of equation (2.18), but $y_2(x)$ is a solution of equation (2.18), therefore

$$y_2''(x) + f_1(x)y'_2(x) + f_2(x)y_2(x) = 0. \tag{2.33}$$

Then from equations (2.32) and (2.33), we obtain that

$S_i^{''}(x) - y''(x) = y_2^{''}(x)$, then

$$||S_i^{''}(x) - y''(x)||_\infty \leq \frac{1753\, h^5}{20160} W_7(f;h) + |\mu|\, ||S_i^{''}(x) - y''(x)||_\infty,$$

And, consequently

$$||S_i^{''}(x) - y''(x)||_\infty \leq \frac{1753\, h^5}{(1-|\mu|)\, 20160} W_7(f;h).$$

Similarly we can prove $||S_i^{(r)}(x) - y^{(r)}(x)||_\infty$, $r = 3,4,5,6$ of the theorem.

Numerical Example

This section allocated for performing numerical result to show the applicability and efficiency of our construction.

Example 1: Consider a boundary value problem

$y''(x)+y(x)+1=0$, where $x\in [0,1]$ and y(0)=0, y(1)=0.

Solution: Since $y''(x)+y(x)+1=0$**,** then$y''(x)+y(x)=$ -1,and from comparing this equation with (2.13) we find $f_1(x)=0$, $f_2(x)=1$, and $r(x)=$ -1.

So transforming the given boundary value problem to initial value problems:

$y''_1(x) + y_1(x) = -1$, $y_1(0)=0$, $y'_1(0)=0$, and

$y''_2(x) + y_2(x)=0$, $y_2(0)=0$, $y'_2(0)=1$.

Now for h = 0.1, n = 10, the maximum absolute error for $S_i^{(r)}(x) - y^{(r)}$, r = 2, 3, 4, 5, at the various point of x in the given intervals [0, 1] shown in the following table:

x	$\|\|S_i^{''}(x)-y^{''}(x)\|\|_\infty$	$\|\|S_i^{'''}(x)-y^{'''}(x)\|\|_\infty$
0.0	0.0	0.0
0.1	$5.35379048429003\times 10^{-2}$	$1.003737228633076\times 10^{-1}$
0.2	$1.045367361115452\times 10^{-1}$	$2.007974687244833\times 10^{-1}$
0.3	$1.524943122600346\times 10^{-1}$	$3.002097805725068\times 10^{-1}$
0.4	$1.969476790505222\times 10^{-1}$	$3.975302295108122\times 10^{-1}$
0.5	$2.374709884892816\times 10^{-1}$	$4.916615738070577\times 10^{-1}$
0.6	$2.736722568115483\times 10^{-1}$	$5.814922573157195\times 10^{-1}$
0.7	$3.051889721898444\times 10^{-1}$	$6.658991721068873\times 10^{-1}$
0.8	$3.316825269431578\times 10^{-1}$	$7.437506089472701\times 10^{-1}$
0.9	$3.528314518907330\times 10^{-1}$	$8.139093181808205\times 10^{-1}$
1.0	$3.683234335360309\times 10^{-1}$	$8.752356025474408\times 10^{-1}$

x	$\|\|S_i^{(4)}(x)-y^{(4)}(x)\|\|_\infty$	$\|\|S_i^{(5)}(x)-y^{(5)}(x)\|\|_\infty$
0.0	0.0	0.0
0.1	$5.353790498429003\times 10^{-2}$	$1.003737228633076\times 10^{-1}$
0.2	$1.045367361115452\times 10^{-1}$	$2.007974687244833\times 10^{-1}$
0.3	$1.524943122600346\times 10^{-1}$	$3.002097805725068\times 10^{-1}$
0.4	$1.969476790505222\times 10^{-1}$	$3.975302295108122\times 10^{-1}$
0.5	$2.374709884892816\times 10^{-1}$	$4.916615738070577\times 10^{-1}$
0.6	$2.736722568115483\times 10^{-1}$	$5.814922573157195\times 10^{-1}$
0.7	$3.051889721898444\times 10^{-1}$	$6.658991721068873\times 10^{-1}$
0.8	$3.316825269431578\times 10^{-1}$	$7.437506089472701\times 10^{-1}$
0.9	$3.528314518907330\times 10^{-1}$	$8.139093181808205\times 10^{-1}$
1.0	$3.683234335360309\times 10^{-1}$	$8.752356025474408\times 10^{-1}$

Example 2: Consider a boundary value problem

y''(x)+y(x)+=-1, where $x\in[0,1]$ and y(0)=0, y(1)=1.

Solution: Since *y''(x)+y(x)*= -1,and from comparing this equation with (2.13) we find $f_1(x)$=0, $f_2(x)$=1, and *r(x)*= -1.

So transforming the given boundary value problem to initial value problems:

$y''_1(x)+y_1(x)$= -1, $y_1(0)$=0, $y'_1(0)$=0, and

$y''_2(x)+y_2(x)$=0, $y_2(0)$=0, $y'_2(0)$=1.

Now for h = 0.1, n = 10, the maximum absolute error for $S_i^{(r)}(x)-y^{(r)}$, r = 2, 3, 4, 5, at the various point of x in the given intervals [0, 1] shown in the following table:

x	$\|\|S_i''(x)-y''(x)\|\|_\infty$	$\|\|S_i'''(x)-y'''(x)\|\|_\infty$
0.0	0.0	0.0
0.1	$9.98318033470838\times 10^{-4}$	$9.982834169437509\times 10^{-2}$
0.2	$3.973157320853875\times 10^{-3}$	$1.986269328801009\times 10^{-1}$
0.3	$8.864682354346639\times 10^{-3}$	$2.953670046023819\times 10^{-1}$
0.4	$1.557486511504746\times 10^{-2}$	$3.890216806984618\times 10^{-1}$
0.5	$2.397002621000374\times 10^{-2}$	$4.785667484885327\times 10^{-1}$
0.6	$3.388438304939456\times 10^{-2}$	$5.629812188558353\times 10^{-1}$
0.7	$4.512459491573953\times 10^{-2}$	$6.412476059044188\times 10^{-1}$
0.8	$5.747529238232568\times 10^{-2}$	$7.123518567188055\times 10^{-1}$
0.9	$7.070557571229692\times 10^{-2}$	$7.752828621691926\times 10^{-1}$
1.0	$8.457646410190961\times 10^{-2}$	$8.290314802085675\times 10^{-1}$
x	$\|\|S_i^{(4)}(x)-y^{(4)}(x)\|\|_\infty$	$\|\|S_i^{(5)}(x)-y^{(5)}(x)\|\|_\infty$
0.0	0.0	0.0
0.1	$9.98318033470838\times 10^{-4}$	$9.982834169437509\times 10^{-2}$

0.2	$3.973157320853875 \times 10^{-3}$	$1.986269328801009 \times 10^{-1}$
0.3	$8.864682354346639 \times 10^{-3}$	$2.953670046023819 \times 10^{-1}$
0.4	$1.557486511504746 \times 10^{-2}$	$3.890216806984618 \times 10^{-1}$
0.5	$2.397002621000374 \times 10^{-2}$	$4.785667484885327 \times 10^{-1}$
0.6	$3.388438304939456 \times 10^{-2}$	$5.629812188558353 \times 10^{-1}$
0.7	$4.512459491573953 \times 10^{-2}$	$6.412476059044188 \times 10^{-1}$
0.8	$5.747529238232568 \times 10^{-2}$	$7.123518567188055 \times 10^{-1}$
0.9	$7.070557571229692 \times 10^{-2}$	$7.752828621691926 \times 10^{-1}$
1.0	$8.457646410190961 \times 10^{-2}$	$8.290314802085675 \times 10^{-1}$

3 Conclusion

In this paper we conclude that this construction which defined in section two of the type of (0, 1, 3, 6) lacunary interpolation by seven degree solved numerically boundary value problem.

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