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Fixed Point Theorems for Expansion Mappings in Cone Rectangular Metric Spaces

Sunanda R. Patil[1] and J.N. Salunke[2]

[1]Department of Mathematics, K.C.E. Society's
College of Engineering and Information Technology,
Jalgaon-425001, India
E-mail: sunanda.patil27@gmail.com
[2]Department of Mathematics,
Swami Ramanand Teerth Marathwada University,
Nanded-431606, India
E-mail: drjnsalunke@gmail.com

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Abstract

In this paper we prove some fixed point theorems for mappings satisfying expansive conditions in cone rectangular metric spaces.

Keywords: *Cone rectangular metric space, fixed point, weakly compatible mapping, expansion mapping.*

1 Introduction

L.G. Huang and X. Zhang in [6]introduced cone metric spaces. Later, Rezapour and Hamlbrani [10] proved results in [6] removing the condition of normality of the underlying cone.

Following A.Branciari[4],cone rectangular metric spaces were introduced by A.Azam,M.Arshad and I.Beg [1]in which they replaced the triangular inequality in a metric by the rectangular inequality.Further Kannan's fixed point theorem,Reich type contraction and more results were proved in [5],[7],[8] and [11]for these spaces.

Many authors,[3],[12],[13],[14] have obtained coincidence point and fixed point results for mappings satisfying expansive type conditions in cone metric spaces. We extend those results to the cone rectangular metric space.

2 Preliminaries

Definition 2.1 [6] *Let E be a real Banach space and P a subset of $E.P$ is called a cone if and only if:*

(i) P *is closed, nonempty,and* $P \neq \{\theta\}$.

(ii) $a, b \in \mathbb{R}, a, b \geq 0, x, y \in P \Rightarrow ax + by \in P$.

(iii) $x \in Pand - x \in P \Rightarrow x = \theta$.

Given a cone $P \subset E$ we define a partial ordering $\leq$ with respect to P by:

$$x \leq y \Leftrightarrow y - x \in P$$

We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}P$,int P denotes the interior of P.

The cone P is called normal if there is a number $k > 0$ such that for all $x, y \in E$,

$$\theta \leq x \leq y \Rightarrow \|x\| \leq k \|y\|$$

where$\|.\|$ is the norm in E.Here number k is called the normal constant of P.

In the following we always suppose that E is a Banach space,P is a solid cone in E with int$P \neq \phi$ and $\leq$ is partial ordering with respect to P.

Definition 2.2 [1] *Let X be a nonempty set.If the mapping $\rho : X \times X \to E$ satisfies:*

(a) $\theta < \rho(x, y)$ *for all* $x, y \in X, x \neq y$ *and* $\rho(x, y) = \theta$ *if and only if* $x = y$.

(b) $\rho(x, y) = \rho(y, x)$ *for all* $x, y \in X$.

(c) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$, *for all* $x, y, z \in X$

Then (X, ρ) *is a cone metric space.*

The following remark will be useful in proving the results which follow:

Remark 2.3 [9] *Let P be a cone in a real Banach space E and let $a, b, c \in P$, then,*
(a)If $a \leq b$ *and* $b \ll c$, *then* $a \ll c$.
(b)If $a \ll b$ *and* $b \ll c$,*then* $a \ll c$.

(c)If $\theta \leq u \ll c$, for each $c \in P^0$, then $u = \theta$
(d)If $c \in P^0$ and $a_n \to \theta$,then there exists,$n_0 \in \mathbb{N}$ such that for all $n > n_0$, we have $a_n \ll c$.
(e)If $\theta \leq a_n \leq b_n$, for each n and $a_n \to a, b_n \to b$, then $a \leq b$.
(f)If $a \leq \lambda a$, where $0 < \lambda < 1$, then $a = \theta$.

The concept of cone metric spaces is more general than that of metric spaces since each metric space is a cone metric space with $E = \mathbb{R}$ and $P = [0, +\infty)$.

Definition 2.4 [1] *Let X be a nonempty set.If the mapping $d : X \times X \to E$ satisfies:*

(a) $\theta < d(x, y)$ for all $x, y \in X, x \neq y$ and $d(x, y) = \theta$ if and only if $x = y$.

(b) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(c) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ for all $x, y \in X$ and for all distinct points $u, v \in X \setminus \{x, y\}$ { rectangular property }.

Here d is called a cone rectangular metric on X, and $(X, d)$is called a cone rectangular metric space.

Example 2.5 [7] *Let $X = \mathbb{R}, E = \mathbb{R}^2$ and $P = \{(x, y) : x, y \geq 0\}$
Define $d : X \times X \to E$ as follows:*

$$d(x,y) = \begin{cases} (0,0) & \text{if } x = y; \\ (3a,3) & \text{if } x \text{ and } y \text{ are both in } \{1,2\}, x \neq y; \\ (a,1) & \text{if } x \text{ and } y \text{ are not both at a time in } \{1,2\}, x \neq y \end{cases}$$

*where $a > 0$ is a constant.Then (X, d) is a cone rectangular metric space.
But it is not a cone metric space since $d(1, 2) = (3a, 3) > d(1, 3) + d(3, 2) = (2a, 2)$,the triangle inequality does not hold true.*

Example 2.6 [9] *Let $X = \mathbb{N}, E = \mathbb{C}^1_{\mathbb{R}}[0, 1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and $P = \{x \in E : x(t) \geq 0\}$for $t \in [0, 1]$.Then this cone is not normal.
Define $d : X \times X \to E$ as follows:*

$$d(x,y) = \begin{cases} 0, & \text{if } x = y; \\ 3e^t & \text{if } x \text{ and } y \text{ are both in } \{1,2\}, x \neq y; \\ e^t & \text{if } x \text{ and } y \text{ are not both at a time in } \{1,2\}, x \neq y \end{cases}$$

Then (X, d) is a cone rectangular metric space but it is not a cone metric space as it does not satisfy the triangular property.

Definition 2.7 [7]*Let* (X, d) *be a cone rectangular metric space.Let* $\{x_n\}$*be a sequence in* X *and* $x \in X$. *If for every* $c \in E, c \gg \theta$ *there is* N *such that for all* $n > N, d(x_n, x) \ll c$, *then* $\{x_n\}$ *is said to be convergent to* x *and* x *is the limit of* $\{x_n\}$.*This is denoted be* $x_n \to x$ *as* $n \to +\infty$.

Definition 2.8 [7]*Let* (X, d) *be a cone rectangular metric space,*$\{x_n\}$ *be a sequence in* X.*If for any* $c \in X$ *with* $\theta \ll c$, *there is* N *such that for all* $n, m > N, d(x_n, x_m) \ll c$, *then* $\{x_n\}$ *is called a Cauchy sequence in* X.

Definition 2.9 [7] *Let* (X, d) *be a cone rectangular metric space.If every Cauchy sequence is convergent in* X *,then* X *is called a complete cone rectangular metric space.*

Definition 2.10 *Let* (X, d) *be a cone rectangular metric space.A mapping* $T : X \to X$ *is called expansive if there exists a real constant* $k > 1$ *such that*

$$d(Tx, Ty) \geq kd(x, y)$$

for all $x, y \in X$.

Definition 2.11 [2] *Let* f *and* g *be two self maps of a nonempty set* X.*If* $fx = gx = y$ *for some* $x \in X$,*then* x *is called the coincidence point of* f *and* g *and* y *is called the point of coincidence of* f *and* g.

Definition 2.12 *Two self mappings* f *and* g *are said to be weakly compatible if they commute at their coincidence points,that is* $fx = gx$ *implies that* $fgx = gfx$.

Proposition 2.13 [2] *If* f *and* g *are weakly compatible self maps of a nonempty set* X *such that they have a unique point of coincidence i.e.*$fx = gx = y$,*then* y *is the unique common fixed point of* f *and* g.

Now, we state our main results.

3 Main Results

Theorem 3.1 *Let* (X, d) *be a cone rectangular metric space and let* $f, g : X \to X$ *be mappings which satisfy,*

$$d(fx, fy) \geq \alpha d(gx, gy) + \beta d(fx, gx) + \gamma d(fy, gy) \tag{1}$$

for all $x, y \in X$,*where* α, β *and* γ *are nonnegative real numbers with* $\alpha + \beta + \gamma > 1, \beta < 1, \gamma < 1$,*and* $\alpha > 1$.*If* $g(X) \subseteq f(X)$ *and either of* $f(X)$ *or* $g(X)$ *is complete,then* f *and* g *have a unique point of coincidence in* X.*If* f *and* g *are weakly compatible then they have a unique common fixed point in* X.

Proof: Let $x_0 \in X$,since $g(X) \subseteq f(X)$, we can choose $x_1 \in X$ such that $gx_0 = fx_1$.Continuing this process we construct a sequence $\{x_n\}$ in X such that $fx_n = gx_{n-1}$, for all $n \geq 1$.
If $gx_{n-1} = gx_n$ for some $n \geq 1$,then $fx_n = gx_n$ and x_n is a coincidence point of f and g.
Hence assume that $x_{n-1} \neq x_n$ for all $n \geq 1$.
By equation (1), we have

$$\begin{aligned} d(gx_{n-1}, gx_n) &= d(fx_n, fx_{n+1}) \\ &\geq \alpha d(gx_n, gx_{n+1}) + \beta d(fx_n, gx_n) + \gamma d(fx_{n+1}, gx_{n+1}) \\ &\geq \alpha d(gx_n, gx_{n+1}) + \beta d(gx_{n-1}, gx_n) + \gamma d(gx_n, gx_{n+1}) \end{aligned}$$

i.e.

$$d(gx_n, gx_{n+1}) \leq \frac{1-\beta}{\alpha+\gamma} d(gx_{n-1}, gx_n)$$

Hence,

$$d(gx_n, gx_{n+1}) \leq \lambda d(gx_{n-1}, gx_n)$$

where $\lambda = \frac{1-\beta}{\alpha+\gamma} \in (0, 1)$.
By induction we get,

$$d(gx_n, gx_{n+1}) \leq \lambda^n d(gx_0, gx_1) \tag{2}$$

for all $n \geq 0$.
Consider,

$$\begin{aligned} d(gx_{n-1}, gx_{n+1}) &= d(fx_n, fx_{n+2}) \\ &\geq \alpha d(gx_n, gx_{n+2}) + \beta d(fx_n, gx_n) + \gamma d(fx_{n+2}, gx_{n+2}) \\ &\geq \alpha d(gx_n, gx_{n+2}) + \beta d(gx_{n-1}, gx_n) + \gamma d(gx_{n+1}, gx_{n+2}) \end{aligned}$$

Therefore,

$$\begin{aligned} \alpha d(gx_n, gx_{n+2}) &\leq d(gx_{n-1}, gx_{n+1}) - \beta d(gx_{n-1}, gx_n) - \gamma d(gx_{n+1}, gx_{n+2}) \\ &\leq d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+2}) + d(gx_{n+2}, gx_{n+1}) \\ &- \beta d(gx_{n-1}, gx_n) - \gamma d(gx_{n+1}, gx_{n+2}) \end{aligned}$$

Hence,

$$d(gx_n, gx_{n+2}) \leq \frac{1-\beta}{\alpha-1} d(gx_{n-1}, gx_n) + \frac{1-\gamma}{\alpha-1} d(gx_{n+1}, gx_{n+2})$$

i.e.

$$d(gx_n, gx_{n+2}) \leq a_1 d(gx_{n-1}, gx_n) + a_2 d(gx_{n+1}, gx_{n+2}) \tag{3}$$

where $a_1 = \frac{1-\beta}{\alpha-1} > 0$, $a_2 = \frac{1-\gamma}{\alpha-1} > 0$
For the sequence $\{gx_n\}$, we consider $d(gx_n, gx_{n+p})$ in two cases,p is even and p is odd.
Suppose p is even,let $p = 2m$,$m \geq 2$,then by (2),(3) and the rectangular inequality, we have,

$$\begin{aligned} d(gx_n, gx_{n+2m}) &\leq d(gx_n, gx_{n+2}) + d(gx_{n+2}, gx_{n+3}) + ... + d(gx_{n+2m-1}, gx_{n+2m}) \\ &\leq a_1 d(gx_{n-1}, gx_n) + a_2 d(gx_{n+1}, gx_{n+2}) + d(gx_{n+2}, gx_{n+3}) + \\ &\quad ... + d(gx_{n+2m-1}, gx_{n+2m}) \\ &\leq a_1 \lambda^{n-1} d(gx_0, gx_1) + a_2 \lambda^{n+1} d(gx_0, gx_1) + \lambda^{n+2} d(gx_0, gx_1) + \\ &\quad ... + \lambda^{n+2m-1} d(gx_0, gx_1) \\ &\leq a_1 \lambda^{n-1} d(gx_{n-1}, gx_n) + a_2 \lambda^{n+1} d(gx_{n+1}, gx_{n+2}) + \frac{\lambda^{n+2}}{1-\lambda} d(gx_0, gx_1) \end{aligned}$$

Suppose p is odd,let $p = 2m + 1$,$m \geq 1$,then by (2) and the rectangular inequality, we have,

$$\begin{aligned} d(gx_n, gx_{n+2m+1}) &\leq d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) + ... + d(gx_{n+2m}, gx_{n+2m+1}) \\ &\leq \lambda^n d(gx_0, gx_1) + \lambda^{n+1} d(gx_0, gx_1) + ... + \lambda^{n+2m} d(gx_0, gx_1) \\ &\leq \frac{\lambda^n}{1-\lambda} d(gx_0, gx_1) \end{aligned}$$

As $a_1, a_2 > 0$ and $\lambda \in (0, 1)$,$a_1 \lambda^{n-1} d(gx_0, gx_1) \to \theta$,$a_2 \lambda^{n+1} d(gx_0, gx_1) \to \theta$, $\frac{\lambda^{n+2}}{1-\lambda} d(gx_0, gx_1) \to \theta$,$\frac{\lambda^n}{1-\lambda} d(gx_0, gx_1) \to \theta$ as $n \to \infty$,so by(a) and (d)of Remark (2.3),for every $c \in E$ with $\theta \ll c$, there exits $n_0 \in \mathbb{N}$ such that $d(gx_n, gx_{n+p}) \ll c$ for all $n > n_0$.

Hence, $\{gx_n\}$ is a Cauchy sequence.Suppose $g(X)$ is a complete subspace of X, there exists $y \in g(X) \subseteq f(X)$ such that $gx_n \to y$ and also $fx_n \to y$, and if $f(X)$ is complete, this holds also with $y \in f(X)$.

Let $u \in X$,be such that $fu = y$.For $\theta \ll c$, we can choose a natural number $n_0 \in \mathbb{N}$,such that $d(y, gx_{n-1}) \ll \frac{c}{3}$,$d(gx_{n-1}, gx_n) \ll \frac{c}{3}$ and $d(fx_n, fu) \ll \frac{\alpha c}{3}$ for all $n > n_0$
We have by (1),

$$\begin{aligned} d(gx_{n-1}, fu) &= d(fx_n, fu) \\ &\geq \alpha d(gx_n, gu) + \beta d(fx_n, gx_n) + \gamma d(fu, gu) \\ &\geq \alpha d(gx_n, gu) \end{aligned}$$

i.e.

$$d(gx_n, gu) \leq \frac{1}{\alpha} d(gx_{n-1}, fu)$$

By the rectangular inequality,

$$\begin{aligned} d(y, gu) &\leq d(y, gx_{n-1}) + d(gx_{n-1}, gx_n) + d(gx_n, gu) \\ &\leq d(y, gx_{n-1}) + d(gx_{n-1}, gx_n) + \frac{1}{\alpha} d(gx_{n-1}, fu) \\ &\leq d(y, gx_{n-1}) + d(gx_{n-1}, gx_n) + \frac{1}{\alpha} d(fx_n, fu) \end{aligned}$$

Thus,

$$d(y, gu) \ll \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c$$

for all $n > n_0$ and $gu = y$,hence $fu = gu = y$, which means that y is a coincidence point of f and g.
Suppose there exists another point of coincidence y^*, such that $gu^* = fu^* = y^*$ for some $u^* \in X$. Then,

$$\begin{aligned} d(y, y^*) &= d(fu, fu^*) \\ &\geq \alpha d(gu, gu^*) + \beta d(fu, gu) + \gamma d(fu^*, gu^*) \\ &\geq \alpha d(y, y^*) + \beta d(y, y) + \gamma d(y^*, y^*) \end{aligned}$$

Hence,

$$d(y, y^*) \leq \frac{1}{\alpha} d(y, y^*)$$

Since $\alpha > 1$, we have by Remark(2.3)(f),$d(y, y^*) = \theta$ i,e,$y = y^*$.Therefore f and g have a unique point of coincidence in X.If f and g are weakly compatible, then by Proposition (2.13), f and g have a unique common fixed point in X. □

Corollary 3.2 *Let (X, d) be a complete cone rectangular metric space and let $f, g : X \to X$ be mappings which satisfy,*

$$d(fx, fy) \geq \alpha d(gx, gy) \tag{4}$$

for all $x, y \in X$,where $\alpha > 1$ is a constant.If $g(X) \subseteq f(X)$ and either of $f(X)$ or $g(X)$ is complete,then f and g have a unique point of coincidence in X.If f and g are weakly compatible then they have a unique common fixed point in X.

Proof: Taking $\beta = \gamma = 0$ in Thm.(3.1), we get the result. $\square$

Example 3.3 *Let* $X = \{1, 2, 3, 4\}, E = R^2$ *and* $P = \{(x, y) : x, y \in X\}$ *be a cone in* E.
Define $d : X \times X \to E$ *as follows:*
$d(1, 2) = d(2, 1) = (3, 6)$
$d(2, 3) = d(3, 2) = d(1, 3) = d(3, 1) = (1, 2)$
$d(1, 4) = d(4, 1) = d(2, 4) = d(4, 2) = d(3, 4) = d(4, 3) = (2, 4)$
then (X, d) *is a cone rectangular metric space but not a cone metric space because it lacks the triangular property as*
$(3, 6) = d(1, 2) > d(1, 3) + d(3, 2) = (1, 2) + (1, 2) = (2, 4)$
since $(3, 6) - (2, 4) = (1, 2) \in P$.
Now define mappings $f, g : X \to X$ *as follows:*

$$fx = x \text{ for all } x \in X.$$

$$g(x) = \begin{cases} 3 & \text{if } x \neq 4; \\ 1 & \text{if } x = 4; \end{cases}$$

All conditions of Thm.(3.1) *hold for* $\alpha \in (1, 2], \beta = 0$ *and* $\gamma = 0$,$3 \in X$ *is the unique common fixed point of* f *and* g. $\square$

Corollary 3.4 *Let* (X, d) *be a complete cone rectangular metric space and let* $f : X \to X$ *be onto mapping which satisfies,*

$$d(fx, fy) \geq \alpha d(x, y) + \beta d(fx, x) + \gamma d(fy, y) \tag{5}$$

for all $x, y \in X$,*where* α, β *and* γ *are nonnegative real numbers with* $\alpha + \beta + \gamma > 1, \beta < 1, \gamma < 1$,*and* $\alpha > 1$.*Then* f *has a unique fixed point in* X.

Proof: It follows by taking $g = I$ in Thm.(3.1). $\square$

Corollary 3.5 *Let* (X, d) *be a complete cone rectangular metric space and let* $f : X \to X$ *be onto mapping which satisfies,*

$$d(fx, fy) \geq \alpha d(x, y) \tag{6}$$

for all $x, y \in X$,*where* $\alpha > 1$ *is a constant.Then* f *has a unique fixed point in* X.

Proof: It follows by taking $g = I$ and $\beta = \gamma = 0$ in Thm.(3.1). $\square$

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