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Lacunary Weak I-Statistical Convergence

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Abstract

In this study, we provide a new approach to $I-$ statistical convergence. We introduce a new concept with $I-$ statistical convergence and weak convergence together and we call it weak $I-$statistical convergence or $WS(I)-$convergence. Then we introduce this concept for lacunary sequences and we obtain lacunary weak I- statistical convergence i.e. $WS_\theta(I)-$convergence. $WN_\theta(I)-$convergence is any other definition in our study. After giving this description, we investigate their relationship and we have some results.

Keywords: *I-statistical convergence, weak statistical convergence, lacunary sequence.*

1 Introduction

In this area, statistical convergence is an important concept and Zygmund [15] gave it in the first edition of his monograph published in Warsaw in 1935. It was formally introduced by Fast and Steinhaus [5, 14] and later was reintroduced by Schoenberg. [13] This concept has a wide application area for example number theory [4], measure theory [10], trigonometric series [15], summability theory [6], etc. Fridy gave important properties about statistical convergence in his study [7], Fridy and Orhan studied statistical convergence with lacunary sequences. [8].

Let K be a subset of the set of all natural numbers N and $K_n = \left|\{k \le n : k \in K\}\right|$ where the vertical bars indicate the number of elements in the enclosed set. The natural density of K is defined by $\delta(K) = \lim_{n\to\infty} \frac{1}{n} \left|\{k \le n : k \in K\}\right|$. If a property $P(k)$ holds for all $k \in A$ with $\delta(A) = 1$ we say that P holds for almost all k that is a.a.k.

Definition 1.1: *[14] A number sequence $x = (x_k)$ is statistically convergent to x provided that for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k \le n : |x_k - x| \ge \varepsilon\}\right| = 0.$$

In this case we write $st - \lim x_k = x$.

Statistical convergence was extended to $I-$convergence in a metric space in Kostyrko, Salát and Wilezyński's study. [9]

Definition 1.2: *A family of sets $I \subseteq 2^{\mathrm{N}}$ is called an ideal if and only if*

- *(i) $\phi \in I$*
- *(ii) For each $A, B \in I$ we have $A \cup B \in I$*
- *(iii) For each $A \in I$ and each $B \subseteq A$ we have $B \in I$*

An ideal is called non-trivial if $\mathrm{N} \notin I$ and a non-trivial ideal is called admissible if $\{n\} \in I$ for each $n \in \mathrm{N}$.

Definition 1.3: *A family of sets $F \subseteq 2^{\mathrm{N}}$ is called a filter in N if and only if*

- *(i) $\phi \notin F$*
- *(ii) For each $A, B \in F$ we have $A \cap B \in F$*
- *(iii) For each $A \in F$ and each $B \supseteq A$ we have $B \in F$*

Proposition 1.1 *I is a non-trivial ideal in N if and only if*

$$F = F(I) = \{M = N \setminus \mathrm{A} : \mathrm{A} \in \mathrm{I}\}$$

is a filter in N.

Throughout the paper, I will be an admissible ideal.

Definition 1.4: *A real sequence* $x=(x_k)$ *is said to be* $I-$ *convergent to* $L\in\Re$ *if and only if for each* $\varepsilon>0$ *the set*

$$A_\varepsilon=\{k\in \mathrm{N}:|x_k-L|\geq\varepsilon\}$$

belongs to I. *The number* L *is called the* $I-$*limit of the sequence x.*

Example 1.1: *Take for I class the* I_f *of all finite subsets of N. Then* I_f *is an admissible ideal and* I_f-*convergence coincides with the usual convergence.*

In 2011, Das, Savas and Ghosal [3] have introduced the concept of $I-$ statistical convergence and $I-$ lacunary statistical convergence.

Definition 1.5: *[3] A sequence* $x=(x_k)$ *is said to be* $I-$ *statistically convergent to* L *for each* $\varepsilon>0$ *and* $\delta>0$,

$$\left\{n\in \mathrm{N}:\frac{1}{n}\left|\{k\leq n:|x_k-L|\geq\varepsilon\}\right|\geq\delta\right\}\in I .$$

Example 1.2: *Let us take the sequence* (y_n) *where* $y_n=\begin{cases}1, & n=1 \text{ to } 10\\ n-10, & n\geq 10\end{cases}$ *and the ideal* I_d *which is the ideal of density zero sets of* N. *Let* $A=\{1^2,2^2,3^2,...\}$. *Define* $x=(x_k)$ *in a normed linear space X by,*

$$x_k=\begin{cases}ku, & \text{for } n-[\sqrt{y_n}]+1\leq k\leq n,\ n\notin A\\ ku, & \text{for } n-y_n+1\leq k\leq n,\ n\in A\\ \theta, & \text{otherwise}\end{cases}$$

where $u\in X$ *is a fixed element with* $\|u\|=1$ *and* θ *is the null element of* X. *Then the sequence* $x=(x_k)$ *is* $I-$ *statistically convergent but it is not statistically convergent.*

Now, we will give the definition of $I-$ lacunary statistically convergent sequences from the paper of Das, Savas and Ghosal. But first, we need to remind lacunary sequence.

Definition 1.6: *A lacunary sequence is an increasing integer sequence* $\theta=(k_r)$ *such that* $k_0=0$ *and* $h_r=k_r-k_{r-1}\to\infty$ *as* $r\to\infty$. *The intervals determined by* θ *will be denoted by* $J_r=(k_{r-1},k_r]$ *and the ratio* $\frac{k_r}{k_{r-1}}$ *will be denoted by* q_r.

Definition 1.7: *[3] Let θ be a lacunary sequence. A sequence $x=(x_k)$ is said to be $I-$ lacunary statistically convergent to L for each $\varepsilon>0$ and $\delta>0$,*

$$\left\{ r\in \mathrm{N} : \frac{1}{h_r}\left|\left\{k\in J_r : \left|x_k - L\right|\geq\varepsilon\right\}\right|\geq\delta\right\}\in I.$$

Let's continue to remind important concepts that we need for our study.

Definition 1.8: *Let B be a Banach space, $x=(x_k)$ be a B-valued sequence and $x\in B$. The sequence $x=(x_k)$ is weakly convergent to x provided that for any f in the continuous dual B^* of B,*

$$\lim_k f(x_k - x) = 0$$

and in this case we write $w-\lim x_k = x$.

Definition 1.9: *Let B be a Banach space, $x=(x_k)$ be a B-valued sequence and $x\in B$. The sequence $x=(x_k)$ is weakly C_1-convergent to x provided that for any f in the continuous dual B^* of B,*

$$\lim_n \frac{1}{n}\sum_{k=1}^{n} f(x_k - x) = 0$$

In 2000, Connor et al. [2], have introduced a new concept of weak statistical convergence and have characterized Banach spaces with seperable duals via statistical convergence. Pehlivan and Karaev [12] have also used the idea of weak statistical convergence in strengthening a result of Gokhberg and Klein on compact operators. Bhardwaj and Bala have investigated some relations between weak convergent sequences and weakly statistically convergent sequences [1].

Following Connor et al. we define weak statistical convergence as follows:

Definition 1.10: *[2] Let B be a Banach space, $x=(x_k)$ be a B-valued sequence and $x\in B$. The sequence $x=(x_k)$ is weakly statistically convergent to x provided that for any f in the continuous dual B^* of B the sequence ($f(x_k - x)$) is statistically convergent to x i.e.*

$$\lim_n \frac{1}{n}\left|\left\{k\leq n : \left|f(x_k - x)\right|\geq\varepsilon\right\}\right| = 0$$

and in this case we write $W-st-\lim x_k = x$.

It is easy to see that the weak statistical limit of a weakly statistically convergent sequence is unique.

In 2011, Nuray [11] studied weak statistical convergence by using lacunary sequences.

Definition 1.11: *Let B be a Banach space, $x=(x_k)$ be a B-valued sequence, θ be a lacunary sequence and $x\in B$. $x=(x_k)$ is weakly lacunary statistically convergent to x or $WS_\theta -$ convergent to x provided that for any f in the continuous dual B^* of B,*

$$\lim_r \frac{1}{h_r}\left|\left\{k\in J_r : \left| f(x_k - x)\right| \geq \varepsilon\right\}\right| = 0.$$

2 Lacunary Weak *I*- Statistical Convergence

Definition 2.1: *Let B be a Banach space, $x=(x_k)$ be a B-valued sequence and $x\in B$. The sequence $x=(x_k)$ is weakly $I-$ convergent to x provided that for any f in the continuous dual B^* of B,*

$$\left\{k\in \mathrm{N} : \left| f(x_k - x)\right| \geq \varepsilon\right\}\in I.$$

The set of all weakly $I-$ convergent sequences is denoted by WI and if we take $I = I_f$ the ideal of all finite subsets of N, we have the usual weak convergence.

Example 2.1: *I_d is an admissible ideal and $WI_d -$convergence coincides with the weak statistical convergence.*

Example 2.2: *Denote by I_θ the class of all $K\subset \mathrm{N}$ with*

$$\lim_r \frac{1}{h_r}\left|\left\{k\in J_r : k\in K\right\}\right| = 0.$$

Then I_θ is an admissible ideal and $WI_\theta -$ convergence coincides with the lacunary weak statistical convergence.

We now introduce our main definitions.

Definition 2.2: *Let B be a Banach space, $x=(x_k)$ be a B-valued sequence and $x\in B$. The sequence $x=(x_k)$ is weakly $I-$ statistically convergent to x provided that for any f in the continuous dual B^* of B and every $\varepsilon>0$ and $\delta>0$,*

$$\left\{n\in \mathrm{N} : \frac{1}{n}\left|\left\{k\leq n : \left| f(x_k - x)\right| \geq \varepsilon\right\}\right| \geq \delta\right\}\in I.$$

The set of all weakly $I-$ *statistically convergent sequences is denoted by* $WS(I)$.

Definition 2.3: *Let B be a Banach space,* $x=(x_k)$ *be a B-valued sequence,* $x\in B$ *and* $\theta=(k_r)$ *be a lacunary sequence. The sequence* $x=(x_k)$ *is lacunary weak* $I-$ *statistically convergent to x provided that for any f in the continuous dual* B^* *of B and every* $\varepsilon>0$ *and* $\delta>0$,

$$\left\{r\in \mathbb{N} : \frac{1}{h_r}\left|\left\{k\in J_r : \left|f(x_k-x)\right|\geq\varepsilon\right\}\right|\geq\delta\right\}\in I.$$

The set of all lacunary weak $I-$ statistically convergent sequences is denoted by $WS_\theta(I)$.

Definition 2.4: *Let B be a Banach space,* $x=(x_k)$ *be a B-valued sequence,* $x\in B$ *and* $\theta=(k_r)$ *be a lacunary sequence. The sequence* $x=(x_k)$ *is* $WN_\theta(I)-$*convergent to x provided that for any f in the continuous dual* B^* *of B and every* $\varepsilon>0$,

$$\left\{r\in \mathbb{N} : \frac{1}{h_r}\sum_{k\in J_r}\left|f(x_k-x)\right|\geq\varepsilon\right\}\in I.$$

Theorem 2.1: *Let* $\theta=(k_r)$ *be a lacunary sequence. Then* (x_k) *is* $WN_\theta(I)-$ *convergent to x if and only if* (x_k) *is* $WS_\theta(I)-$*convergent to x.*

Proof: Assume that (x_k) is $WN_\theta(I)-$ convergent to x and $\varepsilon>0$. We can write,

$$\frac{1}{h_r}\sum_{k\in J_r}\left|f(x_k-x)\right|\geq\frac{1}{h_r}\sum_{k\in J_r \text{ and } |f(x_k-x)|\geq\varepsilon}\left|f(x_k-x)\right|$$

$$\geq\frac{\varepsilon}{h_r}\left|\left\{k\in J_r : \left|f(x_k-x)\right|\geq\varepsilon\right\}\right|$$

Then,

$$\frac{1}{\varepsilon h_r}\sum_{k\in J_r}\left|f(x_k-x)\right|\geq\frac{1}{h_r}\left|\left\{k\in J_r : \left|f(x_k-x)\right|\geq\varepsilon\right\}\right|$$

and for any $\delta>0$,

$$\left\{r\in \mathbb{N} : \frac{1}{h_r}\left|\left\{k\in J_r : \left|f(x_k-x)\right|\geq\varepsilon\right\}\right|\geq\delta\right\}\subseteq\left\{r\in \mathbb{N} : \frac{1}{h_r}\sum_{k\in J_r}\left|f(x_k-x)\right|\geq\varepsilon\delta\right\}.$$

We know that the right side is in ideal. So, the left side is also in ideal.

Now suppose that (x_k) is $WS_\theta(I)-$convergent to x. Since $f \in B^*$, f is bounded. Then there exists a $K \geq 0$ for all $k \in \mathrm{N}$ such that $|f(x_k - x)| \leq K$. Given $\varepsilon > 0$, we get,

$$\frac{1}{h_r}\sum_{k\in J_r}|f(x_k - x)| = \frac{1}{h_r}\sum_{k\in J_r \text{ and } |f(x_k-x)|\geq\frac{\varepsilon}{2}}|f(x_k - x)| + \frac{1}{h_r}\sum_{k\in J_r \text{ and } |f(x_k-x)|<\frac{\varepsilon}{2}}|f(x_k - x)|$$

$$\leq K\frac{1}{h_r}\left|\left\{k \in J_r : |f(x_k - x)| \geq \frac{\varepsilon}{2}\right\}\right| + \frac{\varepsilon}{2}.$$

Consequently we have,

$$\left\{r \in \mathrm{N} : \frac{1}{h_r}\sum_{k\in J_r}|f(x_k - x)| \geq \varepsilon\right\} \subseteq \left\{r \in \mathrm{N} : \frac{1}{h_r}\left|\left\{k \in J_r : |f(x_k - x)| \geq \frac{\varepsilon}{2}\right\}\right| \geq \frac{\varepsilon}{2K}\right\} \in I.$$

Theorem 2.2: *Let $\theta = (k_r)$ be a lacunary sequence with* $\liminf q_r > 1$. *Then $WS(I)-$convergence implies $WS_\theta(I)-$convergence.*

Proof: Assume that $\liminf q_r > 1$. Then there exists an $\alpha > 0$ such that $q_r \geq 1 + \alpha$ for all sufficiently large r. This implies $\frac{h_r}{k_r} \geq \frac{\alpha}{1+\alpha}$. Since (x_k) is $WS(I)-$ convergent to x, for every $\varepsilon > 0$ and sufficiently large r we have,

$$\frac{1}{k_r}\left|\left\{k \leq k_r : |f(x_k - x)| \geq \varepsilon\right\}\right| \geq \frac{1}{k_r}\left|\left\{k \in J_r : |f(x_k - x)| \geq \varepsilon\right\}\right|$$

$$\geq \frac{\alpha}{1+\alpha}\frac{1}{h_r}\left|\left\{k \in J_r : |f(x_k - x)| \geq \varepsilon\right\}\right|.$$

Then for any $\delta > 0$ we get

$$\left\{r \in \mathrm{N} : \frac{1}{h_r}\left|\left\{k \in J_r : |f(x_k - x)| \geq \varepsilon\right\}\right| \geq \delta\right\} \subseteq \left\{r \in \mathrm{N} : \frac{1}{k_r}\left|\left\{k \leq k_r : |f(x_k - x)| \geq \varepsilon\right\}\right| \geq \frac{\delta\alpha}{1+\alpha}\right\} \in I.$$

This proves the theorem.

Theorem 2.3: *Let $\theta = (k_r)$ be a lacunary sequence with* $\limsup q_r < \infty$. Then *$WS_\theta(I)-$convergence implies $WS(I)-$convergence.*

Proof: If $\limsup q_r < \infty$ then there is a $K > 0$ such that $q_r < K$ for all r. Suppose that (x_k) is $WS_\theta(I)-$ convergent to x and $\varepsilon, \delta, \eta > 0$. Define the sets,

$$M = \left\{ r \in \mathrm{N} : \frac{1}{h_r} \left| \left\{ k \in J_r : \left| f(x_k - x) \right| \geq \varepsilon \right\} \right| < \delta \right\}$$

$$R = \left\{ n \in \mathrm{N} : \frac{1}{n} \left| \left\{ k \leq n : \left| f(x_k - x) \right| \geq \varepsilon \right\} \right| < \eta \right\}.$$

Let $F(I)$ be the filter associated with the ideal I. It is obvious that $M \in F(I)$. If we can show that $R \in F(I)$ then we will have the proof. For all $j \in M$ let,

$$A_j = \frac{1}{h_j} \left| \left\{ k \in J_j : \left| f(x_k - x) \right| \geq \varepsilon \right\} \right| < \delta.$$

Choose $n \in \mathrm{N}$ such that $k_{r-1} < n < k_r$ for some $r \in M$. Now,

$$\begin{aligned}
\frac{1}{n} \left| \left\{ k \leq n : \left| f(x_k - x) \right| \geq \varepsilon \right\} \right| &\leq \frac{1}{k_{r-1}} \left| \left\{ k \leq k_r : \left| f(x_k - x) \right| \geq \varepsilon \right\} \right| \\
&= \frac{1}{k_{r-1}} \left\{ k \in J_1 : \left| f(x_k - x) \right| \geq \varepsilon \right\} + \ldots + \frac{1}{k_{r-1}} \left\{ k \in J_r : \left| f(x_k - x) \right| \geq \varepsilon \right\} \\
&= \frac{k_1}{k_{r-1}} \frac{1}{h_1} \left\{ k \in J_1 : \left| f(x_k - x) \right| \geq \varepsilon \right\} + \frac{k_2 - k_1}{k_{r-1}} \frac{1}{h_2} \left\{ k \in J_2 : \left| f(x_k - x) \right| \geq \varepsilon \right\} + \ldots \\
&\quad + \frac{k_r - k_{r-1}}{k_{r-1}} \frac{1}{h_r} \left\{ k \in J_r : \left| f(x_k - x) \right| \geq \varepsilon \right\} \\
&= \frac{k_1}{k_{r-1}} A_1 + \frac{k_2 - k_1}{k_{r-1}} A_2 + \ldots + \frac{k_r - k_{r-1}}{k_{r-1}} A_r \\
&\leq \sup_{j \in M} A_j \frac{k_r}{k_{r-1}} \\
&< K.\delta
\end{aligned}$$

Choosing $\eta = \dfrac{\delta}{K}$ and in view of the fact that $\bigcup \{ n : k_{r-1} < n < k_r,\, r \in M \} \subset R$ then we have $R \in F(I)$.

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