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Some Generalized Difference Sequence Spaces of Non-Absolute Type

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Abstract

In this paper, we introduce the spaces $\ell_\infty(\Delta_\lambda^m)$, $c(\Delta_\lambda^m)$ and $c_0(\Delta_\lambda^m)$, which are BK-spaces of non-absolute type and we prove that these spaces are linearly isomorphic to the spaces ℓ_∞, c and c_0, respectively. Moreover, we give some inclusion relations and compute the $\alpha-$, $\beta-$ and $\gamma-$duals of these spaces. We also determine the Schauder basis of the $c(\Delta_\lambda^m)$ and $c_0(\Delta_\lambda^m)$.

Keywords: *Sequence spaces of non-absolute type, BK-spaces, Difference Sequence Spaces.*

1 Introduction

A sequence space is defined to be a linear space of real or complex sequences. Let w denote the spaces of all complex sequences. If $x \in w$, then we simply write $x = (x_k)$ instead of $x = (x_k)_{k=0}^{\infty}$.

Let X be a sequence space. If X is a Banach space and

$$\tau_k : X \to C,\ \tau_k(x) = x_k \ \ (k = 1, 2, ...)$$

is a continuous for all k, X is called a $BK-$space.

We shall write ℓ_∞, c and c_0 for the sequence spaces of all bounded, convergent and null sequences, respectively, which are $BK-$spaces with the norm given by $\|x\|_\infty = \sup_k |x_k|$ for all $k \in \mathbf{N}$.

For a sequence space X, the matrix domain X_A of an infinite matrix A defined by

$$X_A = \{x = (x_k) \in w : Ax \in X\} \tag{1}$$

which is a sequence space.

We shall denote the collection of all finite subsets of $\mathbf{N}$ by $\mathcal{F}$.

M. Mursaleen and A. K. Noman [9] introduced the sequence spaces ℓ_∞^λ, c^λ and c_0^λ as the sets of all $\lambda-bounded$, $\lambda-convergent$ and $\lambda-null$ sequences, respectively, that is

$$\begin{aligned} \ell_\infty^\lambda &= \{x \in w : \sup_n |\Lambda_n(x)| < \infty\} \\ c^\lambda &= \{x \in w : \lim_{n\to\infty} \Lambda_n(x) \ exists\} \\ c_0^\lambda &= \{x \in w : \lim_{n\to\infty} \Lambda_n(x) = 0\} \end{aligned}$$

where $\Lambda_n(x) = \frac{1}{\lambda_n} \sum\limits_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k, \quad k \in \mathbf{N}$.

M. Mursaleen and A. K. Noman [10] also introduced the sequence spaces $c^\lambda(\Delta)$ and $c_0^\lambda(\Delta)$, respectively, that is

$$\begin{aligned} c^\lambda(\Delta) &= \{x \in w : \lim_{n\to\infty} \bar{\Lambda}_n(x) \ exists\} \\ c_0^\lambda(\Delta) &= \{x \in w : \lim_{n\to\infty} \bar{\Lambda}_n(x) = 0\}. \end{aligned}$$

where $\bar{\Lambda}_n(x) = \frac{1}{\lambda_n} \sum\limits_{k=0}^{n} (\lambda_k - \lambda_{k-1})(x_k - x_{k-1}), \ k \in \mathbf{N}$.

H. Ganie and N. A. Sheikh [2] introduced the spaces $c_0(\Delta_u^\lambda)$ and $c(\Delta_u^\lambda)$ as follows:

$$\begin{aligned} c_0(\Delta_u^\lambda) &= \{x \in w : \lim_{n\to\infty} \widehat{\Lambda}_n(x) = 0\} \\ c(\Delta_u^\lambda) &= \{x \in w : \lim_{n\to\infty} \widehat{\Lambda}_n(x) \ exists\} \end{aligned}$$

where $\widehat{\Lambda}_n(x) = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) u_k (x_k - x_{k-1})$, $k \in \mathbf{N}$.

2 The Sequence Spaces $\ell_\infty(\Delta_\lambda^m)$, $c(\Delta_\lambda^m)$ and $c_0(\Delta_\lambda^m)$ of Non-Absolute Type

We define the sequence spaces $\ell_\infty(\Delta_\lambda^m)$, $c(\Delta_\lambda^m)$ and $c_0(\Delta_\lambda^m)$ as follows;

$$\begin{aligned} \ell_\infty(\Delta_\lambda^m) &= \left\{x \in w : \sup_n \left|\tilde{\Lambda}_n(x)\right| < \infty\right\} \\ c(\Delta_\lambda^m) &= \left\{x \in w : \lim_{n\to\infty} \tilde{\Lambda}_n(x) \ exists\right\} \\ c_0(\Delta_\lambda^m) &= \left\{x \in w : \lim_{n\to\infty} \tilde{\Lambda}_n(x) = 0\right\} \end{aligned}$$

where $\tilde{\Lambda}_n(x) = \frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})\Delta^m x_k,\ k,m \in \mathbf{N}$. Δ denotes the difference operator. i.e., $\Delta^0 x_k = x_k,\ \Delta x_k = x_k - x_{k-1}$ and $\Delta^m x_k = \sum_{v=0}^{m}(-1)^v \begin{pmatrix} m \\ v \end{pmatrix} x_{k-v}$. $\lambda = (\lambda_k)_{k=0}^{\infty}$ is a strictly increasing sequence of positive reals tending to infinity, that is $0 < \lambda_0 < \lambda_1 < ...$ and $\lambda_k \to \infty$ as $k \to \infty$.

Here and in sequel, we use the convention that any term with a negative subscript is equal to naught. e.g. $\lambda_{-1} = 0$ and $x_{-1} = 0$.

If we take $m = 1$ sequence spaces which we defined reduces to $\ell_\infty^\lambda(\Delta)$, $c^\lambda(\Delta)$ and $c_0^\lambda(\Delta)$.

We define the matrix $\tilde{\Lambda} = \left(\tilde{\lambda}_{nk}\right)$ for all $n,k \in \mathbf{N}$ by

$$\tilde{\lambda}_{nk} = \begin{cases} \sum\limits_{i=k}^{n} \begin{pmatrix} m \\ i-k \end{pmatrix} (-1)^{i-k} \frac{\lambda_i - \lambda_{i-1}}{\lambda_n}, & k \leq n \\ 0, & n < k \end{cases}.$$

$\tilde{\Lambda} = \left(\tilde{\lambda}_{nk}\right)$ equality can be eaisly seen from

$$\tilde{\Lambda}_n(x) = \frac{1}{\lambda_n}\sum_{k=0}^{n}\left(\lambda_k - \lambda_{k-1}\right)\Delta^m x_k \tag{2}$$

for all $m,n \in \mathbf{N}$ and every $x = (x_k) \in w$. Then it leads us together with (1) to the fact that

$$\ell_\infty\left(\Delta_\lambda^m\right) = \left(\ell_\infty\right)_{\tilde{\Lambda}},\ c_0\left(\Delta_\lambda^m\right) = \left(c_0\right)_{\tilde{\Lambda}},\ c\left(\Delta_\lambda^m\right) = (c)_{\tilde{\Lambda}}.$$

The matrix $\tilde{\Lambda} = \left(\tilde{\lambda}_{nk}\right)$ is a triangle, i.e., $\tilde{\lambda}_{nn} \neq 0$ and $\tilde{\lambda}_{nk} = 0$ $(k > n)$ for all $n,k \in \mathbf{N}$. Further, for any sequence $x = (x_k)$ we define the sequence $y(\lambda) = \{y_k(\lambda)\}$ as the $\tilde{\Lambda}$-transform of x, i.e., $y(\lambda) = \tilde{\Lambda}(x)$ and so we have that

$$y(\lambda) = \sum_{j=0}^{k}\sum_{i=j}^{k}(-1)^{i-j}\begin{pmatrix} m \\ i-j \end{pmatrix}\left(\frac{\lambda_i - \lambda_{i-1}}{\lambda_k}\right)x_j \tag{3}$$

for $k \in \mathbf{N}$. Here and in what follows, the summation running from 0 to $k-1$ is equal to zero when $k = 0$.

Theorem 2.1 $\ell_\infty(\Delta_\lambda^m), c_0(\Delta_\lambda^m)$ *and* $c(\Delta_\lambda^m)$ *are BK-spaces with the norm*

$$\|x\|_{(\ell_\infty)_{\tilde{\Lambda}}} = \left\|\tilde{\Lambda}_n(x)\right\|_\infty = \sup_n\left|\tilde{\Lambda}_n(x)\right|. \tag{4}$$

Proof: We know that c and c_0 are $BK-$spaces with their natural norms from [5]. (3) holds and $\tilde{\Lambda} = \left(\tilde{\lambda}_{nk}\right)$ is a triangle matrix and from Theorem 4.3.12 of Wilansky [1], we derive that $\ell_\infty(\Delta_\lambda^m)$, $c_0\left(\Delta_\lambda^m\right)$ and $c\left(\Delta_\lambda^m\right)$ are $BK-$spaces. This completes the proof.

Remark 2.2 *The absolute property does not hold on the* $\ell_\infty(\Delta_\lambda^m)$, $c_0(\Delta_\lambda^m)$ *and* $c(\Delta_\lambda^m)$ *spaces. For instance, if we take* $|x| = (|x_k|)$ *we hold* $\|x\|_{(\ell_\infty)_{\tilde{\Lambda}}} \neq \||x|\|_{(\ell_\infty)_{\tilde{\Lambda}}}$ *.Thus, the space* $\ell_\infty(\Delta_\lambda^m)$, $c_0(\Delta_\lambda^m)$ *and* $c(\Delta_\lambda^m)$ *are BK-space of non-absolute type.*

Theorem 2.3 *The sequence spaces* $\ell_\infty(\Delta_\lambda^m)$, $c_0(\Delta_\lambda^m)$ *and* $c(\Delta_\lambda^m)$ *of non-absolute type are linearly isomorphic to the spaces* ℓ_∞, c_0 *and* c, *respectively, that is* $\ell_\infty(\Delta_\lambda^m) \cong \ell_\infty$, $c_0(\Delta_\lambda^m) \cong c_0$ *and* $c(\Delta_\lambda^m) \cong c$.

Proof: We only consider $c_0(\Delta_\lambda^m) \cong c_0$ and others will prove similarly. To prove the theorem we must show the existence of linear bijection operator between $c_0(\Delta_\lambda^m)$ and c_0. Hence, let define the linear operator with the notation (3), from $c_0(\Delta_\lambda^m)$ and c_0 by $x \to y(\lambda) = Tx$.

Then $Tx = y(\lambda) = \tilde{\Lambda}(x) \in c_0$ for every $x \in c_0(\Delta_\lambda^m)$. Also, the linearity of T is clear. Further, it is trivial that $x = 0$ whenever $Tx = 0$. Hence T is injective.

Let $y = (y_k) \in c_0$ and define the sequence $x = \{x(\lambda)\}$ by

$$x_k(\lambda) = \sum_{j=0}^{k} \binom{m+k-j-1}{k-j} \sum_{i=j-1}^{j} (-1)^{j-i} \frac{\lambda_i}{\lambda_j - \lambda_{j-1}} y_i. \tag{5}$$

and we have

$$\Delta^m x_k = \sum_{i=k-1}^{k} (-1)^{k-i} \frac{\lambda_i}{\lambda_k - \lambda_{k-1}} y_i. \tag{6}$$

Thus, for every $k \in \mathbf{N}$, we have by (2) that

$$\tilde{\Lambda}_n(x) = \frac{1}{\lambda_n} \sum_{k=0}^{n} \sum_{i=k-1}^{k} (-1)^{k-i} \lambda_i y_i = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k y_k - \lambda_{k-1} y_{k-1}) = y_n \tag{7}$$

This shows that $\tilde{\Lambda}(x) = y$ and since $y \in c_0$, we obtain that $\tilde{\Lambda}(x) \in c_0$. Thus we deduce that $x \in c_0(\Delta_\lambda^m)$ and $Tx = y$. Hence T is surjective.

Further, we have for every $x \in c_0(\Delta_\lambda^m)$ that

$$\|Tx\|_{c_0} = \|Tx\|_{\ell_\infty} = \|y(\lambda)\|_{\ell_\infty} = \left\|\tilde{\Lambda}(x)\right\|_{\ell_\infty} = \|x\|_{(c_0)_{\tilde{\Lambda}}} \tag{8}$$

which means that $c_0(\Delta_\lambda^m)$ and c_0 is linearly isomorphic.

3 The Inclusion Relations

Theorem 3.1 *The inclusion* $c_0(\Delta_\lambda^m) \subset c(\Delta_\lambda^m)$ *strictly holds.*

Proof: It is clear that $c_0\left(\Delta_\lambda^m\right) \subset c\left(\Delta_\lambda^m\right)$. To show strict, consider the sequence $x=(x_k)$ defined by $x_k = k^m$ for all $k \in \mathbf{N}$. Then we obtain that

$$\tilde{\Lambda}_n(x) = \frac{1}{\lambda_n}\sum_{k=0}^{n}\left(\lambda_k - \lambda_{k-1}\right)\Delta^m x_k = m! \tag{9}$$

for $n \in \mathbf{N}$ which shows that $\tilde{\Lambda}(x) \in c - c_0$. Thus, the sequence x is in $c\left(\Delta_\lambda^m\right)$ but not in $c_0\left(\Delta_\lambda^m\right)$. Hence the inclusion $c_0\left(\Delta_\lambda^m\right) \subset c\left(\Delta_\lambda^m\right)$ is strict and this completes the proof.

Theorem 3.2 *The inclusion $c \subset c_0(\Delta_\lambda^m)$ strictly holds.*

Proof: Let $x \in c$. Then $\tilde{\Lambda}(x) \in c_0$. This shows that $x \in c_0\left(\Delta_\lambda^m\right)$. Hence, the inclusion $c \subset c_0\left(\Delta_\lambda^m\right)$ holds. Then, consider the sequence $y=(y_k)$ defined by $y_k = \sqrt{k+1}$ for $k \in \mathbf{N}$. It is trivial that $y \notin c$. On the other hand, it can easily be seen that $\tilde{\Lambda}(y) \in c_0$ and $y \in c_0\left(\Delta_\lambda^m\right)$.Consequently, the sequence y is in $c_0\left(\Delta_\lambda^m\right)$ but not in c. We therefore deduce that the inclusion $c \subset c_0\left(\Delta_\lambda^m\right)$ is strict. This concludes proof.

Theorem 3.3 *The inclusion $c\left(\Delta_\lambda^{m-1}\right) \subset c\left(\Delta_\lambda^m\right)$ holds.*

Proof: Let $x \in c\left(\Delta_\lambda^{m-1}\right)$. Then we have

$$\tilde{\Lambda}_n(x) = \frac{1}{\lambda_n}\sum_{k=0}^{n}\left(\lambda_k - \lambda_{k-1}\right)\Delta^{m-1} x_k \to l \quad (k \to \infty). \tag{10}$$

Furthermore, we obtain that $x \in c\left(\Delta_\lambda^m\right)$ from the following inequality, hence the inclusion $c\left(\Delta_\lambda^{m-1}\right) \subset c\left(\Delta_\lambda^m\right)$ holds.

$$\begin{aligned}\left|\tfrac{1}{\lambda_n}\textstyle\sum_{k=0}^{n}\left(\lambda_k - \lambda_{k-1}\right)\Delta^m x_k\right| &\leq \left|\tfrac{1}{\lambda_n}\textstyle\sum_{k=0}^{n}\left(\lambda_k - \lambda_{k-1}\right)\Delta^{m-1} x_k - l\right| \\ &+ \left|\tfrac{1}{\lambda_n}\textstyle\sum_{k=0}^{n}\left(\lambda_k - \lambda_{k-1}\right)\Delta^{m-1} x_{k-1} - l\right| \to 0.\end{aligned} \tag{11}$$

Theorem 3.4 *The inclusion $\ell_\infty(\Delta_\lambda^{m-1}) \subset \ell_\infty(\Delta_\lambda^m)$ strictly holds.*

Proof: Let $x \in \ell_\infty(\Delta_\lambda^{m-1})$. Then we have

$$\left|\tilde{\Lambda}_n(x)\right| = \left|\frac{1}{\lambda_n}\sum_{k=0}^{n}\left(\lambda_k - \lambda_{k-1}\right)\Delta^{m-1} x_k\right| \leq K \tag{12}$$

for $K > 0$. We obtain the following equality that $x \in \ell_\infty(\Delta_\lambda^m)$, hence the inclusion $\ell_\infty(\Delta_\lambda^{m-1}) \subset \ell_\infty(\Delta_\lambda^m)$ holds.

$$\left|\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})\Delta^m x_k\right| \leq \left|\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})\Delta^{m-1} x_k\right| + \left|\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})\Delta^{m-1} x_{k-1}\right|. \tag{13}$$

To show strict, we consider $x=(x_k)$ defined by $x=(k^m)$, then we obtain $x \in \ell_\infty(\Delta_\lambda^m) - \ell_\infty(\Delta_\lambda^{m-1})$.

4 The Bases for the Spaces $c\left(\Delta_\lambda^m\right)$ and $c_0\left(\Delta_\lambda^m\right)$

If a normed sequence space X contains a sequence (b_n) with the property that for every $x \in X$ there is a unique sequence (α_n) of scalars such that

$$\lim_n \|x-(\alpha_0 b_0+\alpha_1 b_1+...+\alpha_n b_n)\|=0. \tag{14}$$

Then (b_n) is called a Schauder basis (or briefly basis) for X. The series $\sum \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n) , and written as $x=\sum_k \alpha_k b_k$.

Theorem 4.1 *Define the sequence* $b^{(k)}(\lambda,m)=\left\{b_n^{(k)}(\lambda,m)\right\}_{k=0}^{\infty}$ *for every fixed* $k,m\in\mathbf{N}$ *and by*

$$b_n^{(k)}(\lambda,m)=\begin{cases}\begin{pmatrix} m+n-k-1 \\ n-k\end{pmatrix}\frac{\lambda_k}{\lambda_k-\lambda_{k-1}}-\begin{pmatrix} m+n-k-2 \\ n-k-1\end{pmatrix}\frac{\lambda_k}{\lambda_{k+1}-\lambda_k}, & n>k \\ \frac{\lambda_k}{\lambda_k-\lambda_{k-1}}, & n=k \\ 0, & n<k\end{cases}. \tag{15}$$

Then, the sequence $\left\{b_n^{(k)}(\lambda,m)\right\}_{k=0}^{\infty}$ is a basis for the space $c_0\left(\Delta_\lambda^m\right)$ and every $x\in c_0\left(\Delta_\lambda^m\right)$ has a unique representation of the form

$$x=\sum_k \alpha_k(\lambda)\, b^{(k)}(\lambda,m) \tag{16}$$

where $\alpha_k(\lambda)=\tilde{\Lambda}_k(x)$ for all $k\in\mathbf{N}$.

Theorem 4.2 *The sequence* $\left\{b,b^{(0)}(\lambda,m),b^{(1)}(\lambda,m),...\right\}$ *is a basis for the space* $c\left(\Delta_\lambda^m\right)$ *and every* $x\in c\left(\Delta_\lambda^m\right)$ *has a unique representation of the form*

$$x=lb+\sum_k\left[\alpha_k(\lambda)-l\right]b^{(k)}(\lambda,m); \tag{17}$$

where $\alpha_k(\lambda)=\tilde{\Lambda}_k(x)$ *for all* $k\in\mathbf{N}$, *the sequence* $b=(b_k)$ *is defined by*

$$b_k=\sum_{j=0}^{k}\begin{pmatrix} m+k-j-1 \\ k-j\end{pmatrix}. \tag{18}$$

Corollary 4.3 *The difference sequence spaces* $c\left(\Delta_\lambda^m\right)$ *and* $c_0\left(\Delta_\lambda^m\right)$ *are seperable.*

5 The $\alpha-,\beta-$ and $\gamma-$Duals of the Spaces $c\left(\Delta_{\lambda}^{m}\right)$ and $c_0\left(\Delta_{\lambda}^{m}\right)$

In this section, we introduce and prove the theorems determining the $\alpha-,\beta-$ and $\gamma-$ duals of the difference sequence spaces $c\left(\Delta_{\lambda}^{m}\right)$ and $c_0\left(\Delta_{\lambda}^{m}\right)$ of non-absolute type.

For arbitrary sequence spaces X and Y ,the set $M\left(X,Y\right)$ defined by

$$M\left(X,Y\right)=\left\{a=\left(a_k\right)\in w: ax=\left(a_kx_k\right)\in Y\ for\ all\ x=\left(x_k\right)\in X\right\} \quad (19)$$

is called the multipier space of X and Y.

With the notation of (19); the $\alpha-,\beta-$ and $\gamma-$duals of a sequence space X, which are respectively denoted by X^{α},X^{β} and X^{γ} are defined by

$$X^{\alpha}=M\left(X,\ell_1\right),X^{\beta}=M\left(X,cs\right)\ and\ X^{\gamma}=M\left(X,bs\right). \quad (20)$$

Now, we may begin with lemmas which are needed in proving theorems.

Lemma 5.1 *$A\in\left(c_0:\ell_1\right)=\left(c:\ell_1\right)$ if and only if*

$$\sup_{K\in\mathcal{F}}\sum_n\left|\sum_{k\in K}a_{nk}\right|<\infty. \quad (21)$$

Lemma 5.2 *$A\in\left(c_0:c\right)$ if and only if*

$$\lim_n a_{nk}\ \ exists\ for\ each\ k\in\mathbf{N}, \quad (22)$$

$$\sup_n\sum_k\left|a_{nk}\right|<\infty. \quad (23)$$

Lemma 5.3 *$A\in\left(c:c\right)$ if and only if (22) and (23) hold, and*

$$\lim_n\sum_k a_{nk}\ \ exists. \quad (24)$$

Lemma 5.4 *$A\in\left(c_0:\ell_\infty\right)=\left(c:\ell_\infty\right)$ if and only if (23) holds.*

Lemma 5.5 *$A\in\left(\ell_\infty:c\right)$ if and only if (22) holds and*

$$\lim_{n\to\infty}\sum_k\left|a_{nk}\right|=\sum_k\left|\alpha_k\right|. \quad (25)$$

Theorem 5.6 *The $\alpha-$dual of the space $c_0\left(\Delta_{\lambda}^{m}\right)$ and $c\left(\Delta_{\lambda}^{m}\right)$ is the set*

$$b_1^{\lambda}=\left\{a=\left(a_k\right)\in w:\sup_{K\in\mathcal{F}}\sum_n\left|\sum_{k\in K}b_{nk}\left(\lambda,m\right)\right|<\infty\right\}; \quad (26)$$

where the matrix $B^{\lambda}=\left(b_{nk}^{\lambda m}\right)$ is defined via the sequence $a=\left(a_k\right)$ by

$$b_n^{(k)}(\lambda,m)=\begin{cases}\left[\binom{m+n-k-1}{n-k}\frac{\lambda_k}{\lambda_k-\lambda_{k-1}}-\binom{m+n-k-2}{n-k-1}\frac{\lambda_k}{\lambda_{k+1}-\lambda_k}\right]a_n, & n>k\\ \frac{\lambda_n}{\lambda_n-\lambda_{n-1}}a_n, & n=k\\ 0, & n<k\end{cases}. \tag{27}$$

Proof: Let $a=(a_k)\in w$. Then, we obtain the equality

$$a_kx_k=\sum_{k=0}^{n}\binom{m+n-k-1}{n-k}\sum_{j=k-1}^{k}(-1)^{k-j}\frac{\lambda_j}{\lambda_k-\lambda_{k-1}}y_j=B_n^{\lambda}(y),\quad (n\in\mathbf{N}). \tag{28}$$

Thus, we observe by (28) that $ax=(a_kx_k)\in\ell_1$ whenever $x=(x_k)\in c_0(\Delta_\lambda^m)$ or $c(\Delta_\lambda^m)$ if and only if $B^\lambda y\in\ell_1$ whenever $y=(y_k)\in c_0$ or c. This means that the the sequence $a=(a_k)$ is in the $\alpha-$dual of the spaces $c_0(\Delta_\lambda^m)$ or $c(\Delta_\lambda^m)$ if and only if $B^\lambda\in(c_0:\ell_1)=(c:\ell_1)$. We therefore obtain by Lemma 5.1 with B^λ instead of A that $a\in\{c_0(\Delta_\lambda^m)\}^\alpha=\{c(\Delta_\lambda^m)\}^\alpha$ if and only if

$$\sup_{K\in\mathcal{F}}\sum_n\left|\sum_{k\in K}b_{nk}(\lambda,m)\right|<\infty. \tag{29}$$

Which leads us to the consequence that $\{c_0(\Delta_\lambda^m)\}^\alpha=\{c(\Delta_\lambda^m)\}^\alpha=b_1^\lambda$. This concludes proof.

Theorem 5.7 *Define the sets*

$$b_2^\lambda=\left\{a=(a_k)\in w:\sum_{j=k}^{\infty}\binom{m+n-j-1}{n-j}a_j\ exists\ for\ each\ k\in\mathbf{N}.\right\} \tag{30}$$

$$b_3^\lambda=\left\{a=(a_k)\in w:\sup_{n\in\mathbf{N}}\sum_{k=0}^{n-1}|g_k(n)|<\infty.\right\} \tag{31}$$

$$b_4^\lambda=\left\{a=(a_k)\in w:\sup_{n\in\mathbf{N}}\left|\frac{\lambda_n}{\lambda_n-\lambda_{n-1}}a_n\right|<\infty.\right\} \tag{32}$$

$$b_5^\lambda=\left\{a=(a_k)\in w:\lim_{n\to\infty}\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{m+k-j-1}{k-j}a_k\ exists.\right\} \tag{33}$$

$$b_6^\lambda=\left\{a=(a_k)\in w:\lim_{n\to\infty}\sum_k\left|t_{nk}^\lambda\right|=\sum_k\left|\lim_{n\to\infty}t_{nk}^\lambda\right|\right\} \tag{34}$$

where the matrice $T^\lambda=\left(t_{nk}^\lambda\right)$ is defined as follow:

$$t_{nk}^\lambda=\begin{cases}a_k(n), & k<n\\ \frac{\lambda_n}{\lambda_n-\lambda_{n-1}}a_n, & k=n\\ 0, & k>n\end{cases} \tag{35}$$

for all $k, n \in \mathbf{N}$ and the $a_k(n)$ is defined by

$$a_k(n) = \lambda_k \left(\frac{1}{\lambda_k - \lambda_{k-1}} \sum_{j=k}^{n} \binom{m+j-k-1}{j-k} a_j - \frac{1}{\lambda_{k+1} - \lambda_k} \sum_{j=k}^{n} \binom{m+j-k-2}{j-k-1} a_j \right) y_k \tag{36}$$

for $k < n$. Then $\{c_0(\Delta_\lambda^m)\}^\beta = b_2^\lambda \cap b_3^\lambda \cap b_4^\lambda$, $\{c(\Delta_\lambda^m)\}^\beta = b_2^\lambda \cap b_3^\lambda \cap b_4^\lambda \cap b_5^\lambda$ and $\{\ell_\infty(\Delta_\lambda^m)\}^\beta = b_2^\lambda \cap b_4^\lambda \cap b_6^\lambda$.

Proof: We have from (5)

$$\begin{aligned}
\sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n} \left[\sum_{j=0}^{k} \binom{m+k-j-1}{k-j} \sum_{i=j-1}^{j} (-1)^{j-i} \frac{\lambda_i}{\lambda_j - \lambda_{j-1}} y_i \right] a_k \\
&= \sum_{k=0}^{n-1} \lambda_k \left[\frac{\sum_{j=k}^{n} \binom{m+j-k-1}{j-k} a_j}{\lambda_k - \lambda_{k-1}} - \frac{\sum_{j=k+1}^{n} \binom{m+j-k-2}{j-k-1} a_j}{\lambda_{k+1} - \lambda_k} \right] y_k + \frac{a_n \lambda_n}{\lambda_n - \lambda_{n-1}} y_n \\
&= \sum_{k=0}^{n-1} a_k(n) y_k + \frac{a_n \lambda_n}{\lambda_n - \lambda_{n-1}} y_n = \left(T^\lambda y\right)_n ; (n \in \mathbf{N}).
\end{aligned}$$

Then we derive that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in c_0(\Delta_\lambda^m)$ if and only if $T^\lambda y \in c$ whenever $y = (y_k) \in c_0$. This means that $a = (a_k) \in \{c_0(\Delta_\lambda^m)\}^\beta$ if and only if $T^\lambda \in (c_0 : c)$. Therefore, by using Lemma 5.2, we obtain

$$\sum_{j=k}^{\infty} \binom{m+k-j-1}{k-j} a_j \quad exists\ for\ each\ k \in \mathbf{N}, \tag{37}$$

$$\sup_{n \in \mathbf{N}} \sum_{k=0}^{n-1} |a_k(n)| < \infty \tag{38}$$

and

$$\sup_{k \in \mathbf{N}} \sum_{k=0}^{n-1} \left| \frac{\lambda_k}{\lambda_k - \lambda_{k-1}} a_k \right| < \infty. \tag{39}$$

Hence we conclude that $\{c_0(\Delta_\lambda^m)\}^\beta = b_2^\lambda \cap b_3^\lambda \cap b_4^\lambda$.

Theorem 5.8 $\{c_0(\Delta_\lambda^m)\}^\gamma = \{c(\Delta_\lambda^m)\}^\gamma = \{\ell_\infty(\Delta_\lambda^m)\}^\gamma = b_3^\lambda \cap b_4^\lambda$.

Proof: It can be proved similalry as the proof of the Theorem 5.7 with Lemma 5.4 instead of Lemma 5.2.

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