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Translation Surfaces Generated by Mannheim Curves in Three Dimensional Euclidean Space

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Abstract

In this paper we study translation surfaces which is generated by mannheim curves according to Frenet frame in Euclidean 3- space. Then, we give some characterizetion of these surfaces.

Keywords: *Translation surface, Mannheim curve, minimal surface.*

1 Introduction

A surface M in the Euclidean 3-space is called a translation surface if it is given by the graph $z(x, y) = f(x) + g(y)$, where f and g are smooth functions on some interval of R. In [17], Scherk proved that, besides the planes, the only minimal translation surfaces are given by

$$z(x, y) = \frac{1}{a} \log \left| \frac{\cos(ax)}{\cos(ay)} \right|$$

where a is a non-zero constant. These surfaces are nowreferred as Scherk's minimal surfaces, [6].

Translation surfaces have been investigated from the various viewpoints by many differential geometers. L. Verstraelen, J. Walrave and S. Yaprak have investigated minimal translation surfaces in n-dimensional Euclidean spaces [5]. H. Liu has given the classification of the translation surfaces with constant mean curvature or constant Gauss curvature in 3- dimensional Euclidean space $\mathbb{E}^3$ and 3-dimensional Minkowski space $\mathbb{E}^3_1$, [3]. D. W. Yoon has studied translation surfaces in the 3-dimensional Minkowski space whose Gauss map G satisfies the condition $\Delta G = AG$, $A \in Mat(3, IR)$, where Δ denotes the Laplacian of the surface with respect to the induced metric and $Mat(3, IR)$ the set of $3x3$ real metrics, [1]. M. I. Munteanu and A. I. Nistor have studied the second fundamental form of translation surfaces in $\mathbb{E}^3$, [10]. They have given a non-existence result for polynominal translation surfaces in $\mathbb{E}^3$ with vanishing second Gauss curvature K_{II}. They have classified those translation surfaces for which K_{II} and H are proportional.

The curves are a fundamental structure of differential geometry. A regular curve in the Euclidean 3-space $\mathbb{E}^3$, it is well-known that one of the important problem is the characterization of a regular curve. The curvature functions κ and τ of a regular curve play an important role to determine the shape and size of the curve. For example: If $\kappa = \tau = 0$, then the curve is a geodesic. If $\kappa \neq 0$ (constant) and $\tau = 0$, then the curve is a circle with radius $1/\kappa$. If $\kappa \neq 0$ (constant) and $\tau \neq 0$ (constant), then the curve is a helix in the space, etc., [11]

Another way to classification and characterization of curves is the relationship between the Frenet vectors of the curves. For example, in the plane, a curve α rolls on a straight line, the center of curvature of its point of contact describes a curve β which is the Mannheim of α Mannheim partner curves in three dimensional Euclidean 3-space are studied by Liu and Wang, [4]. They have given the definition of Mannheim offsets as follows: Let C and C_1 be two space curves. C is said to be a Mannheim partner curve of C_1 if there exists a one to one correspondence between their points such that the binormal vector of C is coincident with the principal normal vector of C_1. They showed that C is Mannheim partner curve of C_1 if and only if the following equality holds

$$\frac{d\tau}{ds} = \frac{\kappa}{\tau}(1 + \lambda^2\tau^2)$$

where κ and τ are the curvature and the torsion of the curve C, respectively,and λ is a nonzero constant.

Mannheim curves in the Lorentzian Heisenberg group $Heis^3$ are studied by Turhan and Körpınar, [2]. They have characterize Mannheim curves in terms of its horizontal biharmonic partner curves in the Lorentzian Heisenberg group $Heis^3$.

2 Translation Surfaces With Mannheim Curves

In this section, we will study translation surface which is generated by Mannheim curve and we will give some characterizations of this surface according to Frenet frame in three dimensional Euclidean space.

From the definition of Mannheim curve, we can write

$$\alpha^*(s^*) = \alpha(s) + \lambda \mathbf{B}_\alpha, 2.1 \tag{1}$$
$$\gamma^*(t^*) = \gamma(t) + \mu \mathbf{B}_\gamma 2.2 \tag{2}$$

where λ and μ are non-zero constant.

Theorem 2.1 *Let $\varphi(s,t)$ be a translation surface which is generated by $\alpha^*(s^*)$ and $\gamma(t)$ parametrized as*

$$\begin{aligned}\varphi(s,t) &= \alpha^*(s^*) + \gamma(t) \\ &= \alpha(s) + \lambda \mathbf{B}_\alpha + \gamma(t).2.3\end{aligned} \tag{3}$$

The mean curvature of the translation surface $\varphi(s,t)$

$$\begin{aligned}H = {} & \frac{1}{2\sigma}((-\kappa_\alpha + \lambda^2\kappa_\alpha\tau_\alpha^2 + \lambda\tau_\alpha') < \mathbf{B}_\alpha, \mathbf{T}_\gamma > \\ & + \lambda\tau_\alpha^2 < \mathbf{N}_\alpha, \mathbf{T}_\gamma > - \lambda^2\tau_\alpha^3 < \mathbf{T}_\alpha, \mathbf{T}_\gamma > 2.4 \\ & + \kappa_\gamma(1+\lambda^2\tau_\alpha^2)(< \mathbf{B}_\gamma, \mathbf{T}_\alpha > - \lambda\tau_\alpha < \mathbf{B}_\gamma, \mathbf{N}_\alpha >).\end{aligned} \tag{4}$$

Proof: Differentiating of (2.3) according to s and t, we have

$$\varphi_s(s,t) = \mathbf{T}_\alpha - \lambda\tau_\alpha \mathbf{N}_\alpha, 2.5 \tag{5}$$
$$\varphi_t(s,t) = \mathbf{T}_\gamma.2.6 \tag{6}$$

From (2.5) and (2.6), coefficients of first fundamental form are

$$\begin{aligned}E &= 1 + \lambda^2\tau_\alpha^2, \\ F &= < \mathbf{T}_\alpha, \mathbf{T}_\gamma > - \lambda\tau_\alpha < \mathbf{N}_\alpha, \mathbf{T}_\gamma >, 2.7 \\ G &= 1.\end{aligned} \tag{7}$$

Then, from (2.5) and (2.6) second derivatives are

$$\varphi_{ss}(s,t) = \lambda\tau_\alpha\kappa_\alpha \mathbf{T}_\alpha + (\kappa_\alpha - \lambda\tau_\alpha')\mathbf{N}_\alpha - \lambda\tau_\alpha^2 \mathbf{B}_\alpha, 2.8 \tag{8}$$
$$\varphi_{st}(s,t) = \varphi_{ts}(s,t) = 02.9 \tag{9}$$
$$\varphi_{tt}(s,t) = \kappa_\gamma \mathbf{N}_\gamma.2.10 \tag{10}$$

The unit normal vector field of the surface φ

$$\mathbf{U} = \frac{1}{\sqrt{\|\mathbf{T}_\alpha \times \mathbf{T}_\gamma\|^{1/2} - \lambda^2\tau_\alpha^2 \|\mathbf{N}_\alpha \times \mathbf{T}_\gamma\|^{1/2}}}(\mathbf{T}_\alpha \times \mathbf{T}_\gamma) - \lambda\tau_\alpha(\mathbf{N}_\alpha \times \mathbf{T}_\gamma). \quad (2.11)$$

From (2.8)-(2.10) and (2.11)equations, coefficients of second fundamental form are

$$\begin{aligned} h_{11} &= (-\kappa_\alpha + \lambda^2\kappa_\alpha\tau_\alpha^2 + \lambda\tau_\alpha') < \mathbf{B}_\alpha, \mathbf{T}_\gamma > + \lambda\tau_\alpha^2 < \mathbf{N}_\alpha, \mathbf{T}_\gamma > - \lambda^2\tau_\alpha^3 < \mathbf{T}_\alpha, \mathbf{T}_\gamma > , \\ h_{12} &= h_{21} = 0, 2.12 \\ h_{22} &= \kappa_\gamma < \mathbf{B}_\gamma, \mathbf{T}_\alpha > -\lambda\tau_\alpha\kappa_\gamma < \mathbf{B}_\gamma, \mathbf{N}_\alpha > . \end{aligned} \tag{11}$$

So, from equations (2.7), (2.12), the mean curvature of the translation surface φ

$$\begin{aligned} H &= \frac{1}{2\sigma}((-\kappa_\alpha + \lambda^2\kappa_\alpha\tau_\alpha^2 + \lambda\tau_\alpha') < \mathbf{B}_\alpha, \mathbf{T}_\gamma > \\ &\quad +\lambda\tau_\alpha^2 < \mathbf{N}_\alpha, \mathbf{T}_\gamma > - \lambda^2\tau_\alpha^3 < \mathbf{T}_\alpha, \mathbf{T}_\gamma > \\ &\quad +\kappa_\gamma(1+\lambda^2\tau_\alpha^2)(< \mathbf{B}_\gamma, \mathbf{T}_\alpha > - \lambda\tau_\alpha < \mathbf{B}_\gamma, \mathbf{N}_\alpha >), \end{aligned}$$

where

$$\sigma = 1- < \mathbf{T}_\alpha, \mathbf{T}_\gamma >^2 +\lambda^2\tau_\alpha^2(1- < \mathbf{N}_\alpha, \mathbf{T}_\gamma >^2 .$$

Corollary 2.2 *If $\alpha(s)$ is an geodesic curve and $\gamma(t)$ is an straight line, the translation surface φ is a minimal surface.*

Corollary 2.3 *If $\alpha(s)$ is an geodesic curve, $\mathbf{B}_\gamma$ and $\mathbf{T}_\alpha$ perpendicular each other then, the translation surface φ is a minimal surface.*

Let $\psi(s,t)$ be a translation surface which is generated by $\alpha^*(s)$ and $\gamma(t)$ parametrized as

$$\begin{aligned} \psi(s,t) &= \alpha(s) + \gamma^*(t^*) \\ &= \alpha(s) + \gamma(t) + \mu\mathbf{B}_\gamma.2.13 \end{aligned} \tag{12}$$

The mean curvature of the translation surface $\psi(s,t)$

$$\begin{aligned} H &= \frac{1}{2}(\kappa_\alpha(1+\mu^2\tau_\gamma^2)(< \mathbf{B}_\alpha, \mathbf{T}_\gamma > -\mu\tau_\gamma < \mathbf{B}_\alpha, \mathbf{N}_\gamma >)2.14 \\ &\quad +(-\kappa_\gamma + \mu^2\kappa_\gamma\tau_\gamma^2 + \mu\tau_\gamma') < \mathbf{B}_\gamma, \mathbf{T}_\alpha > +\mu\tau_\gamma^2 < \mathbf{N}_\gamma, \mathbf{T}_\alpha > -\mu^2\tau_\gamma^3 < \mathbf{T}_\gamma, \mathbf{T}_\alpha > . \end{aligned} \tag{13}$$

Proof: Differentiating of (2.13) according to s and t, we have

$$\varphi_s(s,t) = \mathbf{T}_\alpha, 2.15 \tag{14}$$

$$\varphi_t(s,t) = \mathbf{T}_\gamma - \mu\tau_\gamma\mathbf{N}_\gamma.2.16 \tag{15}$$

Coefficients of first fundamental form are

$$\begin{aligned} E &= 1, \\ F &= <\mathbf{T}_\alpha, \mathbf{T}_\gamma> -\mu\tau_\gamma <\mathbf{T}_\alpha, \mathbf{N}_\gamma>, 2.17 \\ G &= 1+\mu^2\tau_\gamma^2. \end{aligned} \tag{16}$$

Then, from (2.15), (2.16) second derivatives of the translation surface ψ are

$$\begin{aligned} \varphi_{ss}(s,t) &= \kappa_\alpha \mathbf{N}_\alpha, \\ \varphi_{st}(s,t) &= \varphi_{ts}(s,t) = 02.18 \\ \varphi_{tt}(s,t) &= \mu\tau_\gamma\kappa_\gamma\mathbf{T}_\gamma + (\kappa_\gamma - \lambda\tau_\gamma')\mathbf{N}_\gamma - \lambda\tau_\gamma^2\mathbf{B}_\gamma. \end{aligned} \tag{17}$$

The unit normal vector field of the surface ψ

$$\mathbf{U} = \frac{1}{\sqrt{\|\mathbf{T}_\alpha \times \mathbf{T}_\gamma\|^{1/2} - \lambda^2\tau_\alpha^2 \|\mathbf{N}_\alpha \times \mathbf{T}_\gamma\|^{1/2}}}(\mathbf{T}_\alpha \times \mathbf{T}_\gamma) - \mu\tau_\gamma(\mathbf{T}_\alpha \times \mathbf{N}_\gamma). \quad (2.19)$$

From equalities (2.18), (2.19), coefficients of second fundamental form are

$$\begin{aligned} h_{11} &= \kappa_\alpha <\mathbf{B}_\alpha, \mathbf{T}_\gamma> -\mu\tau_\gamma\kappa_\alpha <\mathbf{B}_\alpha, \mathbf{N}_\gamma>, \\ h_{12} &= h_{21} = 0, 2.20 \\ h_{22} &= (-\kappa_\gamma + \mu^2\kappa_\gamma\tau_\gamma^2 + \mu\tau_\gamma') <\mathbf{B}_\gamma, \mathbf{T}_\alpha> +\mu\tau_\gamma^2 <\mathbf{N}_\gamma, \mathbf{T}_\alpha> -\mu^2\tau_\gamma^3 <\mathbf{T}_\gamma, \mathbf{T}_\alpha>. \end{aligned} \tag{18}$$

So, the mean curvature of the translation surface ψ is

$$\begin{aligned} H &= \frac{1}{2}(\kappa_\alpha(1+\mu^2\tau_\gamma^2)(<\mathbf{B}_\alpha, \mathbf{T}_\gamma> -\mu\tau_\gamma <\mathbf{B}_\alpha, \mathbf{N}_\gamma>) \\ &\quad +(-\kappa_\gamma + \mu^2\kappa_\gamma\tau_\gamma^2 + \mu\tau_\gamma') <\mathbf{B}_\gamma, \mathbf{T}_\alpha> +\mu\tau_\gamma^2 <\mathbf{N}_\gamma, \mathbf{T}_\alpha> -\mu^2\tau_\gamma^3 <\mathbf{T}_\gamma, \mathbf{T}_\alpha>, \end{aligned}$$

where

$$\rho = 1- <\mathbf{T}_\alpha, \mathbf{T}_\gamma>^2 +\mu^2\tau_\gamma^2(1- <\mathbf{T}_\alpha, \mathbf{N}_\gamma>^2 .$$

Corollary 2.4 *If $\gamma(t)$ is an geodesic curve and $\alpha(s)$ is an straight line, the translation surface ψ is a minimal surface.*

Corollary 2.5 *If $\gamma(t)$ is an geodesic curve, $\mathbf{B}_\alpha$ and $\mathbf{T}_\gamma$ perpendicular each other then, the translation surface φ is a minimal surface.*

Example 2.6 *Let*

$$\alpha^*(s^*) = (\frac{4}{5}\cos s - 3, 1-\sin s, -(\frac{3}{5}\cos s + 4))$$

be Mannheim pair of the curve

$$\alpha(s) = (\frac{4}{5}\cos s, 1-\sin s, -\frac{3}{5}\cos s)$$

and

$$\gamma(t) = (\cos t, \sin t, t).$$

The translation surface $\varphi(s,t)$ *generated by* $\alpha^*(s^*)$ *and* $\gamma(t)$ *is parameterized as*

$$\varphi(s,t) = (\frac{4}{5}\cos s + \cos t - 3, 1 - \sin s + \sin t, t - (\frac{3}{5}\cos s + 4)).$$

Then, the mean curvature of the $\varphi(s,t)$ *is*

$$H = \frac{1}{10\sqrt{2}}((4 - 3\sin t) + 5\cos s\cos t - \frac{4}{5}\sin s\sin t + \frac{3}{5}\sin s).$$

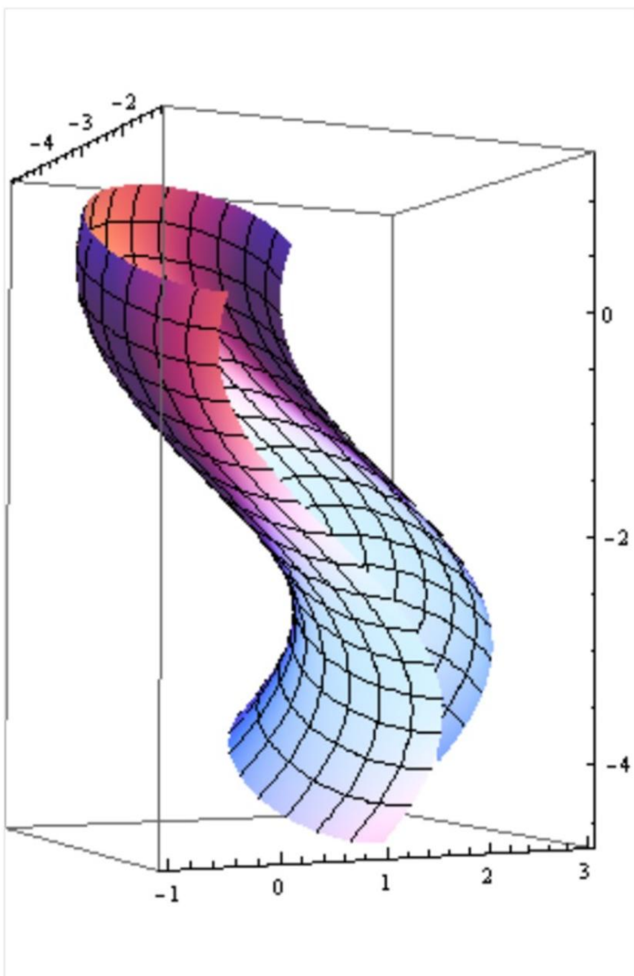

Figure 1: Translation Surface generated by Mannheim curve $\alpha^*(s^*)$ and helix $\gamma(t)$.

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