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Approximate Symmetry Analysis and Optimal System of ϕ^4 Equation with a Small Parameter

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Abstract

In this paper, the problem of approximate symmetries of the nonlinear ϕ^4 equation have been investigated. In order to compute the first-order approximate symmetry, we have applied the method which was proposed by Fushchich and Shtelen [3] and basically based on the expansion of the dependent variables in perturbation series. Especially, an optimal system of one dimensional subalgebras is constructed and some invariant solutions corresponding to the resulted symmetries are obtained.

Keywords: *Approximate symmetry, approximate solution, Lie group theory.*

1 Introduction

The theory of Lie symmetry groups of differential equations was developed by Sophus Lie. These Lie groups are invertible point transformations of both the dependent and independent variables of the differential equations. Symmetry group methods help us in reducing the order of differential equation and constructing invariant solution ([5], [6]). Classic methods for analyzing groups have this ability to survey symmetry properties for all important and applied equations in physics and mathematics, but with any small perturbation in terms of differential equations that usually have physical applications (or even can be artificial) will alter in admitted symmetry groups .so we need a method which admitted symmetry groups of differential equation stay stable under small perturbation and approximate Lie theorem will help us in this case. There are two methods for studying approximate symmetry. The first method, done by Baikov, Gazizov and Ibragimov ([1], [2]) can be summarized as follow. We consider perturbed differential equation in form of:

$$F = F_0 + \varepsilon F_1 = 0 \tag{1}$$

In which F_0 is unperturbed equation and F_1 is perturbed equation.

Theorem 1.1 *EQ (1) is approximately invariant with the generator $X = X^0 + \varepsilon X^1$ if and only if*

$$[XF]_{F\approx 0} = O(\varepsilon) \quad or \quad \left[X^0 F^0 + \varepsilon\left(X^1 F_0 + X^0 F_1\right)\right]_{F\approx 0} = O(\varepsilon).$$

In which X^0 is a generator of Lie symmetry of $F_0 = 0$ and X^1 is a generator of Lie symmetry of F_1 [8].

The exact symmetry of the unperturbed equation F_0 is denoted by X^0 and can be obtained as follows:

$$X^0 F_0 \,|_{F_0=0} = 0$$

Then, by applying the following auxiliary function:

$$H = \frac{1}{\varepsilon}\left[X^0(F_0 + \varepsilon F_1)\,|_{F_0+\varepsilon F_1=0}\right]$$

X^1 will be deduced from the following relation:

$$X^1 F_0 \,|_{F_0=0} + H = 0$$

Finally, after obtaining the approximate symmetries, the corresponding approximate solutions will be obtained via the classical Lie symmetry method [8].

In the second method, due to Fushchich and Shtelen, first of all the dependent variables are expanded in a perturbation series. In the next step, terms are then separated at each order of approximation and as a consequence a system of equations to be solved in a hierarchy is determined. Finally, the approximate symmetries of the original equation is defined to be the exact symmetries of the system of equations resulted from perturbations [3-4,7]. Pakdemirli et al. in a recent paper [9] have compared these above two methods. According to their comparison, the expansion of the approximate operator applied in the first method, does not reflect well an approximation in the perturbation sense; While the second method is consistent with the perturbation theory and results correct terms for the approximate solutions. Consequently, the second method is superior to the first one according to the comparison in [9].In this paper, we will apply the method proposed by Fushchich and Shtelen [3] in order to present a comprehensive analysis of the approximate symmetries of perturbed ϕ^4 equation

$$\phi_{tt} - \phi_{xx} - \varepsilon\phi + \phi^3 = 0. \tag{2}$$

where $0 < \epsilon \leq 1$ is a small parameter.

2 Exact and Approximate Symmetries

In this section, first of all the problem of exact symmetries of ϕ^4 equation with small parameter is investigated. Then the approximate symmetries of perturbed ϕ^4 equation will be determined.

We consider a one-parameter symmetry group of transformations acting on the space of the independent variables (x, t) and one dependant variable ϕ of equation (2), with infinitesimal generator given by this operator:

$$V = \xi(x,t,\phi)\,\partial_x + \tau(x,t,\phi)\,\partial_t + \eta(x,t,\phi)\,\partial_\phi. \tag{3}$$

The prolongation of order two of the operator (3) is

$$\overset{(2)}{\mathrm{Pr}}\,V = V + \eta^t\frac{\partial}{\partial\phi_t} + \eta^x\frac{\partial}{\partial\phi_x} + \eta^{xt}\frac{\partial}{\partial\phi_{xt}} + \eta^{tt}\frac{\partial}{\partial\phi_{tt}} + \eta^{xx}\frac{\partial}{\partial\phi_{xx}}. \tag{4}$$

where

$$\begin{aligned} \eta^t &= \eta_t + (\eta_\phi - \tau_x)\phi_t - \xi_t\phi_x - \tau_\phi\phi_t^2 - \xi_\phi\phi_x\phi_t, \\ \eta^x &= \eta_x + (\eta_u - \xi_x)\phi_x - \tau_x\phi_t - \xi_\phi\phi_x^2 - \tau_\phi\phi_x\phi_t, \end{aligned} \tag{5}$$

and respectively

$$\eta^{xx} = \eta_{xx} + (2\eta_{xu} - \xi_{xx})\phi_x - \ldots - 2\tau_\phi\phi_x\phi_t.$$

The invariance condition [5] for equation (2) is

$$\overset{(2)}{\mathrm{Pr}}\, V\left[\phi_{tt}-\phi_{xx}-\varepsilon\phi+\phi^3\right]=0,\qquad \text{whenever}\qquad \phi_{tt}-\phi_{xx}-\varepsilon\phi+\phi^3=0. \quad (6)$$

Expanding the (6) we obtain the following overdetermined system of partial differential equations:

$$\begin{gathered}\xi_\phi=0,\quad \tau_\phi=0,\quad \xi_{\phi x}-\eta_{\phi\phi}=0,\\ \tau_x-\xi_t=0,\quad \xi_x-\tau_t=0,\quad \eta_{\phi\phi}-2\tau_{\phi t}=0,\\ 2\tau_{x\phi}-\xi_{\phi t}=0,\quad 3\phi^3-3\varepsilon\phi\tau_\phi+2\eta_{\phi t}+\tau_{xx}-\tau_{tt}=0,\\ u^3\xi_u-2\eta_{ux}-\xi_{tt}-\varepsilon u\xi_u+\xi_{xx}=0,\\ \varepsilon\phi\eta_\phi-\eta_{xx}+\eta\phi-u^3\eta_\phi+3\phi^2\eta-\varepsilon\eta+2\phi^3\tau_t-2\varepsilon\phi\tau_t=0.\end{gathered} \quad (7)$$

By solving this system of PDEs, it is deduced that:

$$\xi=c_1t+c_2,\quad \tau=c_1x+c_3,\quad \eta=0. \quad (8)$$

where c_1, c_2 and c_3 are arbitrary constants. Therefore, this equation admits a 3-dimensional Lie algebra with the following generators:

$$X_1=t\,\partial_x+x\,\partial_t,\quad X_2=\partial_x,\quad X_3=\partial_t. \quad (9)$$

we used the method proposed in [3] in order to analyze the problem of approximate symmetries of the equation (2) with an accuracy of order one. First, we expand the dependent variable in perturbation series, and then we separate terms of each order of approximation, so that a system of equations will be formed. The derived system is assumed to be coupled and its exact symmetry will be considered as the approximate symmetry of the original equation. We expand the dependant variable up to order one as follows:

$$\phi=v+\epsilon w,\qquad 0<\epsilon\leq 1. \quad (10)$$

Where v and w are some smooth functions of x, t. After substitution of (10) into equation (2) and equating to zero the coefficients of zero and first power of epsilon , the following system of partial differential equations is resulted:

$$\begin{aligned} O(\epsilon^0)\ &:\ v_{tt}-v_{xx}+v^3=0 \\ O(\epsilon^1)\ &:\ w_{tt}-w_{xx}+3v^2w-v=0.\end{aligned} \quad (11)$$

Now, consider the following symmetry transformation group acting on the PDE system (11):

$$\begin{aligned}\widetilde{x}&=x+a\xi_1(t,x,v,w)+o(a^2), & \widetilde{t}&=t+a\xi_2(t,x,v,w)+o(a^2),\\ \widetilde{v}&=v+a\varphi_1(t,x,v,w)+o(a^2), & \widetilde{w}&=w+a\varphi_2(t,x,v,w)+o(a^2),\end{aligned} \quad (12)$$

where a is the group parameter and ξ_1, ξ_2 and φ_1, φ_2 are the infinitesimals of the transformations for the independent and dependent variables, respectively. The associated vector field is of the form:

$$X = \xi_1 \, \partial_t + \xi_2 \frac{\partial}{\partial x} + \varphi_1 \, \partial_v + \varphi_2 \frac{\partial}{\partial w}. \tag{13}$$

The invariance of the system (11) under the infinitesimal symmetry transformation group (13) leads to the following invariance condition:

$$\overset{(2)}{\mathrm{Pr}}\, X\,[\Delta] = 0, \qquad \text{whenever} \qquad \Delta = 0.$$

Hence, the following set of determining equations is inferred:

$$\begin{gathered} \phi_{2v} = 0, \quad \xi_{1v} = 0, \quad \xi_{1t} - \xi_{2x} = 0, \quad \xi_{1vt} - \varphi_{2vw} = 0, \\ 3v^3 \xi_{1v} + \xi_{1xx} - \xi_{1tt} + 2\varphi_{1vt} = 0, \quad \ldots, \quad \varphi_{1vw} - \xi_{1wt} = 0. \end{gathered}$$

By solving this system of PDEs, we obtain:

$$\xi_1 = c_1 t + c_3 x + c_{4,} \;\; \xi_2 = c_3 t + c_1 x + c_{2,} \; \phi_1 = -c_1 v, \quad \phi_2 = c_1 w. \tag{14}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants.Thus, the Lie algebra of infinitesimal symmetry of system (11) is spanned by these four vector fields:

$$X_1 = x\,\partial_x + t\,\partial_t - v\frac{\partial}{\partial v} + w\frac{\partial}{\partial w}, \; X_2 = t\,\partial_x + x\,\partial_t, \; X_3 = \partial_t, \; X_4 = \frac{\partial}{\partial x}. \tag{15}$$

3 Optimal System and Invariant Solutions

In this section, an optimal system of subalgebras corresponding to the resulted exact and approximate symmetries of the perturbed ϕ^4 equation is constructed. Each s-parameter subgroup corresponds to one of group invariant solutions. Since any linear combination of infinitesimal generators is also an infinitesimal generator, there are always infinitely many different symmetry subgroups for the differential equation. But it's not practical to find the list of all group invariant solutions of system; we just need the invariant solutions which have no relation with transformation in the full symmetry group. We need an effective, systematic means of classifying these solutions, leading to an "optimal system of group-invariant solutions from which every other such solution can be derived. Let G be a Lie group. An optimal system of s-parameter subgroups is a list of conjugacy inequivalent s-parameter subgroups with the property that any other subgroup is conjugate to precisely one subgroup in the list. Similarly, a list of s-parameter subalgebras forms an optimal system if every s-parameter subalgebra of g is equivalent to a unique member of the list under some element of the adjoint representation:

$$\tilde{h} = \mathrm{Ad}g(h), g \in G.$$

Proposition 3.7 of [5] says that the problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras. For one-dimensional subalgebras, this classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation, since each one-dimensional subalgebra is determined by a nonzero vector in g. This problem is attacked by the naive approach of taking a general element V in g and subjecting it to various adjoint transformations so as to "simplify it as much as possible. Thus we will deal with the construction of the optimal system of subalgebras of g.

The adjoint action is given by the Lie series:

$$\mathrm{Ad}(\exp(sX_i, X_j) = X_j - s[X_i, X_j] + \frac{s^2}{2}[X_i, [X_i, X_j]] - ...$$

Where $[X_i, X_j]$ is the commutator for the Lie algebra, s is a parameter and $i, j = 1, 2, 3, 4$. ([5]).

The adjoint representation of is listed in the following table, it consists the separate adjoint actions of each element of g on all other elements. Where the (i, j)-th indicating $Ad(\exp(sX_i)X_j)$. Optimal system of exact symmetries

3.1 Optimal System of Exact Symmetries

As it was shown in the previous section, the Lie algebra of the exact symmetries corresponding to the perturbed ϕ^4 equation is three dimensional and spanned by the following generators:

$$X_1 = t\,\partial_x + x\,\partial_t, \qquad X_2 = \partial_x, \qquad X_3 = \partial_t. \tag{16}$$

The commutation relations corresponding to these vector fields are given in table 1.

Table 1: The Commutator Table

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	$-X_3$	$-X_2$
X_2	X_3	0	0
X_3	X_2	0	0

Consider $\mathbf{g} = \langle X_1, X_2, X_3 \rangle$ and $\mathbf{g}_1 = \langle X_2, X_3 \rangle$ since commutator $\mathbf{g}_1$ is abelian , $\mathbf{g}$ is solvable. Let $F_i^s : \mathbf{g} \longrightarrow \mathbf{g}$ be a linear map defined by $X \longrightarrow$

$\mathrm{Ad}(\exp(s_i X_i)X)$ for $i = 1, \cdots, 3$. The matrices M_i^s of F_i^S, $i = 1, \cdots, 3$ with respect to the basis $\{X_1, X_2, X_3\}$ are given by:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh s_1 & \sinh s_1 \\ 0 & \sinh s_1 & \cosh s_1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -s_2 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ -s_3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $X = \sum_{i=1}^{3} a_i X_i$ is a nonzero vector field in $\mathbf{g}$.In the following, by alternative action of these matrices on a vector field X, the coefficients a_i of X will be simplified. Let $X = (a_1, a_2, a_3)^t$ by acting the product of the adjoint representations M_2^s, M_3^s on X, we see that:

$$M_2^s.M_3^s. \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ -a_1 s_{3+} a_2 \\ -a_1 s_{2+} a_3 \end{pmatrix}.$$

If $a_1 \neq 0$, then we can make the second and third component vanish by setting $s_3 = a_2/a_1, s_2 = a_3/a_1$ respectively. Scaling X if necessary, we can assume that $a_1 = 1$. So X reduce to the X_1.

If $a_1 = 0$, since now adjoint representation M_1^s operates on $\mathbf{g}_1$ by hyperbolic rotations we find the following classes in $\mathbf{g}_1$:

$$cX_2 + X_3, \qquad X_2 + cX_3 \qquad c \in \mathbb{R}.$$

As a result we can state the following proposition:

Proposition 3.1 *An optimal system of one-dimensional subalgebras corresponding of the Lie algebra of exact symmetries of the perturbed ϕ^4 equation is generated by:*

$$(i)\ \ X_1 \qquad (ii)\ \ cX_2 + X_3 \qquad (iii)\ \ X_2 + cX_3,$$

where $c \in \mathbb{R}$ is arbitrary constant.

3.2 Optimal System of Approximate Symmetries

In this section, an optimal system of subalgebras corresponding to the resulted approximate symmetries of the perturbed ϕ^4 equation constructed. As it was shown in the previous sections,the Lie algebra $\mathbf{g}$ of the approximate symmetries corresponding to the perturbed ϕ^4 equation is four-dimensional and spanned by :

$$X_1 = x\,\partial_x + t\,\partial_t - v\,\partial_v + w\,\partial_w,\ X_2 = t\,\partial_x + x\,\partial_t,\ X_3 = \partial_t,\ X_4 = \partial_x. \qquad (17)$$

The commutation relations corresponding to these vector fields are given in table 2.

Table 2: The Commutator Table g

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	0	$-X_3$	$-X_4$
X_2	0	0	$-X_4$	$-X_3$
X_3	X_3	X_4	0	0
X_4	X_4	X_3	0	0

Consider $\mathbf{g} = \langle X_1, X_2, X_3, X_4\rangle$ and $\mathbf{g}_1 = \langle X_1, X_2\rangle$ and $\mathbf{g}_2 = \langle X_3, X_4\rangle$. since commutator $\mathbf{g}_2$ is abelian , $\mathbf{g}$ is solvable. Let $F_i^s : \mathbf{g} \longrightarrow \mathbf{g}$ be a linear map defined by $X \longrightarrow \mathrm{Ad}(\exp(s_i X_i)X)$ for $i = 1, \cdots, 4$. The matrices M_i^s of F_i^S, $i = 1, \cdots, 4$ with respect to the basis $\{X_1, X_2, X_3, X_4\}$ are given by:

$$M_1^s = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{s_1} & 0 \\ 0 & 0 & 0 & e^{s_1} \end{pmatrix}, \qquad M_2^s = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh s_2 & \sinh s_2 \\ 0 & 0 & \sinh s_2 & \cosh s_2 \end{pmatrix},$$
$$M_3^s = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \\ -s_3 & 0 & 1 & 0 \\ 0 & -s_3 & 0 & 1 \end{pmatrix}, \qquad M_4^s = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -s_4 & 1 & 0 \\ -s_4 & 0 & 0 & 1 \end{pmatrix}. \tag{18}$$

Let $X = \sum_{i=1}^{4} a_i X_i$ is a nonzero vector field in $\mathbf{g}$. In the following, by alternative action of these matrices on a vector field X, the coefficients a_i of X will be simplified.Let $X = (a_1, a_2, a_3, a_4)^t$ by acting the product of the adjoint representations M_3^s, M_4^s on X, we have that:

$$M_3^s.M_4^s.\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ -s_3a_1 - s_4a_2 + a_3 \\ -s_4a_1 - s_3a_2 + a_4 \end{pmatrix}. \tag{19}$$

If $a_1^2 - a_2^2 \neq 0$ then we can make the third and fourth component vanish By placing the appropriate amount for s_3 and s_4. So, X is reduced to $(a_1, a_2, 0, 0)^t$ and we have representation $(c, 1, 0, 0)^t$, $(1, c, 0, 0)^t$. Thus $X = cX_1 + X_2$, $X = X_1 + cX_2$ where $c \in \mathbb{R}$ and $c^2 \neq 1$.

If $a_1^2 - a_2^2 = 0$ and $a_1 = \pm a_2 \neq 0$ then we can assume that $a_1 = 1$ and we have representation $X = (1, \pm 1, a_2, a_3)^t$. By acting the product of the adjoint representations M_3^s, M_4^s on X, we have

$$M_1^s.M_2^s.\begin{pmatrix} 1 \\ \pm 1 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 1 \\ \pm 1 \\ e^{s_1}\cosh s_2 a_3 + e^{s_1}\sinh s_2 a_4 \\ e^{s_1}\sinh s_2 a_3 +^{s_1}\cosh s_2 a_4 \end{pmatrix} \tag{20}$$

where the first and second component fixed and operates the third and fourth component by scalings and rotations. So, the following representations are resulted:

$$\begin{pmatrix} 1 \\ \pm 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ \pm 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \pm 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ \pm 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ \pm 1 \\ 0 \\ 1 \end{pmatrix}.$$

Thus, we have:

$$\begin{aligned} &X = X_1 \pm X_2 + X_3 + X_4, && X = X_1 \pm X_2 + X_3, \\ &X = X_1 \pm X_2 + X_3 - X_4, && X = X_1 \pm X_2 + X_4. \end{aligned}$$

If $a_1^2 - a_2^2 = 0$ and $a_1 = \pm a_2 = 0$, then by acting the product of the adjoint representations M_3^s, M_4^s on X (20), we have that:

$$X = X_3 \pm X_4, \quad X = X_3, \quad X = X_4.$$

As a result we can state the following proposition:

Proposition 3.2 *An optimal system of one dimensional subalgebras corresponding to the Lie algebra of approximate symmetries of the perturbed* ϕ^4 *equation is generated by:*

$$\begin{array}{lll} (1) cX_1 + X_2, & (2) X_1 + cX_2, & (3) X_1 + X_2 + X_3 + X_4, \\ (4) X_1 - X_2 + X_3 + X_4, & (5) X_1 + X_2 + X_3, & (6) X_1 - X_2 + X_3, \\ (7) X_1 + X_2 + X_3 - X_4, & (8) X_1 - X_2 + X_3 - X_4, & (9) X_1 + X_2 + X_4, \\ (10) X_1 - X_2 + X_4, & (11) X_3 + X_4, & (12) X_3 - X_4, \\ (13) X_3, & (14) X_4. & \end{array}$$

4 Symmetry Reduction

In this part, the perturbed ϕ^4 equation will be reduced by demonstrating it in the new coordinates. The equation (1) is expressed in the coordinates (x, t, u). We must search for this equation's from in the appropriate coordinates for reducing it.These new coordinates will be constructed by looking for independent invariants (y, v) corresponding to the generators of the symmetry group. Thus, by using the new coordinates and applying the chain rule, we obtain the reduced equation. We remark this procedure for one dimensional subalgebras of perturbed ϕ^4 equation, which have been obtained in proposition 3.1 and proposition 3.2. For instance, consider the case (2) in proposition 3.1 :

$$cX_2 + X_3 = c\partial_x + \partial_t.$$

The characteristic equations are $dx/c = dt/1 = d\phi/0$. So, we can obtain differential invariants as $y = x - ct$ and $\phi = v(y)$. By substituting these new variables in the equation (2) we obtain the reduced equation:

$$\left(c^2 - 1\right) v_{yy} - \varepsilon v + v^3 = 0.$$

Solving this reduced equation we obtain

$$\phi(x,t) = c_2\sqrt{\frac{2\varepsilon}{\alpha}}\text{jacobiSN}\left(\left(\frac{\sqrt{-2\gamma\beta}}{2\gamma}(x-ct)+c_1\right)\sqrt{\frac{2\varepsilon}{\alpha}}, \frac{c^2}{\sqrt{\beta}}\right).$$

where, $\alpha = -1 + 2\varepsilon + c_2^2$, $\beta = -1 + 2\varepsilon$, $\gamma = c^2 - 1$.

In a similar way, we can compute all of the similarity reduction equations corresponding to the optimal system obtained in proposition 3.1 and 3.2, as shown in Tables 3 and 4.

Table 3: Lie Invariants, Similarity Solutions and Reduced Equation

operator	$\{y_i, v_i\}$	similarity reduced equation
X_1	$\{x^2 - t^2, \phi\}$	$4yv_{yy} + \varepsilon v - v^3 = 0$
$cX_2 + X_3$	$\{x - ct, \phi\}$	$(c^2 - 1)\, v_{yy} - \varepsilon v + v^3 = 0$
$X_2 + cX_3$	$\{cx - t, \phi\}$	$(1 - c^2)v_{yy} - \varepsilon v + v^3 = 0$

we now consider the case (1) in proposition3.2:

$$cX_1 + X_2 = (cx + t)\partial_x + (x + ct)\partial_t - cv\partial_v + w\partial_w.$$

The characteristic equation is

$$\frac{dx}{cx+t} = \frac{dt}{x+ct} = \frac{dv}{-cv} = \frac{dw}{w}.$$

if $c \neq \pm 1, 0$, we obtained

$$\frac{d(x+t)}{(c+1)(x+t)} = \frac{d(x-t)}{(c-1)(x+ct)} = \frac{dv}{-cv} = \frac{dw}{w}.$$

By solving above equation, the following approximate Lie invariants are resulted:

$$\zeta = \frac{(x+t)^{1/(c+1)}}{(x-t)^{1/(c-1)}}, \qquad y = v(x+t)^{c/(c+1)}, \qquad z = \frac{w}{(x+t)^{c/(c+1)}}.$$

If $c = 1$ the invariants are $\zeta = x - t$, $y = v(x+t)$, $z = w/(x+t)$ and if $c = -1$ the invariants are $\zeta = x - t$, $y = v(x-t)$, $z = w/(x-t)$.

Table 4: Lie Invariants, Similarity Solutions

optimal system	ζ_i	y_i
$cX_1 + X_2$	$\frac{(x+t)^{1/(c+1)}}{(x-t)^{1/(c-1)}}$	$v(x+t)^{c/(c+1)}$
$X_1 + cX_2$	$\frac{(x+t)^{1/(c+1)}}{(x-t)^{1/(1-c)}}$	$v(x+t)^{1/(c+1)}$
$X_1 + X_2 + X_3 + X_4$	$x - t$	$v(x+t+1)$
$X_1 - X_2 + X_3 + X_4$	$(x-t)e^{-(x+t)}$	$v(x-t)^{1/2}$
$X_1 + X_2 + X_3$	$(2x+2t+1)e^{2(x+t)}$	$v(2x+2t+1)^{1/2}$
$X_1 - X_2 + X_3$	$\frac{e^{2(x+t)}}{2x-2t-1}$	$v(2x-2t-1)^{1/2}$
$X_1 + X_2 + X_3 - X_4$	$(x+t)e^{(x+t)}$	$v(x+t)$
$X_1 - X_2 + X_3 - X_4$	$x + t$	$v(x-t+1)^{1/2}$
$X_1 + X_2 + X_4$	$(2x+2t+1)e^{2(t-x)}$	$v(2x+2t+1)^{1/2}$
$X_1 - X_2 + X_4$	$(2x-2t+1)e^{-2(t+x)}$	$v(2x-2t+1)^{1/2}$
$X_3 + X_4$	$x - t$	v
$X_3 - X_4$	$x + t$	v
X_3	x	v
X_4	t	v

and $z_i = vw/y_i$, $v_i = vf(\zeta)/y_i$ ad $w_i = g(\zeta)y_i/v$.

5 Conclusion

The investigation of the exact solutions of nonlinear PDEs plays an essential role in the analysis of nonlinear phenomena. Lie symmetry method greatly simplifies many nonlinear problems. Exact solutions are nevertheless hard to investigate in general. Furthermore, many PDEs in application depend on a small parameter, hence it is of great significance and interest to obtain approximate solutions. Perturbation analysis method was thus developed and it has a significant role in nonlinear science, particularly in obtaining approximate analytical solutions for perturbed PDEs. This procedure is mainly based on the expansion of the dependent variables asymptotically in terms of a small parameter. The combination of Lie group theory and perturbation theory yields two distinct approximate symmetry methods. In this paper we have comprehensively analyzed the approximate symmetries of the perturbed ϕ^4 equation.It is worth mentioning that in order to calculate the approximate symmetries corresponding to this equation, we have applied the second approximate symmetry method which was proposed by Fushchich and Shtelen. Meanwhile, we have constructed an optimal system of subalgebras. Also, we have obtained the symmetry transformations and some invariant solutions corresponding to the resulted symmetries.

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