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The Weighted KPC–Hypergroups

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Abstract

In this paper, we extend some basic concepts and results from weighted DJS–hypergroups to KPC–hypergroups. Also, we define a new convolution product on the space $L^1(Q, w)$.

Keywords: *DJS–Hypergroup, KPC–Hypergroup, Haar measure, Weight function, Weighted hypergroup.*

1 Introduction

Roughly speaking, a hypergroup is a topological space equipped with an extra structure, which leads to the construction of a Banach algebra on the Banach space of all bounded complex Radon measures on the hypergroup. Locally compact hypergroups, as an extension of locally compact groups, were introduced in a series of papers by Dunkl [4], Jewett [8], and Spector [12] in 70's (we refer to this definition of hypergroup as DJS–hypergroup). For more details about DJS–hypergroups we refer to [1]. Kalyuzhnyi, Podkolzin, and Chapovsky [9] introduced new axioms for hypergroups in 2010. This new concept is an extension of DJS–hypergroups, and also generalizes a normal hypercomplex system with a basis unity to the nonunimodular case. We refer to this notion as KPC–hypergroup. They studied harmonic analysis on KPC–hypergroups and showed that there is an example of a compact KPC–hypergroup related to the generalized Tchebycheff polynomials, which is not a DJS–hypergroup [9].

Some topics which are related to hypergroups and have been initiated based on a similar study on locally compact groups are "weighted hypergroups" and "weighted hypergroup algebras". The first studies on weighted hypergroup algebras may be tracked back to [2], [5], and [6]. The weighted space $L^1(G, w)$ of a locally compact group G was studied extensively ([3], [10], [11]). The basic idea is to consider a continuous weight w and change the norm of $L^1(G)$ by a factor of w. The new space $L^1(G, w)$ then consists of those Borel measurable functions on G, for which fw is in $L^1(G)$.

2 Preliminaries

In this section, we recall the definition and some basic properties of the locally compact cocommutative KPC–hypergroups. For more details we refer to [9].

Let Q be a locally compact Hausdorff space. We denote by $M(Q)$ the space of all complex Radon measures on Q, and by $C(Q)$, the space of all complex-valued continuous functions on Q. The support of a function f is denoted by $\mathrm{supp}(f)$.

Notation: In the sequel definition we have used the following notations:

$$[(\Delta \times \mathrm{id}) \circ \Delta(f)](p, q, r) := \Delta(\Delta f(p, \cdot))(q, r),$$

$$[(\mathrm{id} \times \Delta) \circ \Delta(f)](p, q, r) := \Delta(\Delta f(\cdot, q))(p, r),$$

$$[(\epsilon \times \mathrm{id}) \circ \Delta(f)](p) := \epsilon(\Delta f(p, \cdot)) = \Delta f(p, e),$$

$$[(\mathrm{id} \times \epsilon) \circ \Delta(f)](p) := \epsilon(\Delta f(\cdot, p)) = \Delta f(e, p),$$

$$(f \otimes 1)(p, q) \cdot (\Delta g)(p, q) = f(p)1(q) \cdot \Delta g(p, q),$$

$$(1 \otimes f)(p, q) \cdot (\Delta g)(p, q) := 1(p)f(q) \cdot \Delta g(p, q),$$

where $f \in C(Q)$ and $p, q, r \in Q$.

Definition 2.1. *Let Q be a locally compact second countable Hausdorff space with an involutive homeomorphism $\star : Q \longrightarrow Q$ satisfying the following conditions:*

1. *there is an element $e \in Q$ such that $e^\star = e$;*
2. *there is a $\mathbb{C}$-linear mapping $\Delta : C(Q) \to C(Q \times Q)$ such that*

   i. *Δ is co-associative, that is,*

   $$(\Delta \times id) \circ \Delta = (id \times \Delta) \circ \Delta;$$

   ii. *Δ is positive, that is, $\Delta f \geq 0$ for all $f \in C(Q)$ such that $f \geq 0$;*

iii. *Δ preserves the identity, that is, $(\Delta 1)(p,q) = 1$ for all $p, q \in Q$;*

iv. *For all $f, g \in C_c(Q)$ we have $(1 \otimes f).(\Delta g) \in C_c(Q \times Q)$ and $(f \otimes 1).(\Delta g) \in C_c(Q \times Q)$.*

3. *the homomorphism $\epsilon : C(Q) \to \mathbb{C}$ defined by $\epsilon(f) = f(e)$, satisfies the counit property, that is,*

$$(\epsilon \times id) \circ \Delta = (id \times \epsilon) \circ \Delta = id,$$

in other words, $(\Delta f)(e,p) = (\Delta f)(p,e) = f(p)$ for all $p \in Q$.

4. *the function $\check{f}$ defined by $\check{f}(q) = f(q^\star)$ for $f \in C(Q)$ satisfies*

$$(\Delta \check{f})(p,q) = (\Delta f)(q^\star, p^\star).$$

5. *there exists a positive measure m on Q, supp $m = Q$, such that*

$$\int_Q (\Delta f)(p,q) g(q) dm(q) = \int_Q f(q)(\Delta g)(p^\star, q) dm(q)$$

for all $f \in C_b(Q)$ and $g \in C_c(Q)$, or $f \in C_c(Q)$ and $g \in C_b(Q), p \in Q$; such a measure m will be called a left Haar measure on Q.

Then $(Q, \star, e, \Delta, m)$, or simply Q, is called a locally compact KPC–hypergroup.

Throughout this paper, Q is a locally compact KPC–hypergroup and m is a left Haar measure on Q.

Definition 2.2. *Let $\mu, \nu \in M(Q)$ be such that the linear functional $\mu * \nu$ defined by*

$$(\mu * \nu)(f) = \int_Q \int_Q \Delta(f)(p,q) d\mu(p) d\nu(q), \qquad (f \in C_c(Q))$$

is a measure. Then the measures μ and ν are called convolvable. Specially, we have $(\delta_p * \delta_q)(f) = (\Delta f)(p,q)$, where $p, q \in Q$.

If $\mu, \nu \in M(Q)$ are bounded, then μ and ν are convolvable ([9], Lemma 3.3).

Definition 2.3. *Let m be a left Haar measure on Q. The convolution of complex-valued Borel measurable functions f and g on Q is denoted by $f * g$ and is defined by*

$$(f * g)(q) = \int_Q f(p)(\Delta g)(p^\star, q) dm(p),$$

where $q \in Q$.

3 The Weight Functions on KPC–Hypergroups

Definition 3.1. *A complex-valued bounded continuous function k on Q is called positive definite if for any $n \in \mathbb{N}$, and $q_i \in Q \quad (i = 1, ..., n)$, the matrix*

$$((\Delta k)(q_i^\star, q_j))_{1 \le i,j \le n}$$

is positive semi-definite [7].

Definition 3.2. *A continuous function $w : Q \to [0, \infty)$ is called a weight function, if $w(e) = 1$ and*

$$\Delta w(p, q) \le w(p)w(q) \qquad (p, q \in Q).$$

Throughout this paper, w is a symmetric weight function on Q, i.e. $w(p^\star) = w(p)$, for all $p \in Q$.

Definition 3.3. *A complex-valued function f on Q is called w–bounded, if there is a constant $K > 0$ such that for all $p \in Q$, $|f(p)| \le Kw(p)$.*

Lemma 3.4. *Let f be a positive definite function on Q. Then for all $p \in Q$ we have*

i. $f(p^\star) = \overline{f(p)}$, *where $\bar{f}$ is the complex conjugate of f;*

ii. $f(e) \ge 0$;

iii. $\Delta f(p, p^\star) \ge 0$;

iv. $|f(p)|^2 \le \Delta f(p, p^\star) f(e)$;

v. $|f(p)| \le \frac{1}{2}(\Delta f(p, p^\star) + f(e))$.

Proof: It is well known that if complex numbers a, b, c and d satisfy in $a + bz + c\bar{z} + dz\bar{z} \ge 0$ (for all $z \in \mathbb{C}$), then $a \ge 0$, $b = \bar{c}$, $d \ge 0$, $|b|^2 \le ad$, and $2|b| \le a + d$. Since f is positive definite, we have

$$f(e) + zf(p) + \bar{z}f(p^\star) + z\bar{z}\Delta f(p, p^\star) \ge 0,$$

where $p \in Q$ and $z \in \mathbb{C}$. Thus, $f(p^\star) = \overline{f(p)}$, $f(e) \ge 0$, $\Delta f(p, p^\star) \ge 0$, $|f(p)|^2 \le \Delta f(p, p^\star)f(e)$ and $2|f(p)| \le (\Delta f(p, p^\star) + f(e))$.

Remark 3.5. *Easily, we can see that for each $f \in C(Q)$ and $p, q \in Q$, $|\Delta f(p, q)| \le \Delta|f|(p, q)$.*

Lemma 3.6. *If f is a w–bounded function on Q, then there is a constant $K > 0$ such that $|\Delta f| \le K\Delta w$.*

Proof: Let f be a w– bounded function on Q. Then there is a constant $K > 0$ such that for all $p \in Q$, $|f(p)| \leq Kw(p)$, and so $0 \leq Kw(p) - |f(p)|$. By Definition 2.1, Δ is a $\mathbb{C}$-linear positive mapping, so $0 \leq K\Delta w(p,q) - \Delta|f|(p,q)$, for all $p, q \in Q$. Then by Remark 3.5, $|\Delta f|(p,q) \leq K\Delta w(p,q)$ for all $p, q \in Q$.

Proposition 3.7. *If f is a w–bounded positive definite function, then for all $p \in Q$, $|f(p)| \leq f(e)w(p)$.*

Proof: Let $K := \sup\{w(p)^{-1}|f(p)| : p \in Q \text{ and } w(p) \neq 0\}$. Since f is positive definite, we have

$$\begin{aligned} |f(p)|^2 &\leq \Delta f(p,p^\star)f(e) \quad (\text{ by Lemma 3.4}) \\ &\leq K\Delta w(p,p^\star)f(e) \quad (\text{by Definition 3.2}) \\ &\leq Kw(p)w(p^\star)f(e) \quad (\text{by Definition 3.2}) \\ &\leq K \cdot w(p)^2 f(e), \end{aligned}$$

where $p \in Q$. Therefore, $|f(p)| \leq (Kf(e))^{\frac{1}{2}}w(p)$, for all $p \in Q$. By the choice of K it follows that $K \leq (Kf(e))^{\frac{1}{2}}$ and $K \leq f(e)$.

Proposition 3.8. Let w be a weight function on Q. Then A defined by

$$A(\mu) := \int_Q wd|\mu|$$

is submultiplicative on $M_c(Q)$.

Proof: Let w be a weight function on Q. By Definition 3.2, for each $\mu, \nu \in M_c(Q)$ we have

$$\begin{aligned} A(\mu * \nu) &= \int_Q wd|\mu * \nu| \\ &= |\mu * \nu|(w) \\ &= \left| \int_Q \int_Q \Delta w(s,t)d|\mu|(s)d|\nu|(t) \right| \\ &\leq \int_Q \int_Q |\Delta w(s,t)|d|\mu|(s)d|\nu|(t) \\ &\leq \int_Q \int_Q \Delta|w|(s,t)d|\mu|(s)d|\nu|(t) \\ &= \int_Q \int_Q w(s)w(t)d|\mu|(s)d|\nu|(t) \\ &= A(\mu)A(\nu). \end{aligned}$$

Notation: Let Q be a cocommutative KPC–hypergroup with a Haar measure m, and w be a weight function on Q. We denote the set of all complex-valued Borel measurable functions f on Q such that $\int_Q |f|\, w\, dm < \infty$, by $L^1_w(Q,m)$. For abbreviation, we write $L^1_w(Q)$ instead of $L^1_w(Q,m)$.

Theorem 3.9. *Let $w(p) \geq 1$ for all $p \in Q$. For any $f \in L^1_w(Q)$ we define $\|f\|_{1,w} := \int_Q |f| w dm$. Then $(L^1_w(Q), \|.\|_{1,w}, *)$ is a normed subalgebra of $L^1(Q)$.*

Proof: Let $f, g \in L^1_w(Q)$. Then,

$$\begin{aligned}
\|f * g\|_{1,w} &= \int_Q |f * g(p)| w(p) dm \\
&= \int_Q | \int_Q f(q) \Delta g(q^\star, p) dm(q) | w(p) dm(p) \\
&\leq \int_Q \int_Q |f(q)| |\Delta g(q^\star, p)| dm(q) w(p) dm(p) \\
&\leq \int_Q \int_Q |f(q)| \Delta |g| (q^\star, p) w(p) dm(q) dm(p) \\
&= \int_Q |f(q)| \left(\int_Q \Delta w(q, p) |g|(p) dm(p) \right) dm(q) \\
&\leq \int_Q |f(q)| w(q) dm(q) \int_Q |g|(p) w(p) dm(p) \\
&= \|f\|_{1,w} \|g\|_{1,w}.
\end{aligned}$$

Therefore, $f * g \in L^1_w(Q)$ and $\|f * g\|_{1,w} \leq \|f\|_{1,w} \|g\|_{1,w}$. Since $w \geq 1$, we have $L^1_w(Q) \subseteq L^1(Q)$.

Proposition 3.10. *Let $f \in L^1_w(Q)$. Then $\|f_p\|_{1,w} \leq w(p) \|f\|_{1,w}$, where $p \in Q$ and $f_p(q) = \Delta f(p, q)$.*

Proof: For all $p \in Q$ we have

$$\begin{aligned}
\|f_p\|_{1,w} &= \int_Q |f_p|(q)w(q)dm(q) \\
&= \int_Q |\Delta f(p,q)|w(q)dm(q) \\
&\leq \int_Q \Delta|f|(p,q)w(q)dm(q) \\
&= \int_Q \Delta|f|(p,q)w(q^\star)dm(q) \\
&= \int_Q \Delta|f|(p,q)\check{w}(q)dm(q) \\
&= \int_Q \Delta\check{w}(p^\star,q)|f|(q)dm(q) \\
&= \int_Q \Delta w(q^\star,p)|f|(q)dm(q) \\
&\leq w(p)\int_Q |f|(q)w(q)dm(q) \\
&= w(p)\|f\|_{1,w}.
\end{aligned}$$

Lemma 3.11. *If $w > 0$ is a weight on Q, then $C_c(Q)$ is dense in $L^1_w(Q)$.*

Proof: Clearly, $C_c(Q) \subseteq L^1_w(Q)$. Now, we consider $f \in L^1_w(Q)$ and $\epsilon > 0$. Then $fw \in L^1(Q)$, and so, there is a $g \in C_c(Q)$ such that

$$\int_Q |fw - g|dm < \frac{\epsilon}{2}.$$

Let C be a compact set such that its interior contains $\text{supp}(g)$. Since $w \in C(Q)$, we can choose a constant K such that $w(p) \leq K$ for all $p \in C$. There is a function $h \in C_c(Q)$ such that $\text{supp}(h) \subseteq C$ and

$$\int_Q |gw^{-1} - h|dm < \frac{\epsilon}{2K}.$$

Then

$$\int_Q |f - h|wdm \leq \int_Q |fw - g|dm + \int_Q |gw^{-1} - h|wdm < \epsilon,$$

as required.

Remark 3.12. *The dual space of $L^l_w(Q)$ is $L^\infty_w(Q)$, formed by all complex-valued measurable functions ϕ on Q such that $\frac{\phi}{w} \in L^\infty(Q)$ (with the usual* *convention that two such functions are same if they coincide locally almost everywhere), and we define*

$$\|.\|_{\infty,w} = ess.sup_{p\in Q}\frac{|\phi(p)|}{w(p)}.$$

That is, the bounded linear functionals on $L^1_w(Q)$ are precisely those of the form $f \mapsto \langle f, \phi\rangle$, where

$$\langle f, \phi\rangle = \int_Q f(p)\overline{\phi(p)}dm(p) \qquad (f \in L^1_w(Q),\ \phi \in L^\infty_w(Q)),$$

with the norm $\|\phi\|_{\infty,w}$. If $\phi \in L^\infty_w(Q)$ is continuous, then for all $p \in Q$, $|\phi(p)| \le \|\phi\|_{\infty,w}w(p)$.

4 A New Convolution on Weighted KPC – Hypergroups

In this section, we assume that Q is a locally compact KPC–hypergroup, m is a Haar measure on Q and w is a weight function on Q with addition assumption $w(p)w(p^\star) = 1$. Moreover, $M_w(Q)$ is the space of all complex Radon measures μ on Q such that

$$\|\mu\|_w := \int_Q w(p)d|\mu|(p) < \infty.$$

Definition 4.1. *Let $\mu_1, \mu_2 \in M_w(Q)$. The linear functional $\mu_1 *_w \mu_2$ is defined by*

$$(\mu_1 *_w \mu_2)(f) = \int_Q\int_Q \Delta f(p,q)\frac{w(p)\Delta w(p^\star,q)}{w(q)}d\mu_1(p)d\mu_2(q).$$

Lemma 4.2. *Let $\mu_1, \mu_2 \in M_w(Q)$ be bounded measures. Then $\mu_1 *_w \mu_2$ exists and is bounded.*

Proof: Under the hypothesis, we have

$$
\begin{aligned}
|(\mu_1 *_w \mu_2)(f)| &= |\int_Q \int_Q \Delta f(p,q) \frac{w(p)\Delta w(p^\star,q)}{w(q)} d|\mu_1|(p) d|\mu_2|(q)| \\
&\leq \int_Q \int_Q |\Delta f(p,q)| \frac{w(p)\Delta w(p^\star,q)}{w(q)} d|\mu_1|(p) d|\mu_2|(q) \quad \text{(by [9], Lemma 3.1)} \\
&\leq \int_Q \int_Q \|f\| \frac{w(p)\Delta w(p^\star,q)}{w(q)} d|\mu_1|(p) d|\mu_2|(q) \\
&\leq \int_Q \int_Q \|f\| \frac{w(p)w(p^\star)w(q)}{w(q)} d|\mu_1|(p) d|\mu_2|(q) \\
&\leq \|f\| \int_Q d|\mu_1|(p) \int_Q d|\mu_2|(q) \\
&= \|f\| \cdot \|\mu_1\| \cdot \|\mu_2\|.
\end{aligned}
$$

Definition 4.3. *We define a new convolution product $*_w$ on $L^1(Q,w)$ by*

$$(f *_w g)(q) = \int_Q f(p)(\Delta g)(p^\star,q) \frac{w(p)\Delta w(p^\star,q)}{w(q)} dm(p), \tag{1}$$

where $f,g \in L^1(Q,w)$.

Theorem 4.4. *Let $f,g \in L^1(Q,w)$. Then $f *_w g \in L^1(Q,w)$ and $\|f *_w g\|_{1,w} \leq \|f\|_{1,w} \|g\|_{1,w}$.*

Proof: Let $f,g \in L^1(Q,w)$ and w be a weight function on Q such that $w(p)w(p^\star) = 1$. Then we have

$$
\begin{aligned}
\|f *_w g\|_{1,w} &= \int_Q |(f *_w g)(q)| w(q) dm(q) \\
&\leq \int_Q \int_Q |f(p)||\Delta g(p^\star,q)| \frac{w(p)\Delta w(p^\star,q)}{w(q)} w(q) dm(p) dm(q) \\
&\leq \int_Q \int_Q |f(p)| \Delta |g|(p^\star,q) \frac{w(p)\Delta w(p^\star,q)}{w(q)} w(q) dm(p) dm(q) \\
&= \int_Q |f(p)| w(p) w(p^\star) \left(\int_Q \Delta |g|(p^\star,q) w(q) dm(q) \right) dm(p) \\
&= \int_Q |f(p)| \left(\int_Q |g(q)| \Delta w(p,q) dm(q) \right) dm(p) \\
&\leq \int_Q |f(p)| w(p) dm(p) \int_Q |g(q)| w(q) dm(q) \\
&= \|f\|_{1,w} \|g\|_{1,w}.
\end{aligned}
$$

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