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Somewhat Open Sets

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Abstract

The concept of a somewhat open set is introduced and used to characterize both somewhat continuity and contra-somewhat continuity. These sets are also used to define and develop two generalizations of somewhat continuity.

Keywords: *Somewhat open, somewhat continuous, weakly somewhat continuous, strongly-θ-somewhat continuous.*

1 Introduction

Gentry and Hoyle [4] introduced the class of somewhat continuous functions in 1971. These functions, which are a generalization of continuity requiring nonempty inverse images of open sets to have nonempty interiors instead of being open, have proved to be very useful in topology. In this note we introduce the notion of a somewhat open set and use it to characterize somewhat continuity and a form of contra-somewhat continuity. Closure and interior operators are defined in terms of these sets and are used to develop weaker and stronger forms of somewhat continuity.

2 Preliminaries

The symbols X and Y represent topological spaces with no separation properties assumed unless explicitly stated. All sets are considered to be subsets of topological spaces. The closure and interior of a set A are signified by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A set A is said to be preopen [7] (respectively, semi-open [5]) if $A \subseteq \mathrm{Int}(\mathrm{Cl}(A))$, (respectively, $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$). A set A is preclosed [7] (respectively, semi-closed [2]) provided its complement is preopen (respectively, semi-open). The θ-closure of a subset A of a space X is given by $\mathrm{Cl}_\theta(A) = \{x \in X :$ for every open set U such that $x \in U, \mathrm{Cl}(U) \cap A \neq \emptyset\}$ and the θ-interior of A is given by $\mathrm{Int}_\theta(A) = \{x \in X :$ for some open set U, such that $x \in U, \mathrm{Cl}(U) \subseteq A\}$. A set A is called θ-open if $\mathrm{Int}_\theta(A) = A$ and θ-closed provided that $\mathrm{Cl}_\theta(A) = A$. The complement of a θ-open set is θ-closed and the complement of a θ-closed set is θ-open. The collection of all θ-open sets forms a topology [10].

Definition 2.1 *A function $f : X \to Y$ is said to be somewhat continuous* [4] *if for every open subset V of Y such that $f^{-1}(V) \neq \emptyset$ there exists an open subset U of X such that $U \neq \emptyset$ and $U \subseteq f^{-1}(V)$.*

Definition 2.2 *A function $f : X \to Y$ is said to be contra-1-somewhat continuous* [1] *provided that for every closed set $F \subseteq Y$ such that $f^{-1}(F) \neq \emptyset$, there exists an open set $U \subseteq X$ such that $\emptyset \neq U \subseteq f^{-1}(F)$.*

Definition 2.3 *A function $f : X \to Y$ is said to be strongly-θ-continuous* [8] *if for every $x \in X$ and every open subset V of X containing $f(x)$, there exists an open subset U of X containing x such that $f(Cl(U)) \subseteq V$.*

Definition 2.4 *A function $f : X \to Y$ is said to be contra-semicontinuous* [3]*(respectively, semicontinuous* [5]*) if $f^{-1}(V)$ is semi-closed (respectively, semi-open) for every open subset V of Y.*

3 Somewhat Open Sets

Definition 3.1 *A subset U of a space X is said to be somewhat open if $U = \emptyset$ or if there exist $x \in U$ and an open subset V such that $x \in V \subseteq U$. A set is called somewhat closed if its complement is somewhat open.*

The following characterizations are consequences of the definitions.

Theorem 3.2 *For a function $f : X \to Y$ the following conditions are equivalent:*

(a) *f is somewhat continuous.*

(b) *For every open subset V of Y, $f^{-1}(V)$ is somewhat open.*

(c) *For every closed subset F of Y, $f^{-1}(F)$ is somewhat closed.*

(d) *For every* $x \in X$ *and every open subset* V *of* Y *containing* $f(x)$ *there exists a somewhat open set* U *of* X *containing* x *such that* $f(U) \subseteq V$.

Theorem 3.3 *A function* $f : X \to Y$ *is contra-1 somewhat continuous if and only if* $f^{-1}(V)$ *is somewhat closed for every open subset* V *of* Y.

Obviously somewhat open sets are closed under arbitrary union but, as we see in the following example, not closed under intersection. Also any set containing a nonempty somewhat open set is somewhat open. Semi-open implies somewhat open and the closure of a preopen set is somewhat open.

Theorem 3.4 (Theorem 3.5, [1]) *If* $f : X \to Y$ *is contra-semicontinuous, then* f *is contra-1-somewhat continuous.*

Theorem 3.5 *If* $f : X \to Y$ *is semicontinuous, then* f *is somewhat continuous.*

Example 3.6 *Let* $X = \{a, b, c\}$ *have the topology* $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. *The sets* $\{a, c\}$ *and* $\{b, c\}$ *are somewhat open, but, since* $\{c\}$ *is not somewhat open, somewhat open sets are not closed under intersection.*

Definition 3.7 *Let* A *be a subset of a space* X. *The somewhat closure of* A, *denoted by* $swCl(A)$, *is given by* $swCl(A) = \cap\{F : F$ *is somewhat closed and* $A \subseteq F\}$ *and the somewhat interior of* A, *denoted by* $swInt(A)$, *is given by* $swInt(A) = \cup\{U \subseteq A : U$ *is somewhat open* $\}$.

The following properties of the closure and interior operators for somewhat open sets are stated for completeness. They are special cases of properties of operators defined for minimal structures by Popa and Noiri [9].

Theorem 3.8 *The following statements hold for a subset* A *of a space* X*:*

(a) $swInt(X - A) = X - swCl(A)$.

(b) $swCl(X - A) = X - swInt(A)$.

(c) $swCl(A)$ *is somewhat closed.*

(d) A *is somewhat closed if and only if* $swCl(A) = A$.

(e) $swCl(A) = \{x \in X :$ *for every somewhat open subset* U *containing* $x, U \cap A \neq \emptyset\}$

Theorem 3.9 *Let* A *be a subset of a space* X. *Then* $swCl(A) = X$ *if* A *is dense in* X, *and* $swCl(A) = A$ *if* A *is not dense in* X.

Proof: Assume A is dense in X. Let $x \in X$ and let U be a somewhat open set containing x. Then there exists a nonempty open set V such that $V \subseteq U$. Since A is dense in X, $A \cap V \neq \emptyset$. Then $A \cap U \neq \emptyset$ and hence $x \in \text{swCl}(A)$, which shows that $\text{swCl}(A) = X$.

Assume A is not dense in X. Let $x \in X$ such that $x \notin \text{Cl}(A)$. Then there exists an open set U such that $x \in U$ and $U \cap A = \emptyset$. Thus $x \in U \subseteq X - A$, which proves that $X - A$ is somewhat open and that A is somewhat closed. Thus swCl(A) $= A$. □

Corollary 3.10 *Let A be a subset of a space X. Then swInt$(A) = A$ if $X - A$ is not dense in X, and swInt$(A) = \emptyset$ if $X - A$ is dense in X.*

4 Weak and Strong Forms of Somewhat Continuity

Definition 4.1 *A function $f : X \to Y$ is said to be weakly somewhat continuous if for every $x \in X$ and every open subset V of Y containing $f(x)$, there exists a somewhat open subset U of X containing x such that $f(U) \subseteq$ Cl(V).*

Obviously somewhat continuity implies weak somewhat continuity. The following example shows that these two concepts are not equivalent.

Example 4.2 *Let $X = \{a, b, c\}$ have the topology $\tau = \{X, \emptyset, \{a\}\}$. The function $f : (X, \tau) \to (Y, \tau)$ given by $f(a) = b$, $f(b) = a$, and $f(c) = c$ is not somewhat continuous because $f^{-1}(\{a\})$ is not somewhat open. However, f is weakly somewhat continuous.*

Theorem 4.3 *A function $f : X \to Y$ is weakly somewhat continuous if and only if $f^{-1}(V) \subseteq$ swInt$(f^{-1}(\text{Cl}(V)))$ for every open subset V of Y.*

Proof: Assume f is weakly somewhat continuous. Let V be an open subset of Y and let $x \in f^{-1}(V)$. Then $f(x) \in V$ and there exists a somewhat open set U such that $x \in U$ and $f(U) \subseteq \text{Cl}(V)$. Thus $x \in U \subseteq f^{-1}(\text{Cl}(V))$ and hence $x \in \text{swInt}(f^{-1}(\text{Cl}(V)))$. Therefore $f^{-1}(V) \subseteq \text{swInt}(f^{-1}(\text{Cl}(V)))$.

Assume $f^{-1}(V) \subseteq \text{swInt}(f^{-1}(\text{Cl}(V)))$ for every open subset V of Y. Let $x \in X$ and let V be an open subset of Y containing $f(x)$. Then $x \in f^{-1}(V) \subseteq \text{swInt}(f^{-1}(\text{Cl}(V)))$. So there exists a somewhat open subset U of X such that $x \in U \subseteq f^{-1}(\text{Cl}(V))$. Therefore $f(U) \subseteq \text{Cl}(V)$, which proves that f is weakly somewhat continuous. □

Corollary 4.4 *A function $f : X \to Y$ is weakly somewhat continuous if and only if for every open subset V of Y, $f^{-1}(V) = \emptyset$ whenever Int$(f^{-1}(\text{Cl}(V))) = \emptyset$.*

Proof: Assume $f : X \to Y$ is weakly somewhat continuous. Let V be an open subset of Y and assume $\mathrm{Int}(f^{-1}(\mathrm{Cl}(V))) = \emptyset$. Then $\mathrm{Cl}(X - f^{-1}(\mathrm{Cl}(V))) = X - \mathrm{Int}(f^{-1}(\mathrm{Cl}(V))) = X$ and hence $X - f^{-1}(\mathrm{Cl}(V))$ is dense in X. Thus by Corollary 3.10 $\mathrm{swInt}(f^{-1}(\mathrm{Cl}(V))) = \emptyset$ and, since $f^{-1}(V) \subseteq \mathrm{swInt}(f^{-1}((Cl(V))),\ f^{-1}(V) = \emptyset$.

Assume that for every open subset V of Y, $f^{-1}(V) = \emptyset$ whenever $\mathrm{Int}(f^{-1}(\mathrm{Cl}(V))) = \emptyset$. Let V be an open subset of Y. Assume $f^{-1}(V) \neq \emptyset$. Then $\mathrm{Int}(f^{-1}(\mathrm{Cl}(V))) \neq \emptyset$ and hence $X - f^{-1}(\mathrm{Cl}(V))$ is not dense in X and by Corollary 3.10 $\mathrm{swInt}(f^{-1}(\mathrm{Cl}(V))) = f^{-1}(\mathrm{Cl}(V))$. Thus $f^{-1}(V) \subseteq \mathrm{swInt}(f^{-1}(\mathrm{Cl}(V))$, which proves that f is weakly somewhat continuous. □

Since Levine [6] proved that a function $f : X \to Y$ is weakly continuous if and only if for every open subset V of Y, $f^{-1}(V) \subseteq \mathrm{Int}(f^{-1}(\mathrm{Cl}(V)))$, we have the following result.

Corollary 4.5 *If $f : X \to Y$ is weakly continuous, then f is weakly somewhat continuous.*

The following example shows that weak continuity is not equivalent to weak somewhat continuity.

Example 4.6 *Let X be the real numbers with the usual topology and let $f : X \to X$ be the function given by $f(x) = 1$ if $x \leq 1$ and $f(x) = 2$ if $x > 1$. Levine [6] proved that a weakly continuous function with a regular codomain is continuous. It then follows that f is not weakly continuous. However f is weakly somewhat continuous.*

Theorem 4.7 *If $f : X \to Y$ is weakly somewhat continuous and Y is regular, then f is somewhat continuous.*

Proof: Assume $f : X \to Y$ is weakly somewhat continuous and Y is regular. Let V be an open subset of Y and let $x \in f^{-1}(V)$. Then there exists an open set W such that $f(x) \in W \subseteq \mathrm{Cl}(W) \subseteq V$. Since f is weakly somewhat continuous, there exists a somewhat open subset U of X such that $x \in U$ and $f(U) \subseteq \mathrm{Cl}(W) \subseteq V$ Hence $x \in U \subseteq f^{-1}(V)$, which proves that $f^{-1}(V)$ is somewhat open. □

Definition 4.8 *A function $f : X \to Y$ is said to be strongly-θ-somewhat continuous if for every $x \in X$ and every open subset V of Y containing $f(x)$, there exists a somewhat open subset U of X containing x such that $f(swCl(U)) \subseteq V$.*

Obviously strongly-θ-somewhat continuous implies somewhat continuous. Also, since $\mathrm{swCl}(A) \subseteq \mathrm{Cl}(A)$ for every set A, strongly-θ-continuous implies strongly-θ-somewhat continuous. The following examples show that the reverse implications do not hold.

Example 4.9 *Let* $X = \{a, b, c\}$ *have the topology* $\tau = \{X, \emptyset, \{a\}\}$. *The identity mapping* $f : (X, \tau) \to (Y, \tau)$ *is obviously somewhat continuous. However* f *is not strongly-*θ*-somewhat continuous since for* $V = \{a\}$ *there is no somewhat open set* U *containing* a *for which* $f(swCl(U)) \subseteq V$.

Example 4.10 *Let* $X = \{a, b, c\}$ *have the topology* $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. *The identity mapping* $f : (X, \tau) \to (Y, \tau)$ *is not strongly-*θ*-continuous but is strongly-*θ*-somewhat continuous. In fact, every somewhat continuous function on* X *is strongly-*θ*-somewhat continuous.*

Definition 4.11 *A collection* $\mathcal{B}$ *of somewhat open subsets of a space* X *is said to be a somewhat open base for* X *provided that every somewhat open set is the union of members of* $\mathcal{B}$.

Theorem 4.12 *If* $f : X \to Y$ *is somewhat continuous and* X *has a somewhat open base consisting of non-dense sets, then* f *is strongly-*θ*-somewhat continuous.*

Proof: Let $x \in X$ and let V be an open subset of Y containing $f(x)$. Since f is somewhat continuous, $f^{-1}(V)$ is somewhat open. Let $\mathcal{B}$ be a somewhat open base for X consisting of non-dense sets.Then there exists $B \in \mathcal{B}$ such that $x \in B \subseteq f^{-1}(V)$. Thus, using Theorem 3.9, we have $f(\mathrm{swCl}(B)) = f(B) \subseteq V$, which proves that$f$ is strongly-θ-somewhat continuous. □

Example 4.13 *The collection* $\mathcal{B} = \{\{a\}, \{b\}, \{a, c\}, \{b, c\}\}$ *is a somewhat open base for the space in Example 4.10 which consists of non-dense sets.*

Definition 4.14 *Let* A *be a subset of a space* X. *The* θ*-somewhat closure of* A *is given by* $swCl_\theta(A) = \{x \in X : swCl(V) \cap A \neq \emptyset$ *for every somewhat open subset* $V \subseteq X$ *such that* $x \in V\}$. *A set* A *is called* θ*-somewhat closed if* $swCl_\theta(A) = A$. *The* θ*-somewhat-interior of a set* A *is given by* $swInt_\theta(A) = \{x \in X :$ *there exists a somewhat open set* U *such that* $x \in U$ *and* $swCl(U) \subseteq A\}$ *A set* A *is called* θ*-somewhat open if* $swInt_\theta(A) = A$.

Lemma 4.15 *Let* A *be a subset of a space* X.

(a) $X - swInt_\theta(A) = swCl_\theta(X - A)$.

(b) $X - swCl_\theta(A) = swInt_\theta(X - A)$.

The following result is a direct consequence of Lemma 4.15.

Theorem 4.16 *The complement of a* θ*-somewhat open set is* θ*-somewhat closed and the complement of a* θ*-somewhat closed set is* θ*-somewhat open.*

Theorem 4.17 *Let A be a subset of a space X. Then $swCl_\theta(A) = \{x \in X : U \cap A \neq \emptyset$ for every non dense somewhat open set U containing $x\}$.*

Proof: Let $x \in \mathrm{swCl}_\theta(A)$. Let U be a somewhat open, non-dense set containing x. Then $\mathrm{swCl}(U) \cap A \neq \emptyset$. Since by Theorem 3.9 $\mathrm{swCl}(U) = U$, $U \cap A \neq \emptyset$.

Let $x \in X$ and assume that for every somewhat open non-dense set U containing x that $U \cap A \neq \emptyset$. Let U be a somewhat open dense set containing x. Since Theorem 3.9 implies that $\mathrm{swCl}(U) = X$, obviously $\mathrm{swCl}(U) \cap A \neq \emptyset$. Therefore $\mathrm{swCl}(U) \cap A \neq \emptyset$ for every somewhat open set U containing x and hence $x \in \mathrm{swCl}_\theta(A)$. □

Theorem 4.18 *For a function $f : X \to Y$ the following statements are equivalent;*

(a) *f is strongly-θ-somewhat continuous.*

(b) *For every closed subset F of Y, $f^{-1}(F)$ is θ-somewhat closed.*

(c) *For every open subset V of Y, $f^{-1}(V)$ is θ-somewhat open.*

Proof: $(a) \Rightarrow (b)$ Let F be a closed subset of Y. Assume $f^{-1}(F)$ is not θ-somewhat closed. Then $f^{-1}(F) \neq \mathrm{swCl}_\theta(f^{-1}(F))$. Let $x \in \mathrm{swCl}_\theta(f^{-1}(F)) - f^{-1}(F)$. Then $f(x) \in Y - F$ which is open in Y. However, for every somewhat open set U containing x, $\mathrm{swCl}(U) \cap f^{-1}(F) \neq \emptyset$ and hence for every somewhat open set containing x, $f(\mathrm{swCl}(U)) \nsubseteq Y - F$. Thus f is not strongly-θ-somewhat continuous .

$(b) \Rightarrow (c)$ Let V be an open subset of Y. Then $Y - V$ is closed and by (b) $X - f^{-1}(V) = f^{-1}(Y - V)$ is θ-somewhat closed in X. Thus $f^{-1}(V)$ is θ-somewhat open in X.

$(c) \Rightarrow (a)$ Let $x \in X$ and let V be an open subset of Y containing $f(x)$. Then by (c) $f^{-1}(V)$ is θ-somewhat open and therefore $f^{-1}(V) = \mathrm{swInt}_\theta(f^{-1}(V))$. Hence there exists a somewhat open set U such that $x \in U$ and $\mathrm{swCl}(U) \subseteq f^{-1}(V)$. Thus $f(\mathrm{swCl}(U)) \subseteq V$, which proves that f is strongly-θ-somewhat continuous . □

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