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Some Properties which are Preserved under Localization of Commutative Rings and Modules

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Abstract

In this paper, the effect of localization on the properties of commutative rings and some types of modules are investigated. Certain properties of commutative rings and modules which are preserved under localization are proved and some properties of co-multiplication modules have been extended to those modules the localization of which is co-multiplication.

Keywords: *Multiplicative closed sets, localization of commutative rings and modules, co-multiplication modules and second submodules.*

1 Introduction

In all what follows, R is a commutative ring with identity and M is a (left) R −module. A non-empty subset S of R is called a multiplicatively closed set in R if $0 \notin S$ and whenever, $s, t \in S$, then $st \in S$[5]. If S is a multiplicatively closed set in R, then one can easily make M_S as an R_S −module under the module operations $\frac{a}{s} + \frac{b}{t} = \frac{ta+sb}{st}$ and $\frac{r}{u} . \frac{a}{s} = \frac{ra}{us}$, for $\frac{r}{u} \in R_S$ and $\frac{a}{s}, \frac{b}{t} \in M_S$[6], so that when we say M_S is a module we mean M_S is an R_S −module. It is known that, for any prime ideal P of R, we have $S = R \setminus P$ is a multiplicatively closed set in R, in this case we denote $R_{R\setminus P}$ and $M_{R\setminus P}$ by R_P and M_P respectively. Let N be a submodule $M$and A be an ideal of R, then we define $(N:_M A) = \{m \in M: Am \subseteq N\}$[7], especially if $N = 0$, then $(0:_M A) = \{m \in M: Am = 0\}$and $(N: M)$ is defined as $(N: M) = \{r \in R: rM \subseteq N\}$, which is an ideal of R[8], in case that $N = 0$, then $(0: M)$ is called the annihilator of M and denoted by $Ann(M)$, so that $Ann(M) = (0: M) = \{r \in R: rM = 0\}$. N is called a minimal submodule, if $N \neq 0$ and whenever K is a submodule of M such that $0 \subseteq K \subseteq N$, then $K = 0$ or $N = K$. The Jacobson radical of R, denoted by $J(R)$, is defined as the intersection of all maximal ideals of R[9].An element $r \in R$ is said to be prime to N, if $x \in M$ such that $rx \in N$, then $x \in N$[2], equivalently r is not prime to$N$ if there exists $m \in M \setminus N$ such that $rm \in N$. The set of all elements of R that are not prime to N is denoted by $S_M(N)$[2], so that $S_M(N) = \{r \in R: rx \in N$, for some $x \notin N\}$ and for$N = 0$, we have $S_M(0) = \{r \in R: rx = 0$, for some $0 \neq x \in M\}$. It is obvious that, $S_M(M) = \emptyset$.M is called a co-multiplication module, if for each submodule N of M, there exists an ideal A of R suchthat $N = (0:_M A)$[1] and$N$ is called a second submodule, if $N \neq 0$ and for each $r \in R$, the homomorphism $f_r: N \to N$, defined by $f_r(m) = rm$, for $m \in N$, is either surjective or zero [1].

2 The Main Results

Proposition 2.1: *Let M be an R −module and Sbe a multiplicatively closed set in R. Then any endomorphism $f: M \to M$ of M can be extended to an endomorphism $f_S: M_S \to M_S$ of M_S.*

Proof: $f_S: M_S \to M_S$ defined by $f_S\left(\frac{m}{s}\right) = \frac{f(m)}{s}$, where $m \in M$ and $s \in S$, is the required endomorphism. If $\frac{m}{p} = \frac{n}{t}$, for $m, n \in M$ and $p, t \in S$, then $q(tm - pn) = 0$ for some q$\in S$, which gives that $qtm = qpn$, then $f(qtm) = f(qpn)$, from which we get $qtf(m) = qpf(n)$, that means $q\big(tf(m) - pf(n)\big) = 0$, this gives $\frac{f(m)}{p} = \frac{f(n)}{t}$ and that $f_S\left(\frac{m}{p}\right) = f_S\left(\frac{n}{t}\right)$. Next,$f_S\left(\frac{m}{p} + \frac{n}{t}\right) = f_S\left(\frac{tm+pn}{pt}\right) = \frac{f(tm+pn)}{pt} = \frac{f(tm)+f(pn)}{pt} = \frac{tf(m)+pf(n)}{pt} = \frac{tf(m)}{tp} + \frac{pf(n)}{pt} = \frac{f(m)}{p} + \frac{f(n)}{t} = f_S\left(\frac{m}{p}\right) + f_S\left(\frac{n}{t}\right)$. Now, if $\frac{r}{s} \in R_S$, then $f_S\left(\frac{r}{s}\frac{m}{p}\right) = f_S\left(\frac{rm}{sp}\right) = \frac{f(rm)}{sp} = \frac{rf(m)}{sp} = \frac{r}{s} f_S(\frac{m}{p})$.

Hence, f_S is an endomorphism of M_S.

It is known that, if P is a prime ideal of R, then $S = R \setminus P$ is a multiplicatively closed in R, so that as a corollary to the above proposition we give.

Corollary 2.2: *Let M be an R −module and P be a prime ideal of$R$, then any endomorphism$f: M \to M$can be extended to an endomorphism$f_P: M_P \to M_P$ofM_P.*

Proof: By taking $S = R \setminus P$ inProposition 2.1, the proof will follows at once.

Theorem2.3: *Let M be an R −module and Sbe a multiplicatively closed set in R. If $f$is an endomorphism of M, then $kerf_S = (kerf)_S$.*

Proof: Let $\frac{m}{s} \in kerf_S$, where $m \in M, s \in S$, so that $f_S\left(\frac{m}{s}\right) = 0$, which gives that $\frac{f(m)}{s} = 0 = \frac{0}{s}$, so there exists $t \in S$ such that $tf(m) = 0$, then we get $f(tm) = 0$, thus $tm \in kerf$, this gives that $\frac{m}{s} = \frac{t}{t}\frac{m}{s} = \frac{tm}{ts} \in (kerf)_S$. Hence, we get $kerf_S \subseteq (kerf)_S$. Next, let$\frac{m}{s} \in (kerf)_S$, where $m \in M, s \in S$, then we have $tm \in kerf$, for some $t \in S$. So that, $f(tm) = 0$. Then, we get $f_S\left(\frac{m}{s}\right) = \frac{f(m)}{s} = \frac{t}{t}\frac{f(m)}{s} = \frac{tf(m)}{ts} = \frac{f(tm)}{ts} = 0$. Thus, $\frac{m}{s} \in kerf_S$, so that $(kerf)_S \subseteq kerf_S$.

Hence, $kerf_S = (kerf)_S$.

Corollary 2.4: *Let M be an R −module and P be a prime ideal of R. If f is anendomorphism of M, then $kerf_P = (kerf)_P$.*

Proof: Since, $S = R \setminus P$ is a multiplicatively closed set in R, so that the proof will follows at once by taking $S = R \setminus P$ in Theorem 2.3.

Theorem 2.5: *Let M be an R −module and S be a multiplicatively closed set in R. If $f$is an endomorphism of M, then $Imf_S = (Imf)_S$.*

Proof: Let $\frac{y}{t} \in Imf_S$, where $y \in M, t \in S$, so that $f_S\left(\frac{x}{s}\right) = \frac{y}{t}$, for some $\frac{x}{s} \in M_S$, that is, $\frac{f(x)}{s} = \frac{y}{t}$. Then, there exists $u \in S$ such that $utf(x) = usy$, so that $usy = f(utx) \in Imf$. Hence, we get $\frac{y}{t} = \frac{u}{u}\frac{s}{s}\frac{y}{t} = \frac{usy}{ust} \in (Imf)_S$, so that we get $Imf_S \subseteq (Imf)_S$. Next, let $\frac{y}{t} \in (Im(f))_S$, where $y \in M, t \in S$, then $sy \in Imf$, for some $s \in S$, so that $sy = f(x)$, for some $x \in M$. Then, clearly we have $\frac{x}{st} \in M_S$ and that, $f_S\left(\frac{x}{st}\right) = \frac{f(x)}{st} = \frac{sy}{st} = \frac{y}{t}$, so that $\frac{y}{t} \in Imf_S$. Thus, $(Im(f))_S \subseteq Imf_S$.

Hence, we get $Imf_S = (Imf)_S$.

Corollary 2.6: *Let M be an R −module and P be a prime ideal of R. If f is an endomorphism of M, then $Imf_P = (Imf)_P$.*

Proof: As, $S = R \setminus P$is a multiplicatively closed set in R, the proof will follows atonce by taking $S = R \setminus P$in Theorem 2.5.

Proposition 2.7: *Let M be an R −module and P be a prime ideal of R. If N is a minimal submodule of M such that $S_M(0) \subseteq P$ and $S_M(N) \subseteq P$, then N_P is a minimal submodule of M_P.*

Proof: If $N_P = 0$, then as $S_M(0) \subseteq P$, by [4, Lemma 2.1], we get $N = 0$, that is a contradiction, so that $N_P \neq 0$. Now, suppose that $0 \neq \overline{K}$ be any submodule of M_P such that $0 \subseteq \overline{K} \subseteq N_P$, then by [3, Proposition 2.16], we have $\overline{K} = K_P$, for the submodule $K = \{k \in M : \frac{k}{1} \in \overline{K}\}$. As, $\overline{K} = K_P \neq 0$, we must have $K \neq 0$. Let $x \in K$, then $\frac{x}{1} \in N_P$ and as $S_M(N) \subseteq P$, by [4, Lemma 2.1], we get $x \in N$, so that $0 \subset K \subseteq N$. As N is minimal, we get $N = K$ and hence, $\overline{K} = K_P = N_P$. Thus N_P is a minimal submodule of M_P.

Proposition 2.8: *Let M be an R −module. If N is a submodule of M such that $S_M(0) \subseteq J(R)$ and N_P is a minimal submodule of M_P, for every maximal ideal P of R, then N is a minimal submodule of M.*

Proof: Let K be any submodule of M such that $0 \subseteq K \subseteq N$ and let P be any maximal ideal of R. If $N = 0$, then we get $N_P = 0$, that is a contradiction, so that $N \neq 0$. Now, we have $0 \subseteq K_P \subseteq N_P$. If $K_P = N_P$ for every maximal ideal P of R, then by [3, Corollary2.2], we get $K = N$ and if $K_Q \neq N_Q$, for some maximal ideal Q of R and since N_Q is minimal, we get $K_Q = 0$ and as $S_M(0) \subseteq J(R) \subseteq Q$, by [4, Lemma 2.1], we get $K = 0$, so that N is a minimal submodule of M.

Proposition 2.9: *Let M be an R −module and P be a prime ideal of R such that $S_M(0) \subseteq P$. If N is a submodule of M, then $(Ann(N))_P = Ann(N_P)$* and *$(Ann(M))_P = Ann(M_P)$.*

Proof: One can easily get the result by using the same technique as in the proof of [4, Proposition 2.5].

Proposition 2.10: *Let M be an R −module and P be a prime ideal of R. If $f: M \to M$ is an endomorphism of M such that $S_M(0) \subseteq P$, then $Ann(kerf_P) = (Ann(kerf))_P$.*

Proof: As $kerf$ is a submodule of M, by Proposition 2.9, we have $(Ann(kerf))_P = Ann(kerf)_P$, so by Corollary 2.4, we get $Ann(kerf_P) = Ann(kerf)_P = (Ann(kerf))_P$.

Proposition 2.11: *Let$M$ be an R −module and P be a prime ideal of R such that $S_M(0) \subseteq P$. If $Ann(M)$ is a prime ideal of R, then $Ann(M_P)$is a prime ideal of R_P.*

Proof: Suppose that $Ann(M_P) = R_P$. If $r \in R$, then $\frac{r}{1} \in R_P$, so that $\frac{r}{1} \in Ann(M_P)$, thus $\frac{r}{1} M_P = 0$, so by [3, Proposition 2.8], we get $(rM)_P = 0$ and as $S_M(0) \subseteq P$, by [4, Lemma 2.1], we get $rM = 0$, so that $r \in Ann(M)$, that gives $Ann(M) = R$, which is a contradiction. Hence, we get $Ann(M_P) \neq R_P$. Let $\frac{r}{p}\frac{s}{q} \in Ann(M_P)$, then $\frac{rs}{pq} M_P = 0$ and by [3, Proposition 2.8], we get $(rsM)_P = 0$ and since, $S_M(0) \subseteq P$, by [4, Lemma 2.1], we get $rsM = 0$, that is $rs \in Ann(M)$ and as $Ann(M)$ is prime, we get $r \in Ann(M)$ or $s \in Ann(M)$. Hence, by using Proposition 2.9, we get $\frac{r}{p} \in \big(Ann(M)\big)_P = Ann(M_P)or\frac{s}{q} \in \big(Ann(M)\big)_P = Ann(M_P)$, thus $Ann(M_P)$ is a prime ideal of R_P.

Theorem 2.12: *Let M bean R −module such that M_P is a comultiplication R_P −module for every maximal ideal of R and $S_M(0) \subseteq J(R)$. If $f: M \to M$ is ahomomorphism, then $Imf = (Ann(kerf))M$.*

Proof: Let P be any maximal ideal of R, then as $S_M(0) \subseteq J(R)$, we get $S_M(0) \subseteq P$. Since, $f: M \to M$ is an endomorphism of M, so by Corollary 2.2, we have $f_P: M_P \to M_P$ is an endomorphism of M_P and as M_P is a comultiplication R_P −module, by [1, Theorem 2.1], we get $Imf_P = (Ann(kerf_P))M_P$. Next, by Corollary 2.6 and Proposition 2.10, we have $(Imf)_P = (Ann(kerf))_P M_P = (Ann(kerf)M)_P$, then by [3, Corollary2.2], we get $Imf = (Ann(kerf))M$.

Proposition 2.13: *If M is a second R −module and P is any prime ideal of R, then M_P is a second R_P −module.*

Proof: Let $\frac{r}{p} \in R_P$ and to show that the homomorphism $f_{\frac{r}{p}}: M_P \to M_P$, defined by $f_{\frac{r}{p}}\left(\frac{n}{t}\right) = \frac{rn}{pt}$ is zero or surjective. As M is a second module, the homomorphism $f_r: M \to M$, defined by $f_r(m) = rm$, for $m \in M$, is either zero or surjective. If $f_r = 0$, then for any $\frac{m}{q} \in M_P$, we have $f_{\frac{r}{p}}\left(\frac{m}{q}\right) = \frac{rm}{pq} = \frac{f_r(m)}{pq} = 0$, so that $f_{\frac{r}{p}} = 0$ and if f_r is surjective, then for any $\frac{m}{q} \in M_P$, we have $pm \in M$, so there exists $n \in M$ such that $f_r(n) = pm$. Then, $\frac{n}{q} \in M_P$ and that$f_{\frac{r}{p}}\left(\frac{n}{q}\right) = \frac{rn}{pq} = \frac{f_r(n)}{pq} = \frac{pm}{pq} = \frac{m}{q}$, so that $f_{\frac{r}{p}}$ is surjective. Hence, M_P is a second R_P −module.

Now, we introduce the following definitions.

Definition 2.14: *Let M be an R −module. A function $f: M \to M$ is said to be an almost onto function if for each $m \in M$, there exist $n \in M$ and $a \in R$, such that $f(n) = am$.*

It is obvious that, if $f: M \to M$ is onto, then it is almost onto, since if $m \in M$, then as f is onto, there exists $n \in M$ such that $f(n) = m$, that means $f(n) = 1m$, where $1 \in R$. Hence, f is almost onto. But, an almost onto function need not be onto in general, as we see in the following example.

Example 2.15: *Consider the Z −module Z_6. Define $f: Z_6 \to Z_6$ by $f(x) = x$, for every $x \neq \bar{5}$ and $f(\bar{5}) = \bar{0}$. Clearly, f is an almost onto function and since $\bar{5}$ has not preimage, so that f is not onto.*

Definition 2.16: *A submodule N of an R −module M is called an almost second submodule if for every $r \in R$, the homomorphism $f_r: N \to N$, defined by $f_r(m) = rm$, for $m \in N$, is either the zero homomorphism or it is an almost onto (surjective) homomorphism.*

Since, every onto function is almost onto, so it is obvious that, every second module is an almost second module.

Proposition 2.17: *Let N be a submodule of an R −module M and P be a prime ideal of R such that $S_M(0) \subseteq P$. If N_P is a second submodule, then N is an almost second module.*

Proof: Let $r \in R$ and $f_r: N \to N$ be the homomorphism defined by $f_r(m) = rm$, for $m \in N$ and let $f_r \neq 0$. Let $m \in N$, then as N_P is a second submodule, the homomorphism $f_{\frac{r}{1}}: N_P \to N_P$ defined by $f_{\frac{r}{1}}\left(\frac{m}{p}\right) = \frac{rm}{p}$, is either zero or surjective. Let $f_{\frac{r}{1}} = 0$, then for any $m \in N$, we have $f_{\frac{r}{1}}\left(\frac{m}{1}\right) = \frac{rm}{1} = 0$ and as $S_M(0) \subseteq P$, by [4, Lemma 2.1], we get $rm = 0$, so that $f_r(m) = rm = 0$, that means $f_r = 0$. Next, let $f_{\frac{r}{1}}$ be surjective. Then, for any $m \in N$, there exists $\frac{n}{t} \in N_P$ such that $f_{\frac{r}{1}}\left(\frac{n}{t}\right) = \frac{m}{1}$, so that $\frac{rn}{t} = \frac{m}{1}$ and $un \in N$, for some $u \notin P$. Hence, $qrn = qtm$, for some $q \notin P$, then we get $q(rn - tm) = 0$. If $rn - tm \neq 0$, then $q \in S_M(0) \subseteq P$, which is a contradiction, so we get $rn - tm = 0$ and then $u(rn - tm) = 0$, so that $urn = utm$, that is $run = utm$, so that $f_r(un) = utm$, where $ut \in R$ and $un \in N$, so that f_r is almost onto. Hence, N is an almost second module.

Proposition 2.18: *Let M be an R −module and S be a multiplicative closed set in R. If I is an ideal of R and $x \in M$, then:*

(1) Ix is a submodule of M and $(Ix)_S = I_S \frac{x}{s}$ for every $s \in S$.

(2) If N is a submodule of M, then $(N:_M I)_S \subseteq (N_S:_{M_S} I_S)$. Furthermore, if $S_M(N) \cap S = \emptyset$, then $(N_S:_{M_S} I_S) \subseteq (N:_M I)_S$ and that $(N:_M I)_S = (N_S:_{M_S} I_S)$.

Proof: (1) The proof that Ix is a submodule of M is easy, so we only prove the second part. Let $s \in S$ and $\frac{a}{t} \in (Ix)_S$, for $a \in M$ and $t \in S$, then $qa \in Ix$, for some $q \in S$, this implies that $qa = rx$, for some $r \in I$. Now we have, $\frac{a}{t} = \frac{s}{s}\frac{q}{q}\frac{a}{t} = \frac{sqa}{sqt} =$

$\frac{srx}{sqt} = \frac{sr}{qt}\frac{x}{s} \in I_S\frac{x}{s}$. Hence, $(Ix)_S \subseteq I_S\frac{x}{s}$. By a similar technique as in (1), one can easily show that $I_S\frac{x}{s} \subseteq (Ix)_S$, so that $(Ix)_S = I_S\frac{x}{s}$.

(2) Let, $\frac{x}{s} \in (N:_M I)_S$, where $x \in M$ and $s \in S$. Then, $px \in (N:_M I)$, for some $p \in S$, so that $Ipx \subseteq N$, from which we get $(Ipx)_S \subseteq N_S$. As $ps \in S$, by using (1), we get $I_S\frac{px}{ps} = (Ipx)_S \subseteq N_S$, this gives that $\frac{x}{s} = \frac{px}{ps} \in (N_S:_{M_S} I_S)$. Hence, $(N:_M I)_S \subseteq (N_S:_{M_S} I_S)$. To prove the second part, let $\frac{x}{s} \in (N_S:_{M_S} I_S)$, for $x \in M$ and $s \in S$, then $I_S\frac{x}{s} \subseteq N_S$. By using (1), we get $(Ix)_S = I_S\frac{x}{s} \subseteq N_S$. If $Ix \nsubseteq N$, then there exists $a \in I$ such that $ax \notin N$. In this case, $\frac{ax}{s} \in (Ix)_S \subseteq N_S$, so that $qax \in N$, for some $q \in S$, this gives $q \in S_M(N)$ and as $q \in S$, we have $S_M(N) \cap S \neq \emptyset$, that is a contradiction, so we must have $Ix \subseteq N$, that is, $x \in (N:_M I)$, then $\frac{x}{s} \in (N:_M I)_S$. So that, $(N_S:_{M_S} I_S) \subseteq (N:_M I)_S$. Hence, we get $(N:_M I)_S = (N_S:_{M_S} I_S)$.

As a corollary to Proposition 2.18, we give the following.

Corollary 2.19: *Let M be an $R-$module and P be a prime ideal of R. If I is an ideal of R and $x \in M$, then:*

(1) $(Ix)_P = I_P\frac{x}{s}$, *for every $s \notin P$.*

(2) *If N is a submodule of M, then $(N:_M I)_P \subseteq (N_P:_{M_P} I_P)$. Furthermore, if $S_M(N) \subseteq P$, then $(N_P:_{M_P} I_P) \subseteq (N:_M I)_P$ and that $(N:_M I)_P = (N_P:_{M_P} I_P)$.*

Proof: By taking the multiplicative closed set $S = R\backslash P$ in Proposition 2.18, the result follows.

Proposition 2.20: *Let M be an $R-$module and A be an ideal of R. If P is a prime ideal of R, then $(0:_M A)_P \subseteq (0:_{M_P} A_P)$.*

Proof: Let $\frac{m}{p} \in (0:_M A)_P$, where $m \in M, p \notin P$. Then, $qm \in (0:_M A)$, for some $q \notin P$, which implies that $Aqm = 0$. By using Corollary 2.19, we get $A_P\frac{m}{p} = A_P\frac{q}{q}\frac{m}{p} = (Aqm)_P = 0$, so that $\frac{m}{p} \in (0:_{M_P} A_P)$. Hence, $(0:_M A)_P \subseteq (0:_{M_P} A_P)$.

Proposition 2.21: *Let M be an $R-$module and A be an ideal of R. If P is a prime ideal of R such that $S_M(0) \subseteq P$, then $(0:_{M_P} A_P) \subseteq (0:_M A)_P$.*

Proof: Let $\frac{m}{p} \in (0:_{M_P} A_P)$, where $m \in M, p \notin P$. Then, $A_P\frac{m}{p} = 0$ and by Corollary 2.19, we get $(Am)_P = A_P\frac{m}{p} = 0$ and since $S_M(0) \subseteq P$, so by [4, Lemma 2.1], we get $Am = 0$, this gives $m \in (0:_M A)$, which yields $\frac{m}{p} \in (0:_M A)_P$. Hence, $(0:_{M_P} A_P) \subseteq (0:_M A)_P$.

Combining Proposition 2.20 and Proposition 2.21, we get the following corollary.

Corollary 2.22: *Let M be an R −module and A be an ideal of R. If P is a prime ideal of R such that $S_M(0) \subseteq P$, then $\left(0:_{M_P} A_P\right) = (0:_M A)_P$.*

Proposition 2.23: *Let M be a comultiplication module and P be a prime ideal of R such that $S_M(0) \subseteq P$, then M_P is a comultiplication R_P −module.*

Proof: Let $\bar{N}$ be a submodule of M_P, then by [3, Proposition 2.16],$\bar{N} = N_P$, for the submodule $N = \{x \in M: \frac{x}{1} \in \bar{N}\}$. As M is comultiplication, $N = (0:_M A)$, for some ideal A of R. Since, $S_M(0) \subseteq P$, by Corollary 2.22, we get $\bar{N} = N_P = (0:_M A)_P = \left(0:_{M_P} A_P\right)$, where A_P is an ideal of R_P. Hence, M_P is a comultiplication R_P −module.

Proposition 2.24: *Let M be an R −module and P be a prime ideal of R. If N is a minimal submodule of M such that $S_M(0), S_M(N) \subseteq P$ and X, Y are any two submodules of M with $X \cap N = Y \cap N = 0$, then $N \cap (X + Y) = 0$.*

Proof: As N is a minimal submodule of M and $S_M(N) \subseteq P$, by Proposition 2.7, we get that N_P is a minimal submodule of M_P. Now, $X \cap N = Y \cap N = 0$ gives that $(X \cap N)_P = (Y \cap N)_P = 0$, that is $X_P \cap N_P = Y_P \cap N_P = 0$, so that by [1, Theorem 2.1], we get $N_P \cap (X_P + Y_P) = 0$, that is $(N \cap (X + Y))_P = 0$, and as $S_M(0) \subseteq P$, by [4, Lemma 2.1], we get $N \cap (X + Y) = 0$.

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