Gen. Math. Notes, Vol. 36, No. 1, September 2016, pp.34-47
ISSN 2219-7184;
www.i-csrs.org
Available free online at http://www.geman.in

Properties of k-Jacobsthal and k-Jacobsthal Lucas Sequences

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(Received: 17-3-16 / Accepted: 20-8-16)

Abstract

In this study, two sequences called k-Jacobsthal, k-Jacobsthal Lucas are defined by considering the usual Jacobsthal and Jacobsthal Lucas numbers. After that, we establish some properties of these sequences and some important relationships between k-Jacobsthal sequence and k-Jacobsthal Lucas sequence and the generatig functions, some sum formulas.

Keywords: *Jacobsthal numbers, Jacobsthal Lucas numbers, Binet's formula, Generating functions.*

1 Introduction

In the last years, we have seen a great many studies on the on the different number sequences because of abundant applications in science and art, and etc. For instance, the ratio of two consecutive elements of Fibonacci sequence is always golden ratio, is very important number almost every area of science and art. And it is well known that computers use conditional directives to change the flow of execution of a program. In addition to branch instructions, some microcontrollers use skip instructions which conditionally bypass the next instruction.This brings out being useful for one case out of the four possibilities on 2 bits, 3 cases on 3 bits, 5 cases on 4 bits, 11 cases on 5 bits, 21 cases on 6 bits,..., which are exactly the Jacobsthal numbers. There are a lot of identities of number sequences described in [2,3]. From these sequences, Jacobsthal and Jacobsthal Lucas numbers are given by the recurrence relations $j_n = j_{n-1} + 2j_{n-2}$, $j_0 = 0$, $j_1 = 1$ and $c_n = c_{n-1} + 2c_{n-2}$, $c_0 = 2$, $c_1 = 1$ for $n \geq 2$, respectively. We can see any properties of these numbers in all references of us. The second order recurrence sequence has been generalized in two ways mainly, first by preserving the initial conditions and second by preserving the recurrence relation. We used the first way for the following definition. In this paper, a new generalization of the Jacobsthal and Jacobsthal Lucas numbers is introduced. It should be noted that the recurrence formulas of these numbers depend on one real parameter. For example, Uygun defined (s,t) Jacosthal and (s,t) Jacosthal Lucas sequences depending two real parameters and by using them found some properties of Jacobsthal numbers in [4].

2 k-Jacobsthal and k-Jacobsthal Lucas Sequences

Definition 2.1 *For any positive real numbers k; the k-Jacobsthal $\{\hat{j}_{k,n}\}_{n\in\mathbb{N}}$ and the k-Jacobsthal Lucas $\{\hat{c}_{k,n}\}_{n\in\mathbb{N}}$ number sequences are defined recurrently by*

$$\hat{j}_{k,n} = k\hat{j}_{k,n-1} + 2\hat{j}_{k,n-2}, \quad \hat{j}_{k,0} = 0, \ \hat{j}_{k,1} = 1, \quad n \geq 2, \tag{1}$$

and

$$\hat{c}_{k,n} = k\hat{c}_{k,n-1} + 2\hat{c}_{k,n-2}, \qquad \hat{c}_{k,0} = 2, \ \hat{c}_{k,1} = k, \quad n \geq 2, \tag{2}$$

respectively.

From (1) and (2) we thus have the following tabulation for k-Jacobsthal numbers $\hat{j}_{k,n}$ and k-Jacobsthal Lucas numbers $\hat{c}_{k,n}$:

Table 1: k-Jacobsthal numbers for $0 \leq n \leq 10$

$\hat{j}_{k,0} = 0$	$\hat{j}_{k,6} = k^5 + 8k^3 + 12k$
$\hat{j}_{k,1} = 1$	$\hat{j}_{k,7} = k^6 + 10k^4 + 24k^2 + 8$
$\hat{j}_{k,2} = k$	$\hat{j}_{k,8} = k^7 + 12k^5 + 40k^3 + 32k$
$\hat{j}_{k,3} = k^2 + 2$	$\hat{j}_{k,9} = k^8 + 14k^6 + 60k^4 + 80k^2 + 16$
$\hat{j}_{k,4} = k^3 + 4k$	$\hat{j}_{k,10} = k^9 + 16k^7 + 84k^5 + 160k^3 + 80k$
$\hat{j}_{k,5} = k^4 + 6k^2 + 4$	

Table 2: k-Jacobsthal Lucas numbers for $0 \le n \le 10$

$\hat{c}_{k,0} = 2$	$\hat{c}_{k,6} = k^6 + 12k^4 + 36k^2 + 16$
$\hat{c}_{k,1} = k$	$\hat{c}_{k,7} = k^7 + 14k^5 + 56k^3 + 56k$
$\hat{c}_{k,2} = k^2 + 4$	$\hat{c}_{k,8} = k^8 + 16k^6 + 80k^4 + 128k^2 + 32$
$\hat{c}_{k,3} = k^3 + 6k$	$\hat{c}_{k,9} = k^9 + 18k^7 + 108k^5 + 140k^3 + 144k$
$\hat{c}_{k,4} = k^4 + 8k^2 + 8$	$\hat{c}_{k,10} = k^{10} + 20k^8 + 140k^6 + 300k^4 + 272k^2 + 64$
$\hat{c}_{k,5} = k^5 + 10k^3 + 20k$	

Particular cases of the previous definition are:

- If $k = 1$ and $\hat{\jmath}_0 = 0,\ \hat{\jmath}_1 = 1,$ the classic Jacobsthal sequence is obtained,
- If $k = 1$ and $\hat{c}_0 = 2,\ \hat{c}_1 = 1,$ the classic Jacobsthal-Lucas sequence is obtained.

Recurrences (1) and (2) involve the characteristic equation

$$x^2 - kx - 2 = 0$$

with roots

$$\alpha = \frac{k + \sqrt{k^2+8}}{2}, \quad \beta = \frac{k - \sqrt{k^2+8}}{2}, \tag{3}$$

so that

$$\alpha + \beta = k, \quad \alpha\beta = -2, \quad \alpha - \beta = \sqrt{k^2+8}. \tag{4}$$

2.1 Binet Forms

Binet's formulas allow us to express the k-Jacobsthal numbers and k-Jacobsthal Lucas numbers in function of the roots α, β are defined by

$$\hat{\jmath}_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad \hat{c}_n = \alpha^n + \beta^n. \tag{5}$$

In the following theorem, we can see another way of obtaining k-Jacobsthal numbers and k-Jacobsthal Lucas numbers by using sum formula.

Theorem 2.2 *Explicit closed form expressions for $\hat{\jmath}_{k,n}$, $\hat{c}_{k,n}$ are $(n \ge 1)$*

$$\hat{\jmath}_{k,n} = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-i}{i} k^{n-1-2i} 2^i, \tag{6}$$

and

$$\hat{c}_{k,n} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-i} \binom{n-i}{i} k^{n-2i} 2^i. \tag{7}$$

Proof: Induction on n provides the required proofs. In ref [7] it is shown another explicit expression for calculating the general term of the k-Jacobsthal sequence

$$\hat{\jmath}_{k,n} = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1} k^{n-1-2i} (k^2+8)^i. \tag{8}$$

In ref [6] it is shown another explicit expression for calculating the general term of the k-Jacobsthal Lucas sequence

$$\hat{c}_{k,n} = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} (1+4k)^i \tag{9}$$

where $\lfloor a \rfloor$ is the floor function of a.

We can see a lot of identities of k-Jacobsthal numbers and k-Jacobsthal Lucas numbers and relations between them.

2.2 Important Relationships

$$\hat{\jmath}_{k,n} \hat{c}_{k,n} = \hat{\jmath}_{k,2n}, \tag{10}$$

$$\hat{c}_{k,n} = \hat{\jmath}_{k,n+1} + 2\hat{\jmath}_{k,n-1}, \tag{11}$$

$$(k^2+8)\hat{\jmath}_{k,n} = \hat{c}_{k,n+1} + 2\hat{c}_{k,n-1}, \tag{12}$$

$$\hat{c}_{k,n}^2 = (k^2+8)\hat{\jmath}_{k,n}^2 + 4(-2)^n, \tag{13}$$

$$\hat{c}_{k,n} = k\hat{\jmath}_{k,n} + 4\hat{\jmath}_{k,n-1}, \tag{14}$$

$$k\hat{\jmath}_{k,n} + \hat{c}_{k,n} = 2\hat{\jmath}_{k,n+1}, \tag{15}$$

$$(k^2+8)\hat{\jmath}_{k,n} + k\hat{c}_{k,n} = 2\hat{c}_{k,n+1}, \tag{16}$$

$$\sqrt{k^2+8}\,\hat{\jmath}_{k,n} + \hat{c}_{k,n} = 2\alpha^n, \qquad \sqrt{k^2+8}\,\hat{\jmath}_{k,n} - \hat{c}_{k,n} = -2\beta^n, \tag{17}$$

$$2\hat{\jmath}_{k,m+n} = \hat{c}_{k,n}\hat{\jmath}_{k,m} + \hat{\jmath}_{k,m}\hat{c}_{k,n}, \tag{18}$$

$$\hat{\jmath}_{k,m+n+1} = \hat{\jmath}_{k,m+1}\hat{\jmath}_{k,n+1} + 2\hat{\jmath}_{k,m}\hat{\jmath}_{k,n}, \tag{19}$$

$$\hat{c}_{k,m+n+1} = \hat{\jmath}_{k,m+1}\hat{c}_{k,n+1} + 2\hat{\jmath}_{k,m}\hat{c}_{k,n}. \tag{20}$$

Theorem 2.3 *(Summation Formulas)*

For $k-$ Jacobsthal numbers, we can write the following sum formula by using Binet formula:

$$\sum_{p=0}^{n} \hat{\jmath}_{k,pi} = \frac{\hat{\jmath}_{k,i} - \hat{\jmath}_{k,(n+1)i} + (-2)^i \hat{\jmath}_{k,ni}}{1 - \hat{c}_{k,i} + (-2)^i}. \tag{21}$$

If $i = 1$, then we have

$$\sum_{p=0}^{n} \hat{\jmath}_{k,p} = \frac{1 - \hat{\jmath}_{k,n+1} - 2\hat{\jmath}_{k,n}}{1 - k - 2} = \frac{\hat{\jmath}_{k,n+1} + 2\hat{\jmath}_{k,n} - 1}{k+1}.$$

If $i = 2$, then we have

$$\sum_{p=0}^{n} \hat{\jmath}_{k,2p} = \frac{k - \hat{\jmath}_{k,2n+2} + 4\hat{\jmath}_{k,2n}}{1 - k^2}.$$

If $i = 3$, then we have

$$\sum_{p=0}^{n} \hat{\jmath}_{k,3p} = \frac{\hat{\jmath}_{k,3} - \hat{\jmath}_{k,3n+3} - 8\hat{\jmath}_{k,3n}}{1 - \hat{c}_{k,3} - 8} = \frac{\hat{\jmath}_{k,3n+3} + 8\hat{\jmath}_{k,3n} - \hat{\jmath}_{k,3}}{k^3 + 6k + 7}.$$

For $k-$ Jacobsthal Lucas numbers, we can write the following sum formula by using Binet formula:

$$\sum_{p=0}^{n} \hat{c}_{k,pi} = \frac{2 - \hat{c}_{k,(n+1)i} - \hat{c}_{k,i} + (-2)^i \hat{c}_{k,ni}}{1 - \hat{c}_{k,i} + (-2)^i} \tag{22}$$

If $i = 1$, then we have

$$\sum_{p=0}^{n} \hat{c}_{k,p} = \frac{2 - \hat{c}_{k,n+1} - k - 2\hat{c}_{k,n}}{1 - k - 2} = \frac{\hat{c}_{k,n+2} + k - 2}{k+1}.$$

If $i = 2$, then we have

$$\sum_{p=0}^{n} \hat{c}_{k,2p} = \frac{-\hat{c}_{k,2n+2} + 4\hat{c}_{k,2n} - k^2 - 2}{1 - k^2}.$$

If $i = 3$, then we have

$$\sum_{p=0}^{n} \hat{c}_{k,3p} = \frac{2\hat{c}_{k,3n+3} + 8\hat{c}_{k,3n} + \hat{c}_{k,3} - 2}{k^3 + 6k + 7}.$$

From the above theorems we can have the sum of k-Jacobsthal numbers and k-Jacobsthal Lucas numbers with equidistant elements. The distance can be chosen any number.

Theorem 2.4 *(D'ocagne's Property)*
We get with this theorem one of the most important property for numbers. For $n \geq m$ and $n, m \in \mathbb{Z}$, we have

$$\hat{\jmath}_{k,m+1}\hat{\jmath}_{k,n} - \hat{\jmath}_{k,m}\hat{\jmath}_{k,n+1} = (-2)^m \hat{\jmath}_{k,n-m}. \tag{23}$$

And for $m \geq n$ and $n, m \in \mathbb{Z}^+$, we have

$$\hat{c}_{k,m+1}\hat{c}_{k,n} - \hat{c}_{k,m}\hat{c}_{k,n+1} = (k^2+8)(-2)^n \hat{\jmath}_{k,m-n}. \tag{24}$$

Proof: By Binet formula, we have

$$\begin{aligned}
\hat{\jmath}_{k,m+1}\hat{\jmath}_{k,n} - \hat{\jmath}_{k,m}\hat{\jmath}_{k,n+1} &= \frac{\alpha^{m+1}-\beta^{m+1}}{\alpha-\beta}\frac{\alpha^n-\beta^n}{\alpha-\beta} - \frac{\alpha^m-\beta^m}{\alpha-\beta}\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta} \\
&= \frac{1}{(\alpha-\beta)^2}\left[-\alpha^{m+1}\beta^n - \beta^{m+1}\alpha^n + \alpha^m\beta^{n+1} + \alpha^{n+1}\beta^m\right] \\
&= \frac{1}{(\alpha-\beta)^2}\left[\alpha^n\beta^m(\alpha-\beta) - \alpha^m\beta^n(\alpha-\beta)\right] \\
&= \frac{1}{\alpha-\beta}\left[(-2)^m\left(\alpha^{n-m}-\beta^{n-m}\right)\right] = (-2)^m\hat{\jmath}_{k,n-m}.
\end{aligned}$$

The D'ocagne's property was shown as in [6].

$$(-2)^m\sqrt{k^2+8}(\hat{c}_{k,n-m} - 2^{m-n+1}(k+\sqrt{k^2+8})^{n-m}).$$

It can be proved for $k-$ Jacobsthal Lucas numbers by using the same procedure as $k-$ Jacobsthal numbers.

Theorem 2.5 *(Catalan's Property)*
With this theorem we have another important property for integer sequences. For $n, r \in \mathbb{Z}^+$, we have for $k-$ Jacobsthal numbers

$$\hat{\jmath}_{k,n+r}\hat{\jmath}_{k,n-r} - \hat{\jmath}^2_{k,n} = -(-2)^{n-r}\hat{\jmath}^2_{k,r}, \tag{25}$$

and for $k-$ Jacobsthal Lucas numbers

$$\hat{c}_{k,n+r}\hat{c}_{k,n-r} - \hat{c}^2_{k,n-} = (-2)^{n-r}\hat{\jmath}^2_{k,r}(k^2+8) = (-2)^{n-r}\left(\hat{c}^2_{k,r} - (-2)^{r+2}\right). \tag{26}$$

The following theorem is a special form $r = 1$ for Catalan's property

Theorem 2.6 *(Cassini's Property or Simpson Property)*
For $n \in \mathbb{Z}^+$, we have

$$\hat{\jmath}_{k,n+1}\hat{\jmath}_{k,n-1} - \hat{\jmath}^2_{k,n} = -(-2)^{n-1}, \tag{27}$$

and

$$\hat{c}_{k,n+1}\hat{c}_{k,n-1} - \hat{c}^2_{k,n} = (-2)^{n-1}(k^2+8). \tag{28}$$

With the following property we have another relation between k-Jacobsthal numbers and k-Jacobsthal Lucas numbers.

Theorem 2.7 *We get*

$$\hat{j}_{k,4n+p} - 2^{2n}.\hat{j}_{k,p} = \hat{j}_{k,2n}\hat{c}_{k,2n+p} \quad (29)$$

where $n \geq 1$, $p \geq 0$.

For different values of p this theorem can be expressed in the following:
If $p = 0$, *then* $\hat{j}_{k,4n} = \hat{j}_{k,2n}\hat{c}_{k,2n}$ *where* $n \geq 1$,
If $p = 1$, *then* $\hat{j}_{k,4n+1} - 2^{2n} = \hat{j}_{k,2n}\hat{c}_{k,2n+1}$ *where* $n \geq 1$,
If $p = 2$, *then* $\hat{j}_{k,4n+2} - 2^{2n}k = \hat{j}_{k,2n}\hat{c}_{k,2n+2}$ *where* $n \geq 1$ *and so on.*

Proof: It can be proved by using Binet formulas.
If we use $+$ sign instead of $-$ sign, we have the following theorem

Theorem 2.8 *We get*

$$\hat{j}_{k,4n+p} + 2^{2n}.\hat{j}_{k,p} = \hat{c}_{k,2n}\hat{j}_{k,2n+p} \quad (30)$$

where $n \geq 1$, $p \geq 0$.
For different values of p this theorem can be expressed in the following:
If $p = 0$, *then* $\hat{j}_{k,4n} = \hat{c}_{k,2n}\hat{j}_{k,2n}$ *where* $n \geq 1$,
If $p = 1$, *then* $\hat{j}_{k,4n+1} + 2^{2n} = \hat{c}_{k,2n}\ \hat{j}_{k,2n+1}$ *where* $n \geq 1$,
If $p = 2$, *then* $\hat{j}_{k,4n+2} + 2^{2n}k = \hat{c}_{k,2n}\hat{j}_{k,2n+2}$ *where* $n \geq 1$ *and so on.*

By making some little changes in (29) and (30), we obtain the following properties:

Theorem 2.9 *We get*

$$\hat{c}_{k,4n+p} - 2^{2n}.\hat{c}_{k,p} = (k^2+8)\hat{j}_{k,2n}\hat{j}_{k,2n+p}, \quad (31)$$

$$\hat{c}_{k,4n+p} + 2^{2n}.\hat{c}_{k,p} = \hat{c}_{k,2n}\hat{c}_{k,2n+p}, \quad (32)$$

$$\hat{j}_{k,3n+p} - (-2)^n.\hat{j}_{k,n+p} = \hat{j}_{k,n}\hat{c}_{k,2n+p}, \quad (33)$$

$$\hat{j}_{k,3n+p} + (-2)^n.\hat{j}_{k,n+p} = \hat{c}_{k,n}\hat{j}_{k,2n+p}, \quad (34)$$

$$\hat{c}_{k,3n+p} - (-2)^n.\hat{c}_{k,n+p} = (k^2+8)\hat{j}_{k,n}\hat{j}_{k,2n+p}, \quad (35)$$

$$\hat{c}_{k,3n+p} + (-2)^n.\hat{c}_{k,n+p} = \hat{c}_{k,n}\hat{c}_{k,2n+p}. \quad (36)$$

where $n \geq 1$, $p \geq 0$.

Theorem 2.10 *By this theorem we get a different relation between the roots α, β and k-Jacobsthal numbers or k-Jacobsthal Lucas numbers.*

$$\alpha^n = \alpha \hat{j}_{k,n} + 2\hat{j}_{k,n-1}, \tag{37}$$

$$\beta^n = \beta \hat{j}_{k,n} + 2\hat{j}_{k,n-1}, \tag{38}$$

$$\sqrt{k^2+8}\alpha^n = \alpha \hat{c}_{k,n} + 2\hat{c}_{k,n-1}, \tag{39}$$

$$-\sqrt{k^2+8}\beta^n = \beta \hat{c}_{k,n} + 2\hat{c}_{k,n-1}. \tag{40}$$

Proof: We prove (38) and (39) by Binet formula and the product of the roots (4), we have

$$\begin{aligned}
\beta \hat{j}_{k,n} + 2\hat{j}_{k,n-1} &= \beta \frac{\alpha^n - \beta^n}{\alpha - \beta} + 2\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} \\
&= \frac{1}{\alpha - \beta}\left[\beta\left(\alpha^n - \beta^n\right) + 2\left(\alpha^{n-1} - \beta^{n-1}\right)\right] \\
&= \frac{1}{\alpha - \beta}\left(-2\alpha^{n-1} - \beta^{n+1} + 2\alpha^{n-1} - 2\beta^{n-1}\right) \\
&= \frac{1}{\alpha - \beta}\left[-\beta^{n-1}\left(\beta^2 + 2\right)\right] = \beta^n
\end{aligned}$$

$$\begin{aligned}
\alpha \hat{c}_{k,n} + 2\hat{c}_{k,n-1} &= \alpha\left(\alpha^n + \beta^n\right) + 2\left(\alpha^{n-1} + \beta^{n-1}\right) \\
&= \alpha^{n+1} - 2\beta^{n-1} + 2\alpha^{n-1} + 2\beta^{n-1} \\
&= \alpha^{n-1}\left(\beta^2 + 2\right) \\
&= \alpha^n\left(\alpha - \beta\right) = \sqrt{k^2+8}\alpha^n
\end{aligned}$$

other proofs can be done in a similar way.

Theorem 2.11 *The limit of the quotient of two consecutive terms of $k-$ Jacobsthal and $k-$ Jacobsthal Lucas sequences are*

$$\lim_{n\to\infty} \frac{\hat{j}_{k,n+1}}{\hat{j}_{k,n}} = \alpha, \qquad \lim_{n\to\infty} \frac{\hat{c}_{k,n+1}}{\hat{c}_{k,n}} = \alpha. \tag{41}$$

Proof: By Binet formula, we have

$$\lim_{n\to\infty} \frac{\hat{j}_{k,n+1}}{\hat{j}_{k,n}} = \lim_{n\to\infty} \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha^n - \beta^n} = \lim_{n\to\infty} \frac{1 - \left(\frac{\beta}{\alpha}\right)^{n+1}}{\frac{1}{\alpha} - \left(\frac{\beta}{\alpha}\right)^{n+1}\frac{1}{\beta}}$$

and taking into account that $|\beta| < \alpha$ and since $\lim_{n\to\infty}\left(\frac{\beta}{\alpha}\right)^n = 0$, so the proof is completed.

Theorem 2.12 *(The Generating Functions of Jacobsthal and Jacobsthal Lucas Sequences)*

With this theorem, we have the generating functions for k-Jacobsthal numbers and k-Jacobsthal Lucas numbers.

$$\sum_{i=0}^{n} \hat{\jmath}_{k,i} x^i = \frac{x}{1-kx-2x^2}, \tag{42}$$

$$\sum_{i=0}^{n} \hat{c}_{k,i} x^i = \frac{2-kx}{1-kx-2x^2}. \tag{43}$$

You can see the proofs in [6,7].

If we take the power of x as $-i$ instead of i in the above theorem, we obtain the following property for k-Jacobsthal and k-Jacobsthal Lucas numbers.

Theorem 2.13 *Let $n \geq 0$ any integer. Then we have*

$$\sum_{i=0}^{n} \hat{\jmath}_{k,i} x^{-i} = \frac{-1}{x^n.(x^2-kx-2)} \left[-x^{n+1} + x\hat{\jmath}_{k,n+1} + 2\hat{\jmath}_{k,n}\right]. \tag{44}$$

Proof: By using (3), (4), (5), for k-Jacobsthal sequences, we have

$$\begin{aligned}
\sum_{i=0}^{n} \hat{\jmath}_{k,i} x^{-i} &= \frac{1}{\alpha-\beta}\left[\frac{1-\left(\frac{\alpha}{x}\right)^{n+1}}{1-\frac{\alpha}{x}} - \frac{1-\left(\frac{\beta}{x}\right)^{n+1}}{1-\frac{\beta}{x}}\right] \\
&= \frac{1}{(\alpha-\beta)x^n}\left[\frac{x^{n+1}-\alpha^{n+1}}{x-\alpha} - \frac{x^{n+1}-\beta^{n+1}}{x-\beta}\right] \\
&= \frac{-1}{(\alpha-\beta)x^n}\left[\frac{-x^{n+1}(\alpha-\beta) + x(\alpha^{n+1}-\beta^{n+1}) + 2(\alpha^n-\beta^n)}{x^2-kx-2}\right] \\
&= \frac{-1}{x^n.(x^2-kx-2)}\left[-x^{n+1} + x\hat{\jmath}_{k,n+1} + 2\hat{\jmath}_{k,n}\right].
\end{aligned}$$

Conclusion 2.14 *If we take $n \to \infty$ in the above theorem, for k-Jacobsthal sequences, we get*

$$\sum_{i=0}^{\infty} \hat{\jmath}_{k,i} x^{-i} = \frac{x}{(x^2-kx-2)}.$$

Theorem 2.15 *Let $n \geq 0$ any integer. Then we have*

$$\sum_{i=0}^{n} \hat{c}_{k,i} x^{-i} = \frac{-1}{x^n(x^2-kx-2)} \left[x\hat{c}_{k,n+1} + 2\hat{c}_{k,n}\right] + \frac{2x^2-kx}{(x^2-kx-2)}. \tag{45}$$

Proof: By using (3), (4), (5), for k-Jacobsthal Lucas sequences, we have

$$\begin{aligned}
\sum_{i=0}^{n}\hat{c}_{k,i}x^{-i} &= \left[\frac{1-\left(\frac{\alpha}{x}\right)^{n+1}}{1-\frac{\alpha}{x}}+\frac{1-\left(\frac{\beta}{x}\right)^{n+1}}{1-\frac{\beta}{x}}\right]=\frac{1}{x^n}\left[\frac{x^{n+1}-\alpha^{n+1}}{x-\alpha}+\frac{x^{n+1}-\beta^{n+1}}{x-\beta}\right] \\
&= \frac{-1}{x^n}\left[\frac{-2x^{n+2}+x^{n+1}(\alpha+\beta)+x(\alpha^{n+1}+\beta^{n+1})+2(\alpha^n+\beta^n)}{x^2-kx-2}\right] \\
&= \frac{-1}{x^n(x^2-kx-2)}\left[x\hat{c}_{k,n+1}+2\hat{c}_{k,n}\right]+\frac{2x^2-kx}{(x^2-kx-2)}.
\end{aligned}$$

Conclusion 2.16 *If we take $n\to\infty$ in the above theorem, we get*

$$\sum_{i=0}^{\infty}\hat{c}_{k,i}x^{-i}=\frac{2x^2-kx}{x^2-kx-2}.$$

Theorem 2.17 *For $\left|\alpha^k\beta^{r-k}x\right|<1$, we get the generating function for the power of r of k-Jacobsthal sequences*

$$\sum_{i=0}^{\infty}\hat{j}_{k,i}^{r}x^i=\sum_{k=0}^{r}\binom{r}{k}\frac{1}{(\alpha-\beta)^k}\frac{1}{(\beta-\alpha)^{r-k}}\frac{1}{1-\alpha^k\beta^{r-k}x}. \tag{46}$$

Proof: By using geometric series and Binet formula, we have

$$\begin{aligned}
\sum_{i=0}^{\infty}\hat{j}_{k,i}^{r}x^i &= \sum_{i=0}^{\infty}\sum_{k=0}^{r}\binom{r}{k}\left(\frac{\alpha^i}{\alpha-\beta}\right)^k\left(\frac{-\beta^i}{\alpha-\beta}\right)^{r-k}x^i \\
&= \sum_{k=0}^{r}\binom{r}{k}\frac{1}{(\alpha-\beta)^k}.\frac{1}{(\beta-\alpha)^{r-k}}\sum_{i=0}^{\infty}\left[\alpha^k\beta^{r-k}x\right]^i \\
&= \sum_{k=0}^{r}\binom{r}{k}\frac{1}{(\alpha-\beta)^k}.\frac{1}{(\beta-\alpha)^{r-k}}\frac{1}{1-\alpha^k\beta^{r-k}x}.
\end{aligned}$$

Theorem 2.18 *For $\left|\alpha^k\beta^{r-k}x\right|<1$, we get the generating function for the power of r of k-Jacobsthal Lucas sequences*

$$\sum_{i=0}^{\infty}\hat{c}_{k,i}^{r}x^i=\sum_{k=0}^{r}\binom{r}{k}\frac{1}{1-\alpha^k\beta^{r-k}x}. \tag{47}$$

Proof: By using geometric series and Binet formula, we have

$$\begin{aligned}
\sum_{i=0}^{\infty}\hat{c}_{k,i}^{r}x^i &= \sum_{i=0}^{\infty}\sum_{k=0}^{r}\binom{r}{k}(\alpha^i)^k(\beta^i)^{r-k}x^i=\sum_{k=0}^{r}\binom{r}{k}\sum_{i=0}^{\infty}\left[\alpha^k\beta^{r-k}x\right]^i \\
&= \sum_{k=0}^{r}\binom{r}{k}\frac{1}{1-\alpha^k\beta^{r-k}x}.
\end{aligned}$$

Theorem 2.19 *Let $n \geq 0$ any integer and $|\alpha^i x| < 1$ and $|\beta^i x| < 1$. Then we have the generating function for the equidistant elements of k-Jacobsthal sequences*

$$\sum_{n=0}^{\infty} \hat{j}_{k,in} x^n = \frac{\hat{j}_{k,i} x}{1 - \hat{c}_{k,i} x + (-2)^i x^2}, \tag{48}$$

For example $i = 2$ we obtain

$$\sum_{n=0}^{\infty} \hat{j}_{k,2n} x^n = \frac{kx}{1 - (k^2+4)x + 4x^2}.$$

Proof: By using Binet formula and geometric series we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} \hat{j}_{k,in} x^n &= \sum_{n=0}^{\infty} \frac{\alpha^{in} - \beta^{in}}{\alpha - \beta} x^n = \frac{1}{\alpha - \beta} \sum_{n=0}^{\infty} \left[\left(\alpha^i x\right)^n - \left(\beta^i x\right)^n \right] \\
&= \frac{1}{\alpha - \beta} \left[\frac{1}{1 - \alpha^i x} - \frac{1}{1 - \beta^i x} \right] \\
&= \frac{\left(\alpha^i - \beta^i\right) x}{(\alpha - \beta)\left(1 - x\left(\alpha^i + \beta^i\right) + x^2 (-2)^i\right)} \\
&= \frac{\hat{j}_{k,i} x}{1 - \hat{c}_{k,i} x + (-2)^i x^2}.
\end{aligned}$$

Theorem 2.20 *Let $n \geq 0$ any integer and $|\alpha^i x| < 1$ and $|\beta^i x| < 1$. Then we have the generating function for the equidistant elements of k-Jacobsthal Lucas sequences*

$$\sum_{n=0}^{\infty} \hat{c}_{k,in} x^n = \frac{2 + x(\alpha - \beta)\hat{j}_{k,i}}{1 - \hat{c}_{k,i} x + (-2)^i x^2}, \tag{49}$$

For example $i = 2$ we obtain

$$\sum_{n=0}^{\infty} \hat{c}_{k,2n} x^n = \frac{2 + k\sqrt{k^2+8}x}{1 - (k^2+4)x + 4x^2}.$$

Proof: By using Binet formula and geometric series we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} \hat{c}_{k,in} x^n &= \sum_{n=0}^{\infty} \left(\alpha^{in} + \beta^{in}\right) x^n = \sum_{n=0}^{\infty} \left[\left(\alpha^i x\right)^n + \left(\beta^i x\right)^n \right] \\
&= \left[\frac{1}{1 - \alpha^i x} + \frac{1}{1 - \beta^i x} \right] \\
&= \frac{2 + \left(\alpha^i - \beta^i\right) x}{\left(1 - x\left(\alpha^i + \beta^i\right) + x^2 (-2)^i\right)} \\
&= \frac{2 + x(\alpha - \beta)\hat{j}_{k,i}}{1 - \hat{c}_{k,i} x + (-2)^i x^2}.
\end{aligned}$$

Theorem 2.21 *(The Exponential Generating Functions of Jacobsthal and Jacobsthal Lucas Sequences)*

The relation between k-Jacobsthal, k-Jacobsthal Lucas sequences and exponential function is given by

$$\begin{aligned}\sum_{n=0}^{\infty}\hat{\jmath}_{k,n}\frac{x^n}{n!} &= \sum_{n=0}^{\infty}\frac{\alpha^n-\beta^n}{\alpha-\beta}\frac{x^n}{n!}=\frac{1}{\sqrt{k^2+8}}\sum_{n=0}^{\infty}\frac{(\alpha x)^n-(\beta x)^n}{n!}50 \qquad (1)\\ &= \frac{1}{\sqrt{k^2+8}}\left(e^{\alpha x}-e^{\beta x}\right).\\ \sum_{n=0}^{\infty}\hat{c}_{k,n}\frac{x^n}{n!} &= \sum_{n=0}^{\infty}(\alpha^n+\beta^n)\frac{x^n}{n!}=\sum_{n=0}^{\infty}\frac{(\alpha x)^n+(\beta x)^n}{n!}=\left(e^{\alpha x}+e^{\beta x}\right).\end{aligned}$$

Theorem 2.22 *By this theorem we can see another sum property, equals to* $2n$*. th element of k-Jacobsthal, k-Jacobsthal Lucas sequence respectively*

$$\sum_{i=0}^{n}\binom{n}{i}2^{n-i}k^i\hat{\jmath}_{k,i}=\hat{\jmath}_{k,2n}, \qquad (51)$$

$$\sum_{i=0}^{n}\binom{n}{i}2^{n-i}k^i\hat{c}_{k,i}=\hat{c}_{k,2n}. \qquad (52)$$

Proof: By using the property of (37) and (38) we have

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}2^{n-i}k^i\hat{\jmath}_{k,i} &= \sum_{i=0}^{n}\binom{n}{i}2^{n-i}k^i\frac{\alpha^i-\beta^i}{\alpha-\beta}\\ &= \frac{1}{\alpha-\beta}\left(\sum_{i=0}^{n}\binom{n}{i}2^{n-i}(\alpha k)^i-\sum_{i=0}^{n}\binom{n}{i}2^{n-i}(\beta k)^i\right)\\ &= \frac{1}{\alpha-\beta}\left[(2+\alpha k)^n-(2+\beta k)^n\right]=\hat{\jmath}_{k,2n}.\end{aligned}$$

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}2^{n-i}k^i\hat{c}_{k,i} &= \sum_{i=0}^{n}\binom{n}{i}2^{n-i}k^i\left(\alpha^i+\beta^i\right)\\ &= \left(\sum_{i=0}^{n}\binom{n}{i}2^{n-i}(\alpha k)^i+\sum_{i=0}^{n}\binom{n}{i}2^{n-i}(\beta k)^i\right)\\ &= \left[(2+\alpha k)^n+(2+\beta k)^n\right]=\hat{c}_{k,2n}.\end{aligned}$$

Theorem 2.23 *The square of elements of k-Jacobsthal sequence is obtained by the following:*

$$\sum_{n=0}^{\infty}\hat{\jmath}_{k,i}^2=\frac{1}{k^2+8}\left(\frac{4\hat{c}_{k,2n-2}-\hat{c}_{k,2n}-\hat{c}_{k,2}+2}{5-\hat{c}_{k,2}}+2(-1)^n\jmath_n\right). \qquad (53)$$

Proof: By using Binet formulas and the sum of geometric series we have

$$\begin{aligned}\sum_{i=0}^{n-1}\hat{j}_{k,i}^2 &= \sum_{i=0}^{n-1}\left(\frac{\alpha^i-\beta^i}{\alpha-\beta}\right)^2 = \frac{1}{k^2+8}\sum_{i=0}^{n-1}\left(\alpha^{2i}+\beta^{2i}-2(-2)^i\right)\\ &= \frac{1}{k^2+8}\left(\frac{\alpha^{2n}-1}{\alpha^2-1}+\frac{\beta^{2n}-1}{\beta^2-1}+2\frac{(-2)^n-1}{3}\right)\\ &= \frac{1}{k^2+8}\left(\frac{4\hat{c}_{k,2n-2}-\hat{c}_{k,2n}-\hat{c}_{k,2}+2}{5-\hat{c}_{k,2}}+2(-1)^n j_n\right).\end{aligned}$$

Theorem 2.24 *The square of elements of k-Jacobsthal Lucas sequence is obtained by the following*

$$\sum_{n=0}^{\infty}\hat{c}_{k,i}^2 = \frac{1}{k^2+8}\left(\frac{4\hat{c}_{k,2n-2}-\hat{c}_{k,2n}-\hat{c}_{k,2}+2}{5-\hat{c}_{k,2}}+2(-1)^n j_n\right). \tag{54}$$

Proof: The proof is made by the same method with the theorem

$$\begin{aligned}\sum_{i=0}^{n-1}\hat{c}_{k,i}^2 &= \sum_{i=0}^{n-1}\left(\alpha^i+\beta^i\right)^2 = \sum_{i=0}^{n-1}\left(\alpha^{2i}+\beta^{2i}+2(-2)^i\right)\\ &= \left(\frac{\alpha^{2n}-1}{\alpha^2-1}+\frac{\beta^{2n}-1}{\beta^2-1}-2\frac{(-2)^n-1}{3}\right)\\ &= \left(\frac{4\hat{c}_{k,2n-2}-\hat{c}_{k,2n}-\hat{c}_{k,2}+2}{5-\hat{c}_{k,2}}-2(-1)^n j_n\right).\end{aligned}$$

3 Conclusion

In this paper we have obtained a lot of interesting properties, are satisfied by k-Jacobsthal and k-Jacobsthal Lucas sequence. We think that it can be found many more properties of these sequences. Jacobsthal and Jacobsthal Lucas sequences must be given importance as Fibonacci and Lucas sequences because of interesting properties and usefulness in mathematics and use in science.

Acknowledgements: Thank you very much for the valuable comments of the referees.

References

[1] A.F. Horadam, Jacobsthal representation numbers, *The Fibonacci Quarterly*, 34(1) (1996), 40-54.

[2] T. Koshy, *Fibanacci and Lucas Numbers with Applications*, John Wiley and Sons Inc., NY, (2001).

[3] N.J.A. Sloane, *The Online Encyclopedia of Integer Sequences*, (2006).

[4] Ş. Uygun, The (s,t)-Jacobsthal and (s,t)-Jacobsthal Lucas sequences, *Applied Mathematical Sciences*, 70(9) (2015), 3467-3476.

[5] D. Jhala, Some properties of the k-Jacobsthal Lucas sequence, *International Journal of Modern Sciences and Engineering Technology (IJMSET)*, 1(3) (2014), 87-92.

[6] H. Campos, P. Catarino, A.P. Aires, P. Vasco and A. Borges, On some identities of k-Jacobsthal-Lucas numbers, *Int. Journal of Math. Analysis*, 8(10) (2014), 489-494.

[7] D. Jhala, K. Sisodiya and G.P.S. Rathore, On some identities for k-Jacobsthal numbers, *Int. Journal of Math. Analysis*, 7(12) (2013), 551-556.

[8] P. Catarino, On some identities for k-Fibonacci sequence, *Int. Journal of Contemporary Mathematical Sciences*, 9(1) (2014), 37-42.

[9] S. Falcon and A. Plaza, On the Fibonacci k-numbers, *Chaos, Solitions & Fractals*, 32(5) (2007), 1615-1624.

[10] Z. Cerin, Sums of squares and products of Jacobsthal numbers, *Journal of Integer Sequences*, 10(Article 07.2.5) (2007), 1-15.

[11] E.W. Weisstein, Jacobsthal number, *Wolfram Mathworld (Online)*, http://mathworld.wolfram.com/JacobsthalNumber.html, Retrieved (2007).