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On the Generalized Hyers-Ulam Stability of an Euler-Lagrange-Rassias Functional Equation

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Abstract

In this paper, the general solution and the generalized Hyers-Ulam-Rassias stability of the following Euler-Lagrange type quadratic functional equation

$$f(x+ky)+f(y+kz)+f(z+kx)-kf(x+y+z) = (k^2-k+1)\big(f(x)+f(y)+f(z)\big),$$

for all $k \in \mathbb{N}$, is investigated.

Keywords: *Quadratic functional equation, Hyers-Ulam-Rassias stability.*

1 Introduction

The stability problem for the functional equations was first raised by S. M. Ulam [21]. He proposed the following famous question concerning the stability of homomorphisms:

Let G be a group and let G' be a metric group with metric d. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \longrightarrow G'$ satisfies

$$d(f(xy), f(x)f(y)) < \delta \quad \textit{for all} \quad x, y \in G,$$

then there exists a homomorphism $F : G \longrightarrow G'$ with

$$d(f(x), F(x)) < \varepsilon \quad \textit{for all} \quad x \in G \ ?$$

In 1941, Hyers [6] considered the case of approximately additive mappings $f : X \longrightarrow Y$, where X and Y are Banach spaces and f satisfies

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$$

for all $x, y \in X$. It was shown that the limit

$$F(x) = \lim_{n \longrightarrow \infty} 2^{-n} f(2^n x),$$

exists for all $x \in X$ and that $F : X \longrightarrow Y$ is the unique additive mapping satisfying

$$\|f(x) - F(x)\| \le \varepsilon.$$

In 1950, T. Aoki [1] gave the generalized Hyers' theorem. Afterwards, in 1978, a generalization of Hyers' theorem provided by Th. M. Rassias [19].

The quadratic function $f(x) = cx^2$ satisfies the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y),$$

and therefore the above equation is called the quadratic functional equation.

In 1982-1994, J. M. Rassias (see [11-18]) solved the Ulam problem for different mappings and for many Euler-Lagrange type quadratic mappings, by involving a product of different powers of norms. In addition, J. M. Rassias considered the mixed product-sum of powers of norms control function [20]. In 1994, a generalization of the Rassias' theorem was obtained by Gavruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. For more details about the results concerning such problems the reader is referred to [2, 3, 4, 9, 10] and [22].

Consider the following functional equations:

$$f(x+y) + f(y+z) + f(z+x) = f(x+y+z) + f(x) + f(y) + f(z), \quad (1)$$

and

$$f(x+2y) + f(y+2z) + f(z+2x) = 2f(x+y+z) + 3\big(f(x) + f(y) + f(z)\big). \quad (2)$$

The functional equation (1) was solved by Pl. Kannappan in [8]. Recently, the author investigated in his paper [22] the general solution and generalized Hyers-Ulam stability of the equation (2).

In the present paper we consider the quadratic functional equation

$$f(x+ky) + f(y+kz) + f(z+kx) - kf(x+y+z) = (k^2-k+1)\big(f(x)+f(y)+f(z)\big),$$

for all $k \in \mathbb{N}$, which is a generalization of equations (1) and (2), and determine the general solution and generalized Hyers-Ulam stability of this functional equation.

2 The General Solution and Hyers-Ulam Stability

The following theorem provide the general solution of the proposed functional equation by establishing a connection with the classical quadratic functional equation.

For convenience, we use the following abbreviations:

$$\begin{aligned} Df(x,y,z) &= f(x+ky)+f(y+kz)+f(z+kx) \\ &\quad - kf(x+y+z)-(k^2-k+1)\big(f(x)+f(y)+f(z)\big). \end{aligned} \tag{3}$$

Theorem 2.1 *Let X and Y be real vector spaces. A function $f : X \longrightarrow Y$ satisfies the functional equation*

$$Df(x,y,z)=0, \tag{4}$$

for all $x,y,z \in X$ and all $k \in \mathbb{N}$ if and only if it satisfies

$$f(x+y)+f(x-y)=2f(x)+2f(y), \quad (x,y \in X). \tag{5}$$

Proof: The result is proved for the case $k=1$ and $k=2$, in [8] and [22], respectively. So we give the proof for $k \geq 3$. Assume that a function $f : X \longrightarrow Y$ satisfies (4). Letting $x=y=z$ in (4), we get

$$3f((k+1)x)-kf(3x)=3(k^2-k+1)f(x)$$

for all $x \in X$, which implies that $f(0)=0$. Letting $y=z=0$ in (4), we have

$$f(x)+f(kx)=kf(x)+(k^2-k+1)f(x),$$

which yields

$$f(kx)=k^2f(x) \qquad (\ddagger),$$

for all $x \in X$ and all $k \in \mathbb{N}$. Letting $z=0$ in (4), we obtain

$$f(x+ky)+f(y)+f(kx)=kf(x+y)+(k^2-k+1)\big(f(x)+f(y)\big).$$

Applying Eq. ($\ddagger$), then we have

$$f(x+ky)-kf(x+y)=(1-k)f(x)+(k^2-k)f(y). \tag{6}$$

Replacing x by y and y by x in (6), so

$$f(y+kx)-kf(y+x)=(1-k)f(y)+(k^2-k)f(x). \tag{7}$$

Letting $y = z$ in (4), we get

$$f(x+ky)+f((k+1)y)+f(y+kx) = kf(x+2y)+(k^2-k+1)\big(f(x)+2f(y)\big).$$

Using Eq.(‡) for $k+1$, the above equation simplifies to

$$f(x+ky)+f(y+kx)-kf(x+2y) = \\ k^2\big(f(x)+f(y)\big)+(1-k)f(x)+(1-4k)f(y). \quad (8)$$

Eliminating $f(x+ky)$ and $f(y+kx)$ from (8) by applying (6) and (7), we get

$$2kf(x+y)+2kf(y) = kf(x)+kf(x+2y). \quad (9)$$

Replacing x by $x-y$ in above equation, thus the classical quadratic functional equation (5) follows.

Conversely, assume that a function $f : X \longrightarrow Y$ satisfies (5), and suppose the result is establish for each $s < k$, where $k \geq 3$. Replacing x by $x+(k-1)y$ and all cyclic permutations of the variables in (5), then

$$\begin{aligned} f(x+ky)+f(x+(k-2)y) &= 2f(x+(k-1)y)+2f(y), \\ f(y+kz)+f(y+(k-2)z) &= 2f(y+(k-1)z)+2f(z), \\ f(z+kx)+f(z+(k-2)x) &= 2f(z+(k-1)x)+2f(x). \end{aligned} \quad (10)$$

By ammunitions we have

$$f(x+(k-1)y)+f(y+(k-1)z)+f(z+(k-1)x) = \\ (k-1)f(x+y+z)+(k^2-3k+3)\big(f(x)+f(y)+f(z)\big). \quad (11)$$

and

$$f(x+(k-2)y)+f(y+(k-2)z)+f(z+(k-2)x) = \\ (k-2)f(x+y+z)+(k^2-5k+7)\big(f(x)+f(y)+f(z)\big). \quad (12)$$

Applying Eq. (11) and (12), to eliminate $f(x+(k-1)y)$, $f(x+(k-2)y)$ and all cyclic permutations of the variables in the sum of all equations in (10), then the quadratic functional equation (4) follows, so the induction argument finishes the proof.

Theorem 2.2 *Suppose X is a real vector space and Y is a Banach space. Let $k \geq 3$ and $\varphi : X^3 \longrightarrow [0,\infty)$ be a function such that*

$$\sum_{n=0}^{\infty} k^{-2n}\varphi(k^n x, k^n y, k^n z) \quad (13)$$

be convergent. Let $f : X \longrightarrow Y$ be a mapping satisfying $f(0) = 0$ and

$$\|Df(x, y, z)\| \leq \varphi(x, y, z) \tag{14}$$

for all $x, y, z \in X$, then there exists a unique function $F : X \longrightarrow Y$ which satisfies (4) and

$$\|f(x) - F(x)\| \leq \frac{1}{k^2} \sum_{n=0}^{\infty} k^{-2n} \varphi(k^n x, 0, 0) \quad (x \in X). \tag{15}$$

Proof: Letting $y = z = 0$ in (14), we get

$$\|f(kx) - k^2 f(x)\| \leq \varphi(x, 0, 0).$$

Dividing the above inequality by k^2, we obtain

$$\|\frac{f(kx)}{k^2} - f(x)\| \leq \frac{1}{k^2} \varphi(x, 0, 0). \tag{16}$$

Make the induction hypothesis

$$\|\frac{f(k^n x)}{k^{2n}} - f(x)\| \leq \frac{1}{k^2} \sum_{i=0}^{n-1} k^{-2i} \varphi(k^i x, 0, 0), \tag{17}$$

which is true for $n = 1$ by (16). Replacing x by $k^m x$ in (17) and divide the result by k^{2m}, then we have

$$\|\frac{f(k^{n+m} x)}{k^{2(n+m)}} - \frac{f(k^m x)}{k^{2m}}\| \leq \frac{1}{k^2} \sum_{i=m}^{n+m-1} k^{-2i} \varphi(k^i x, 0, 0) \quad (x \in X).$$

It follows that the sequence $\{\frac{1}{k^{2n}} f(k^n x)\}$ is Cauchy sequence for all $x \in X$. Since Y is complete, we may define a function $F : X \longrightarrow Y$ by

$$F(x) := \lim_{n \longrightarrow \infty} \frac{1}{k^{2n}} f(k^n x), \qquad (x \in X).$$

Then by the definition of F, we can see that (15) holds. To show that F satisfies in (4), replacing x, y and z in (14) by $k^n x$, $k^n y$ and $k^n z$, respectively, and divide the result by k^{2n}, we get

$$\|\frac{1}{k^{2n}} Df(k^n x, k^n y, k^n z)\| \leq \frac{\varphi(k^n x, k^n y, k^n z)}{k^{2n}} \to 0, \qquad as \ \ n \to \infty,$$

which implies F satisfies (4). The uniqueness of F follows from Theorem 2.1.

Corollary 2.3 *Let $k \geq 3$ and $f : X \longrightarrow Y$ be a function such that*

$$\|Df(x, y, z)\| \leq \varepsilon$$

for some $\varepsilon > 0$ and for all $x, y, z \in X$. Then there exists a unique function $F : X \longrightarrow Y$ which satisfies (4), and

$$\|f(x) - F(x)\| \leq \frac{\varepsilon}{k^2 - 1} \quad (x \in X).$$

Proof: Apply Theorem 2.2 for $\varphi(x, y, z) = \varepsilon$.

Corollary 2.4 *Let $k \geq 3$ and $f : X \longrightarrow Y$ be a function such that satisfies*

$$Df(x, y, z)\| \leq \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p),$$

with $p < 2$ and for some $\varepsilon > 0$ and for all $x, y, z \in X$. Then there exists a unique quadratic function $F : X \longrightarrow Y$ which satisfies (4), and

$$\|f(x) - F(x)\| \leq \frac{\varepsilon}{|k^2 - k^p|}\|x\|^p \qquad (x \in X).$$

Proof: Apply Theorem 2.2 for $\varphi(x, y, z) = \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p)$.

3 Conclusion

This paper generalized some well-known results in the area of Hyers-Ulam stability of the Euler-Lagrange-Rassias type quadratic functional equation in three variables, in fact, the proposed quadratic functional equations which are given in [8] and [22], can be obtained of the proposed functional equation in the present paper, for $k = 1$ and $k = 2$, respectively.

Concluding remarks, the results of the paper is also true for all $k \in \mathbb{Z}$, but the paper discussed for the case $k \in \mathbb{N}$.

If we take $k = -1$ in the proposed quadratic functional equation we get

$$f(x - y) + f(y - z) + f(z - x) + f(x + y + z) = 3\big(f(x) + f(y) + f(z)\big),$$

that Hyers-Ulam stability of it investigated by Jung in [7]. Thus, the paper is also generalized the Jung's work.

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