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On a New Class of Multivalent Functions With Missing Coefficients

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Abstract

In this paper, we investigate a new class $\Theta^{p,\lambda}_{\xi_1,\xi_2}$ *of analytic functions in the open unit disk. By using the geometry function theory, we discuss the radius problems between the* $\Theta^{p,\lambda}_{\xi_1,\xi_2}$ *and the convex functions or close-to-convex functions. Several properties as the sufficient and necessary conditions and modified-Hadamard product are given.*

Keywords: *Multivalent function, Convex function, Cauchy-schwarz inequality, Modified-Hadamard product.*

1 Introduction

Let $\mathcal{A}_p$ be the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad p \in \mathbb{Z}^+ = \{1, 2, 3, ...\}, \tag{1}$$

that are p-valently analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. If two functions $f_1(z) \in \mathcal{A}_p$, $f_2(z) \in \mathcal{A}_p$ and

$$f_i(z) = z^p + \sum_{n=p+1}^{\infty} a_{n,i} z^n, i = 1, 2, z \in \mathbb{U},$$

then we define the $f_1 \oplus f_2(z)$ as

$$f_1 \oplus f_2(z) = z^p + \sum_{n=p+1}^{\infty} (a_{n,1} + a_{n,2}) z^n, z \in \mathbb{U}.$$

Also, let $\mathcal{K}_p(\alpha)$ denote the subclass of $\mathcal{A}_p$ consisting of $f(z)$ which satisfy

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad (z \in \mathbb{U}) \tag{2}$$

for some real $\alpha (0 \leq \alpha < p)$. A function $f(z) \in \mathcal{K}_p(\alpha)$ is said to be p-valently convex of order α in $\mathbb{U}$. We note that $\mathcal{K}_1(\alpha) \equiv \mathcal{K}$ is usual convex class. Moreover, a function $f(z) \in \mathcal{A}_p$ is in the class $\mathcal{C}_p(\alpha)$ if

$$\Re\left(\frac{f'(z)}{pz^{p-1}}\right) > \alpha, \quad z \in \mathbb{U} \tag{3}$$

for some real $\alpha (0 \leq \alpha < 1)$. $\mathcal{C}_1(0) \equiv \mathcal{C}$ is the close-to-convex class. These are many results on the classes $\mathcal{K}_p(\alpha)$ and $\mathcal{C}_p(\alpha)$(See [1, 2, 8, 9, 10, 13]).

Let $\mathcal{A}_p(\theta)$ denote the subclass of $\mathcal{A}_p$ consisting of functions $f(z)$ with the coefficients $a_n = |a_n| e^{i((n-p)\theta + \pi)}$ $(n \geq p+1)$. Here, we introduce the subclasses $\mathcal{C}_p(\theta, \alpha)$ and $\mathcal{K}_p(\theta, \alpha)$ as follows: $\mathcal{C}_p(\theta, \alpha) = \mathcal{A}_p(\theta) \cap \mathcal{C}_p(\alpha)$, $\quad \mathcal{K}_p(\theta, \alpha) = \mathcal{A}_p(\theta) \cap \mathcal{K}_p(\alpha)$. In fact, The $\mathcal{C}_1(\theta, \alpha)$ was introduced by Uyanik, Owa [12] and the $\mathcal{K}_1(\theta, \alpha) \equiv \mathcal{K}(\theta, \alpha)$ was introduced by Frasin [7].

In some earlier investigations, various interesting subclasses of the class $\mathcal{A}_p$ and $\mathcal{A}_p(\theta)$ have been studied with different view points(see [3, 4]). Motivated by the aforementioned works done by Uyanik et al.[11, 12] and Frasin et al.[5, 6, 7], we now introduce the following subclass $\Theta^{p,\lambda}_{\xi_1,\xi_2,\xi_3}$ of analytic functions:

Definition 1.1 *For the functions $f(z) \in \mathcal{A}_p$ given by (1), we say that $f(z) \in \Theta^{p,\lambda}_{\xi_1,\xi_2}$, if there exists a function $g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n \in \mathcal{G}$ such that*

$$\left| \xi_1 z \left(\frac{f(z) \oplus g(z)}{z^p} \right)' + \xi_2 z^2 \left(\frac{f(z) \oplus g(z)}{z^p} \right)'' \right| \leq \lambda, z \in \mathbb{U}, \tag{4}$$

where ξ_1, $\xi_2 \in \mathbb{C}$, $\lambda > 0$, $p \in \mathbb{Z}^+$ and

$$\mathcal{G} = \Big\{ g(z) \in \mathcal{A}_p : b_{p+1} = 0, b_{p+2} = -\frac{1}{2} a_{p+2}, \\ b_{p+3} = -\frac{2}{3} a_{p+3}, ..., b_n = (\frac{1}{n-p} - 1) a_n, ... \Big\}. \tag{5}$$

In the present paper, some properties for $\Theta^{p,\lambda}_{\xi_1,\xi_2}$ are given. We discuss the radius problems for $f(z)$ belonging to $\mathcal{C}_p(\theta, \alpha)$ or $\mathcal{K}_p(\theta, \alpha)$ to be in the class $\Theta^{p,\lambda}_{\xi_1,\xi_2}$, and obtain the modified-Hadamard product results.

2 Sufficient and Necessary Conditions

Theorem 2.1 *If the function $f(z)$ given by (1) satisfies the condition*

$$\sum_{n=p+1}^{\infty}[|\xi_1|+|\xi_2|(n-p-1)]|a_n|\leq\lambda, \tag{6}$$

then $f(z)\in\Theta^{p,\lambda}_{\xi_1,\xi_2}$ with a function

$$g(z)=z^p+\sum_{n=p+1}^{\infty}b_nz^n\in\mathcal{G},$$

where ξ_1, $\xi_2\in\mathbb{C}$, $\lambda>0$ and $p\in\mathbb{Z}^+=\{1,2,3,...\}$.

Proof For $f(z)\in\mathcal{A}_p$ and $g(z)\in\mathcal{G}$, using the (5), then we have

$$\begin{aligned}
&\left|\xi_1z\left(\frac{f(z)\oplus g(z)}{z^p}\right)'+\xi_2z^2\left(\frac{f(z)\oplus g(z)}{z^p}\right)''\right| \qquad (7)\\
&=\left|\sum_{n=p+1}^{\infty}[\xi_1(n-p)+\xi_2(n-p)(n-p-1)](a_n+b_n)z^{n-p}\right|\\
&\leq\sum_{n=p+1}^{\infty}[|\xi_1|(n-p)+|\xi_2|(n-p)(n-p-1)]|a_n+b_n|\\
&=\sum_{n=p+1}^{\infty}[|\xi_1|+|\xi_2|(n-p-1)]|a_n|.
\end{aligned}$$

It follows from(4), (6) and (7), then $f(z)\in\Theta^{p,\lambda}_{\xi_1,\xi_2}$. The proof of the theorem is complete.

Theorem 2.2 *If $f(z)=z^p+\sum\limits_{n=p+1}^{\infty}a_nz^n\in\Theta^{p,\lambda}_{\xi_1,\xi_2}$ with a function*

$$g(z)=z^p+\sum_{n=p+1}^{\infty}b_nz^n\in\mathcal{G},$$

and $\arg\xi_1=\arg\xi_2=\gamma$ and $a_n=|a_n|e^{i((n-p)\theta)-\gamma)}$, then we have

$$\sum_{n=p+1}^{\infty}[|\xi_1|+|\xi_2|(n-p-1)]|a_n|\leq\lambda.$$

Proof If $f(z) \in \Theta^{p,\lambda}_{\xi_1,\xi_2}$ with $\arg \xi_1 = \arg \xi_2 = \gamma$ and $a_n = |a_n| e^{i((n-p)\theta)-\gamma)}$, applying the (5), then we get

$$\left| \xi_1 z \left(\frac{f(z) \oplus g(z)}{z^p} \right)' + \xi_2 z^2 \left(\frac{f(z) \oplus g(z)}{z^p} \right)'' \right| = \tag{8}$$

$$= \left| \sum_{n=p+1}^{\infty} [\xi_1 (n-p) + \xi_2 (n-p)(n-p-1)](a_n + b_n) z^{n-p} \right|$$

$$= \left| \sum_{n=p+1}^{\infty} [\xi_1 + \xi_2 (n-p-1)] a_n z^{n-p} \right|$$

$$= \left| \sum_{n=p+1}^{\infty} [|\xi_1| + |\xi_2|(n-p-1)] e^{i\gamma} |a_n| e^{i((n-p)\theta - \gamma)} z^{n-p} \right|$$

$$= \left| \sum_{n=p+1}^{\infty} [|\xi_1| + |\xi_2|(n-p-1)] |a_n| e^{i(n-p)\theta} z^{n-p} \right| \leq \lambda$$

for all $z \in \mathbb{U}$. Letting $z \in \mathbb{U}$ such that $z = |z| e^{-i\theta}$, then we have that

$$\left| \sum_{n=p+1}^{\infty} [|\xi_1| + |\xi_2|(n-p-1)] |a_n| e^{i(n-p)\theta} z^{n-p} \right| \tag{9}$$

$$= \sum_{n=p+1}^{\infty} [|\xi_1| + |\xi_2|(n-p-1)] |a_n| |z|^{n-p}$$

Now, taking $|z| \to 1^-$, form (8) and (9), it gives the required result. The proof of the theorem is complete.

3 Radius Problems with Convex and Close-to-Convex Functions

Working in a similar way as in Uyanìk, Owa [11, Lemma 3.1] and Frasin [6, Lemma 4.1], we give the following Lemma 3.1 and Lemma 3.2:

Lemma 3.1 *Suppose* $f(z) = z^p + \sum\limits_{n=p+1}^{\infty} a_n z^n \in \mathcal{C}_p(\theta, \alpha)$, *then we have*

$$\sum_{n=p+1}^{\infty} n |a_n| \leq p(1-\alpha), (0 \leq \alpha < 1).$$

Lemma 3.2 *Suppose* $f(z) = z^p + \sum\limits_{n=p+1}^{\infty} a_n z^n \in \mathcal{K}_p(\theta, \alpha)$, *then we have*

$$\sum_{n=p+1}^{\infty} \frac{n}{p}(n-\alpha)|a_n| \leqslant p - \alpha, (0 \leq \alpha < p).$$

Theorem 3.3 *Let* $f(z) = z^p + \sum\limits_{n=p+1}^{\infty} a_n z^n \in \mathcal{C}_p(\theta, \alpha)$ *and* $\delta(0 < |\delta| < 1)$ *is a complex number, then* $\frac{1}{\delta^p} f(\delta z) \in \Theta_{\xi_1,\xi_2}^{p,\lambda}$ *with a function* $g(z) \in \mathcal{G}$ *for* $0 < |\delta| \leq |\delta_0(\lambda)|$, *where* $|\delta_0(\lambda)|$ *is the smallest positive root of the equation*

$$\begin{aligned} &|\xi_1||\delta|\sqrt{p(1-\alpha)}(1-|\delta|^2) \\ &+ |\xi_2|\sqrt{1+|\delta|^2}|\delta|^2\sqrt{p(1-\alpha)-|a_{p+1}|^2} - \lambda(1-|\delta|^2)^{\frac{3}{2}} = 0. \end{aligned}$$

Proof If $f(z) \in \mathcal{C}_p(\theta, \alpha)$, then we have that

$$\frac{1}{\delta^p} f(\delta z) = z^p + \sum_{n=p+1}^{\infty} a_n \delta^{n-p} z^n.$$

Applying Theorem 2.1, we need to show that

$$\sum_{n=p+1}^{\infty} [|\xi_1| + |\xi_2|(n-p-1)]|a_n||\delta|^{n-p} \leq \lambda.$$

By using the Cauchy–Schwarz inequality, we can obtain

$$\begin{aligned} &\sum_{n=p+1}^{\infty} [|\xi_1| + |\xi_2|(n-p-1)]|a_n||\delta|^{n-p} \\ &\leq \frac{|\xi_1|}{|\delta|^p}\left(\sum_{n=p+1}^{\infty} |\delta|^{2n}\right)^{\frac{1}{2}}\left(\sum_{n=p+1}^{\infty} |a_n|^2\right)^{\frac{1}{2}} \\ &+ \frac{|\xi_2|}{|\delta|^p}\left(\sum_{n=p+2}^{\infty} (n-p-1)^2|\delta|^{2n}\right)^{\frac{1}{2}}\left(\sum_{n=p+2}^{\infty} |a_n|^2\right)^{\frac{1}{2}} \end{aligned} \tag{10}$$

In fact, Lemma 3.1 implies that

$$\begin{aligned} \sum_{n=p+1}^{\infty} |a_n|^2 &\leq \sum_{n=p+1}^{\infty} |a_n| \\ &\leq \sum_{n=p+1}^{\infty} n|a_n| \leq p(1-\alpha), \end{aligned} \tag{11}$$

So we also have

$$\sum_{n=p+1}^{\infty} |a_n|^2 \leq p(1-\alpha) - |a_{n+1}|^2. \tag{12}$$

Moreover, putting $x = |\delta|^2$, then we have

$$\sum_{n=p+1}^{\infty} |\delta|^{2n} = \sum_{n=p+1}^{\infty} x^n = \frac{x^{p+1}}{1-x} \tag{13}$$

and

$$\begin{aligned} &\sum_{n=p+2}^{\infty} (n-p-1)^2 |\delta|^{2n} \\ &= \sum_{n=p+2}^{\infty} (n-p-1)^2 x^n = \frac{1+x}{(1-x)^3} x^{p+2}. \end{aligned} \tag{14}$$

Following (10)-(14), we can obtain that

$$\begin{aligned} &\sum_{n=p+1}^{\infty} [|\xi_1| + |\xi_2|(n-p-1)]|a_n||\delta|^{n-p} \\ &\leq \frac{|\xi_1|}{|\delta|^p} \left(\sum_{n=p+1}^{\infty} |\delta|^{2n} \right)^{\frac{1}{2}} \left(\sum_{n=p+1}^{\infty} |a_n|^2 \right)^{\frac{1}{2}} \\ &+ \frac{|\xi_2|}{|\delta|^p} \left(\sum_{n=p+2}^{\infty} (n-p-1)^2 |\delta|^{2n} \right)^{\frac{1}{2}} \left(\sum_{n=p+2}^{\infty} |a_n|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{|\xi_1|}{|\delta|^p} \left(\frac{x^{p+1}}{1-x} \right)^{\frac{1}{2}} \left(p(1-\alpha) \right)^{\frac{1}{2}} \\ &+ \frac{|\xi_2|}{|\delta|^p} \left(\frac{1+x}{(1-x)^3} x^{p+2} \right)^{\frac{1}{2}} \left(p(1-\alpha) - |a_{p+1}|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{|\xi_1|}{|\delta|^p} \left(\frac{x^{p+1}}{1-x} \right)^{\frac{1}{2}} \left(p(1-\alpha) \right)^{\frac{1}{2}} \\ &+ \frac{|\xi_2|}{|\delta|^p} \left(\frac{1+x}{(1-x)^3} x^{p+2} \right)^{\frac{1}{2}} \left(p(1-\alpha) - |a_{p+1}|^2 \right)^{\frac{1}{2}} \\ &= |\xi_1| \frac{|\delta| \sqrt{p(1-\alpha)}}{(1-|\delta|^2)^{\frac{1}{2}}} + |\xi_2| \frac{\sqrt{1+|\delta|^2}|\delta|^2 \sqrt{p(1-\alpha) - |a_{p+1}|^2}}{(1-|\delta|^2)^{\frac{3}{2}}}. \end{aligned} \tag{15}$$

We need to consider the complex number $\delta(0<|\delta|<1)$ such that

$$|\xi_1|\frac{|\delta|\sqrt{p(1-\alpha)}}{(1-|\delta|^2)^{\frac{1}{2}}}+|\xi_2|\frac{\sqrt{1+|\delta|^2}|\delta|^2\sqrt{p(1-\alpha)-|a_{p+1}|^2}}{(1-|\delta|^2)^{\frac{3}{2}}}=\lambda.$$

Hence, we definite the following function with $|\delta(\lambda)|$ by

$$\begin{aligned}F(|\delta(\lambda)|)&=|\xi_1||\delta|\sqrt{p(1-\alpha)}(1-|\delta|^2)\\&\quad+|\xi_2|\sqrt{1+|\delta|^2}|\delta|^2\sqrt{p(1-\alpha)-|a_{p+1}|^2}-\lambda(1-|\delta|^2)^{\frac{3}{2}}.\end{aligned}$$

It is easily to know that $F(0)=-\lambda<0$ and

$$F(1)=\sqrt{2}|\xi_2|\sqrt{p(1-\alpha)-|a_{p+1}|^2}>0,$$

which implies that there exists some $\delta_0(\lambda)$ such that $F(|\delta_0(\lambda)|)=0(0<|\delta_0(\lambda)|<1)$. The proof of the theorem is complete.

Theorem 3.4 *Let* $f(z)=z^p+\sum\limits_{n=p+1}^{\infty}a_nz^n\in\mathcal{K}_p(\theta,\alpha)$ *and* $\delta(0<|\delta|<1)$ *is a complex number. Then* $\frac{1}{\delta^p}f(\delta z)\in\Theta^{p,\lambda}_{\xi_1,\xi_2}$ *with a function* $g(z)\in\mathcal{G}$ *for* $0<|\delta|\leq|\delta_0(\lambda)|$, *where* $|\delta_0(\lambda)|$ *is the smallest positive root of the equation*

$$|\xi_1||\delta|\sqrt{p-\alpha}(1-|\delta|^2)+|\xi_2|\sqrt{1+|\delta|^2}|\delta|^2\sqrt{p-\alpha-|a_{p+1}|^2}-\lambda(1-|\delta|^2)^{\frac{3}{2}}=0.$$

Proof Since $f(z)\in\mathcal{K}_p(\theta,\alpha)$, using Lemma 3.2, we have that

$$\sum_{n=p+1}^{\infty}\frac{n}{p}(n-\alpha)|a_n|\leqslant p-\alpha,$$

which leads to

$$\begin{aligned}\sum_{n=p+1}^{\infty}|a_n|^2&\leqslant\sum_{n=p+1}^{\infty}(n-p)|a_n|^2\leqslant\sum_{n=p+1}^{\infty}\frac{n}{p}(n-\alpha)|a_n|^2\\&\leqslant\sum_{n=p+1}^{\infty}\frac{n}{p}(n-\alpha)|a_n|\leqslant p-\alpha.\end{aligned}\tag{16}$$

Hence, from (15), we can also note that

$$\begin{aligned}
&\sum_{n=p+1}^{\infty}[|\xi_1|+|\xi_2|(n-p-1)]|a_n||\delta|^{n-p} \qquad (17)\\
&\leq \frac{|\xi_1|}{|\delta|^p}\left(\sum_{n=p+1}^{\infty}|\delta|^{2n}\right)^{\frac{1}{2}}\left(\sum_{n=p+1}^{\infty}|a_n|^2\right)^{\frac{1}{2}}\\
&+\frac{|\xi_2|}{|\delta|^p}\left(\sum_{n=p+2}^{\infty}(n-p-1)^2|\delta|^{2n}\right)^{\frac{1}{2}}\left(\sum_{n=p+2}^{\infty}|a_n|^2\right)^{\frac{1}{2}}\\
&\leq \frac{|\xi_1|}{|\delta|^p}\left(\frac{x^{p+1}}{1-x}\right)^{\frac{1}{2}}\left(p-\alpha)\right)^{\frac{1}{2}}\\
&+\frac{|\xi_2|}{|\delta|^p}\left(\frac{1+x}{(1-x)^3}x^{p+2}\right)^{\frac{1}{2}}\left(p-\alpha-|a_{p+1}|^2\right)^{\frac{1}{2}}\\
&=|\xi_1|\frac{|\delta|\sqrt{p-\alpha}}{(1-|\delta|^2)^{\frac{1}{2}}}+|\xi_2|\frac{\sqrt{1+|\delta|^2}|\delta|^2\sqrt{p-\alpha-|a_{p+1}|^2}}{(1-|\delta|^2)^{\frac{3}{2}}}
\end{aligned}$$

Using the same technique as in the proof of Theorem 3.3, we derive the result. The proof of the theorem is complete.

4 Modified-Hadamard Product

Let $f(z)=z^p+\sum\limits_{n=p+1}^{\infty}|a_n|e^{i((n-p)\theta)-\gamma)}z^n$, $g(z)=z^p+\sum\limits_{n=p+1}^{\infty}|b_n|e^{i((n-p)\theta)-\gamma)}z^n$. We define modified Hadamard product for the functions f, g as follows:

$$(f*g)(z)=z^p+\sum_{n=p+1}^{\infty}|a_n||b_n|e^{i((n-p)\theta)-\gamma)}z^n, z\in\mathbb{U}.$$

Theorem 4.1 *If* $f_1(z)=z^p+\sum\limits_{n=p+1}^{\infty}|a_{n,1}|e^{i((n-p)\theta)-\gamma)}z^n\in\Theta^{p,\lambda_1}_{\xi_1,\xi_2}$ *with* $g_1(z)\in\mathcal{G}$, $f_2(z)=z^p+\sum\limits_{n=p+1}^{\infty}|a_{n,2}|e^{i((n-p)\theta)-\gamma)}z^n\in\Theta^{p,\lambda_2}_{\xi_1,\xi_2}$ *with a function* $g_2(z)\in\mathcal{G}$ *and* $\arg\xi_1=\arg\xi_2=\gamma$ *, then we have*

$$(f_1*f_2)(z)\in\Theta^{p,\lambda^*}_{\xi_1,\xi_2}$$

with a function $g(z)\in\mathcal{G}$*, where*

$$\lambda^*=\frac{1}{|\xi_1|}\lambda_1\lambda_2.$$

Proof Suppose $f_1(z) = z^p + \sum_{n=p+1}^{\infty} |a_{n,1}| e^{i((n-p)\theta)-\gamma)} z^n \in \Theta_{\xi_1,\xi_2}^{p,\lambda_1}$, $f_2(z) = z^p + \sum_{n=p+1}^{\infty} |a_{n,2}| e^{i((n-p)\theta)-\gamma)} z^n \in \Theta_{\xi_1,\xi_2}^{p,\lambda_2}$ and $\arg \xi_1 = \arg \xi_2 = \gamma$, then from Theorem 2.2, we have

$$\sum_{n=p+1}^{\infty} \frac{[|\xi_1| + |\xi_2|(n-p-1)]|a_{n,1}|}{\lambda_1} \leq 1 \tag{18}$$

and

$$\sum_{n=p+1}^{\infty} \frac{[|\xi_1| + |\xi_2|(n-p-1)]|a_{n,2}|}{\lambda_2} \leq 1. \tag{19}$$

Moreover, (18) and (19) imply that

$$\Big\{ \sum_{n=p+1}^{\infty} \frac{[|\xi_1| + |\xi_2|(n-p-1)]|a_{n,1}|}{\lambda_1} \Big\}^{\frac{1}{2}} \leq 1 \tag{20}$$

and

$$\Big\{ \sum_{n=p+1}^{\infty} \frac{[|\xi_1| + |\xi_2|(n-p-1)]|a_{n,2}|}{\lambda_2} \Big\}^{\frac{1}{2}} \leq 1. \tag{21}$$

By using the Holder inequality with (20) and (21), we get

$$\sum_{n=p+1}^{\infty} \Big\{ \frac{[|\xi_1| + |\xi_2|(n-p-1)]}{\lambda_1} \Big\}^{\frac{1}{2}} \Big\{ \frac{[|\xi_1| + |\xi_2|(n-p-1)]}{\lambda_2} \Big\}^{\frac{1}{2}} \sqrt{|a_{n,1}||a_{n,2}|} \leq 1,$$

so

$$\sum_{n=p+1}^{\infty} |\xi_1| + |\xi_2|(n-p-1)] \Big\{ \frac{1}{\lambda_1} \Big\}^{\frac{1}{2}} \Big\{ \frac{1}{\lambda_2} \Big\}^{\frac{1}{2}} \sqrt{|a_{n,1}||b_{n,2}|} \leq 1. \tag{22}$$

In order to obtain the $(f * g)(z) \in \Theta_{\xi_1,\xi_2}^{p,\lambda^*}$ with a function $g(z) \in \mathcal{G}$, we have to find the corresponding λ^* such that

$$\sum_{n=p+1}^{\infty} \frac{[|\xi_1| + |\xi_2|(n-p-1)]|a_{n,1}||b_{n,2}|}{\lambda^*} \leq 1. \tag{23}$$

Following (22), then (23) hold true if for any $n \geq p+1$,

$$\frac{1}{\lambda^*} \leq (\frac{1}{\lambda_1})^{\frac{1}{2}} (\frac{1}{\lambda_2})^{\frac{1}{2}} \frac{1}{\sqrt{|a_{n,1}||b_{n,2}|}}$$

or

$$\lambda^* \geq (\lambda_1)^{\frac{1}{2}} (\lambda_2)^{\frac{1}{2}} \sqrt{|a_{n,1}||b_{n,2}|}. \tag{24}$$

In fact, (24) implies that

$$\lambda^* = \max\{\mathscr{L}(n)|\mathscr{L}(n) = (\lambda_1)^{\frac{1}{2}}(\lambda_2)^{\frac{1}{2}}\sqrt{|a_{n,1}||b_{n,1}|}, \forall n \geq 1+p\}.$$

Furthermore, from (22), it is easy to know that

$$\sqrt{|a_{n,1}||b_{n,1}|} \leq \frac{1}{|\xi_1| + |\xi_2|(n-p-1)}(\lambda_1\lambda_2)^{\frac{1}{2}}, \tag{25}$$

since $|\xi_1| + |\xi_2|(n-p-1)$ is increasing in n, following (25), then we can see that

$$\begin{aligned}\mathscr{L}(n) = (\lambda_1)^{\frac{1}{2}}(\lambda_2)^{\frac{1}{2}}\sqrt{|a_{n,1}||b_{n,1}|} &\leq \frac{1}{|\xi_1| + |\xi_2|(n-p-1)}\lambda_1\lambda_2 \\ &\leq \frac{1}{[|\xi_1| + |\xi_2|(n-p-1)]|_{n=p+1}}\lambda_1\lambda_2 = \frac{1}{|\xi_1|}\lambda_1\lambda_2.\end{aligned}$$

The proof of the theorem is complete.

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