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Adding a General Union of a Prescribed Number of Curves with High Sum of their Degrees Improve the Hilbert Function of any Scheme

Edoardo Ballico

Department of Mathematics, University of Trento
38123 Povo (TN), Italy
E-mail: ballico@science.unitn.it

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Abstract

Let $Z \subset \mathbb{P}^r$, $r \geq 4$, *be a closed subscheme with* $\dim(Z) \leq r-4$. *Fix integers* $c > 0$ *and* $g_i \geq 0$, $i = 1, \dots, c$. *We prove that the general union of* Z *and* c *smooth curves* $Y_i \subset \mathbb{P}^r$ *with genus* g_i *and* $\deg(Y_i) \geq r + gr$ *as maximal rank (i.e. the expected postulation) if* $\deg(Y_1) + \cdots + \deg(Y_c) \gg 0$.

Keywords: *Postulation, Hilbert function, Disjoint unions of curves*

1 Introduction

Let $W \subset \mathbb{P}^r$ be any closed subscheme. We say that W has *maximal rank* if for every integer t either $h^0(\mathcal{I}_W(t)) = 0$ or $h^1(\mathcal{I}_W(t)) = 0$, i.e. if for every integer t the restriction map $\rho_{W,t} : H^0(\mathcal{O}_{\mathbb{P}^r}(t)) \to H^0(\mathcal{O}_W(t))$ is either injective or surjective, i.e. it is a linear map with maximal rank, i.e. for all $t \geq 0$ we have $h^0(\mathcal{I}_W(t)) = \max\{0, \binom{r+t}{r} - h^0(\mathcal{O}_W(t))\}$. Usually the integer $h^0(\mathcal{O}_W(t))$ is known and hence maximal rank for the integer t means that the set of all degree t hypersurfaces containing W has the “ expected ” dimension, i.e. $\rho_{W,t}$ has maximal rank if and only if the set of all degree t hypersurfaces containing W has the minimal dimension which a priori may have. Set $\delta_r(0) := 1$ and $\delta_r(1) := r + 1$. For all integers $r \geq 3$ and $g \geq 2$ set $\delta_r(g) := r + gr$.

In this paper we prove the following results, easily seen to be equivalent (we prove Theorem 2 and then prove, assuming it for fixed data $r, Z, c, g_1, \dots, g_c$ that Theorem 1 is true for the same data, but it is very easy to see the other implication).

Theorem 1 *Fix integers $r \geq 4$, $c > 0$ and $g_i \geq 0$, $1 \leq i \leq c$. Let $Z \subset \mathbb{P}^r$ be a closed subscheme with $\dim(Z) \leq r-4$. Then there in an integer Δ (depending only on r, c, $g_1, \dots, g_c$ and the Hilbert polynomial of Z) with the following property. Fix integers $d_i \geq \delta_r(g_i)$, $1 \leq i \leq c$, such that $d_1 + \cdots + d_c \geq \Delta$. Let $Y \subset \mathbb{P}^r \setminus Z$ be a general union of c smooth curves $Y_1, \dots, Y_c$ with $\deg(Y_i) = d_i$ and $p_a(Y_i) = g_i$ for all i. Then $Z \cup Y$ has maximal rank.*

Theorem 2 *Fix integers $r \geq 4$, $c > 0$ and $g_i \geq 0$, $1 \leq i \leq c$. Let $Z \subset \mathbb{P}^r$ be a closed scheme with $\dim(Z) \leq r-4$. Then there in an integer k_0 (depending only on r, c, $g_1, \dots, g_c$ and the Hilbert polynomial of k) with the following property. Fix integers $d_i \geq \delta_r(g_i)$, $1 \leq i \leq c$. Let $Y \subset \mathbb{P}^r \setminus Z$ be a general union of c smooth curves $Y_1, \dots, Y_c$ with $\deg(Y_i) = d_i$ and $p_a(Y_i) = g_i$ for all i. Then for all integers $k \geq k_0$ either $h^0(\mathcal{I}_{Z\cup Y}(k)) = 0$ or $h^1(\mathcal{I}_{Z\cup Y}(k)) = 0$.*

The generality assumptions in Theorems 1 and 2 implies that $Y_i \cap Y_j$ for all $i \neq j$ and that $Z \cap Y = \emptyset$, so that if $p(t)$ is the Hilbert polynomial of Z, then $p(t) + c - g_1 - \cdots - g_c + (d_1 + \cdots + d_c)t$ is the Hilbert polynomial of $Z \cup Y$. As in [7] and [5] we use the so-called Horace method. As in [7] and in several later papers we use embedded deformation of reducible curves inside a projective space ([9],[6]).

We work over an algebraically closed field $\mathbb{K}$

2 Preliminaries

For all integers $r \geq 3$, $g \geq 0$ and $d \geq \max\{2g+1, g+r\}$ let $Z(r,d,g)$ denote the set of all smooth and non-degenerate curves $X \subset \mathbb{P}^r$ with degree d, genus g and $h^1(X, \mathcal{O}_X(1)) = 0$. Each set $Z(r,d,g)$ is a non-empty and irreducible open subset of the Hilbert scheme of $\mathbb{P}^r$. This observation explains the word " general " in the statement of Theorems 1 and 2. Indeed, each c-ple $(Y_1, \dots, Y_c)$ must be a general element of the irreducible algebraic set $\prod_{i=1}^{c} Z(r, g_i, d_i)$. Let $Z'(r,d,g)$ denote the closure of $Z(r,d,g)$ in the Hilbert scheme of $\mathbb{P}^r$.

Fix $X \in Z(r,d,c)$. Let $E \subset \mathbb{P}^r$ be a smooth rational curve such that the curve $X \cup E$ is nodal and connected and $x := \sharp(X \cap E) \leq 2$. Set $e := \deg(E)$. The curve $X \cup E$ has arithmetic genus $g + x - 1$ and degree $d + e$. Since $X \cup E$ is nodal, its normal sheaf is locally free. We claim that $X \cup E \in Z'(r, d+e, g+e-1)$; this statement is almost contained in both [6] and [9] (both papers handle more general and difficult cases); it can be proved first degenerating E to a connected union E' of e lines with $E' \cup X$ nodal and then applying to $X \cup E'$ either [9, Theorem 5.2] or [6, Theorem 4.1, Remark 4.1.2 and Corollary 4.2]. A Mayer-Vietoris exact sequence shows that $h^1(\mathcal{O}_{X\cup E}(1)) = 0$. Hence for any $Y \in Z(r, d+e, g+e-1)$ and any integer $t \geq 0$ we have $h^0(\mathcal{O}_{X\cup E}(t)) = td + 1 - (g + x - 1) = h^0(\mathcal{O}_Y(t))$. Let $Z \subset \mathbb{P}^r$ be any closed subscheme with $\dim(Z) \leq r - 2$ and $Z \cap (X \cup E) = \emptyset$. Fix $t \in \mathbb{N}$. By the semicontinuity theorem for cohomology ([4, Theorem III.12.8]) to prove that either $h^1(\mathcal{I}_{Z\cup Y}(t)) = 0$ or $h^0(\mathcal{I}_{Z\cup Y}(t)) = 0$ for a general $Y \in Z(r, d+e, g+x-1)$ it is sufficient to prove that either $h^1(\mathcal{I}_{Z\cup X\cup E}(t)) = 0$ or $h^0(\mathcal{I}_{Z\cup X\cup E}(t)) = 0$

For each smooth curve $X \subset \mathbb{P}^r$ let N_X denote its normal bundle.

Remark 1 *Fin integers $n \geq 3$, $g \geq 0$, $d \geq g + n$ and $x > 0$. Fix any $C \in Z(n, d, g)$ and any $S \subset C$ such that $\sharp(S) = x$. To prove that for a general $B \subset \mathbb{P}^n$ there is $X \in Z(n, d, g)$ with $B \subset X$ and that the set of all such curves X has the expected dimension $\dim(Z(n, d, g)) - x(n-1)$ it is sufficient to prove that $h^1(N_C(-S)) = 0$ (use the proof of [8, Theorem 1.5] or [3]). Since C is smooth, N_C is a quotient of $T\mathbb{P}^r|C$ and hence it is a quotient of $\mathcal{O}_C(1)^{\oplus(n+1)}$ by the Euler's sequence of $T\mathbb{P}^n$. Hence $h^1(N_C(-S)) = 0$ if $h^1(\mathcal{O}_C(1)(-S)) = 0$. This is true if $d - x \geq 2g - 1$. In particular if $g = 0$, then it is sufficient to assume $d \geq x + 1$. In the case $g = 1$ we may use Atiyah's classification of vector bundles on elliptic curves and conclude, since $N_C(-1)$ is spanned and with no trivial factor if $x \leq d$. In the case $g = 0$ we may do a proof which works for all $d \geq x$ in the following way. We first reduce to the case $x = d = n$. Then we use that any two $(n+2)$-ples of points of $\mathbb{P}^n$ in linearly general position are projectively normal and that any $n + 2$ general points of an integral and non-degenerate curve $C \subset \mathbb{P}^n$ are linearly independent.*

Lemma 1 *Fix a smooth and projective curve of genus g and integer $d \geq \delta_r(g)$. Let $X \subset \mathbb{P}^r$ be a general embedding of degree d of C. Then $h^1(X, N_X(-1)) = 0$.*

Proof. Since $d > 2g - 2$, we are looking at non-special embeddings of C and hence the set of all such embeddings (for fixed d and r) is parametrized by an irreducible variety. Hence the word " general " is allowed. If $g = 0$, then we use Remark 1 (notice that in the case $g = 0$ we allow degenerate embeddings if $1 \leq d < r$). Now assume $g \geq 1$. By the universal property of the Grassmannian there is a bijection between morphisms $f : C \to \mathbb{P}^r$ and pairs (E, h), where E is a rank r vector bundle on C and $h : \mathbb{K}^{r+1} \to H^0(C, E)$ is a linear map with its image $h(\mathbb{K}^{r+1})$ spanning E. In this bijection $E \cong f^*(T\mathbb{P}^r(-1))$ and $f(C)$ is non-degenerate if and only h is injective. Since $\dim(C) = 1$, a dimensional count shows that every rank r vector bundle F on C is spanned by a linear subspace of $H^0(C, F)$ with dimension at most $r + 1$. Since $r \geq 3$ and $d \geq r(g+1)$, it is easy to find a rank r spanned vector bundle E on C for which the map f is an embedding and $h^1(C, E) = 0$. For instance, take $E = R_1 \oplus \cdots \oplus R_r$ with $(R_1, \dots, R_r)$ general in $\mathrm{Pic}^{d-(r-1)(g+1)}(C) \times \mathrm{Pic}^{g+1}(C)^{r-1}$ and then take a general $(r+1)$-dimensional linear subspace of $H^0(C, E)$ to define the injective map h. Hence for all integers $r \geq 3$ and $d \geq r(g+1) = \delta_r(g)$ the general degree d embedding $X \subset \mathbb{P}^r$ of C has the property that $h^1(X, T\mathbb{P}^r(-1)|X) = 0$. Let $\mathcal{K} \cong TX(-1)$ be the kernel of the surjection $T\mathbb{P}^r(-1) \to N_X(-1)$. Since $\dim(X) = 1$, we have $h^2(X, \mathcal{K}) = 0$ and hence $h^1(X, N_X(-1)) = 0$.

3 The Inductive Set-up

Set $\eta = c - g_1 - \cdots - g_c$. Let $p(t)$ be the Hilbert polynomial of Z. For any hyperplane $M \subset \mathbb{P}^r$ and any closed subscheme $W \subset \mathbb{P}^r$ the residual scheme $\mathrm{Res}_M(W)$ is the closed subscheme of $\mathbb{P}^r$ with $\mathcal{I}_W : \mathcal{I}_M$ as its ideal sheaf. If $W = A \sqcup B$ with A, B closed in W and $A \cap B = \emptyset$, then $\mathrm{Res}_M(A \sqcup B) = \mathrm{Res}_M(A) \sqcup \mathrm{Res}_M(B)$. If W is reduced, then $\mathrm{Res}_M(W)$ is the closure in $\mathbb{P}^r$ of the union of all irreducible components of W not contained in M.

Let $H \subset \mathbb{P}^r$ be a hyperplane which does not contain the support of any component of the scheme Z (i.e. of the sheaf $\mathcal{O}_Z$), not even an embedded one. By the primary decomposition theorem only finitely many subvarieties of Z_{red} are the support of a component of the sheaf $\mathcal{O}_Z$ and hence, after fixing Z, we may take as H a general hyperplane. In particular H contains no irreducible component of Z_{red}. Therefore $Z \cap H = \emptyset$ if $\dim(Z) \leq 0$, while $\dim(Z \cap H) = \dim(Z) - 1$ if $\dim(Z) > 0$. The condition that H contains no component (not even an embedded one) of Z is equivalent to assuming that the equation of H is a non-zero-divisor of $\mathcal{O}_Z$ at each point of Z. Hence $\mathrm{Res}_H(Z) = Z$ and for each $t \in \mathbb{Z}$ the multiplication by a linear form ℓ with $H = \{\ell = 0\}$ induces an exact sequence

$$0 \to \mathcal{I}_Z(t-1) \to \mathcal{I}_Z(t) \to \mathcal{I}_{Z \cap H, H}(t) \to 0 \tag{1}$$

Let $p(t)$ be the Hilbert polynomial of Z. By (1) the Hilbert polynomial of the scheme $Z \cap H$ is the first difference of p, i.e. the polynomial p_1 defined by the formula $p_1(t) = p(t) - p(t-1)$ for all t. For all integers $r \geq 4$, all integer-valued polynomiasl $q(t)$ with $\deg(q(t)) \leq r-3$ and all integers $k > 0$ define the integers $u_{r,q(k),k}$ and $v_{r,q(k),k}$ by the relations

$$q(k) + k u_{r,q(k),k} + v_{r,q(k),k} = \binom{r+k}{r}, 0 \leq v_{r,q(k),k} \leq k-1 \tag{2}$$

From (2) and the same equation for the integer $k-1$ and another polynomial $q_1(t)$ we get

$$q(k) - q_1(k-1) + k(u_{r,q(k),k} - u_{r,q_1(k-1),k-1}) + \tag{3}$$

$$v_{r,q(k),k} - v_{r,q(k-1),k-1)} = \binom{r+k-1}{r-1}$$

Since $\deg(q(t)) \le r-3$, for $k \gg 0$ (depending only on $q(t)$ and r) we have $u_{r,q(k),k} \sim \frac{k^{r-1}}{r!}$ and $u_{r,q(k),k} - u_{r,q(k-1),k-1} \sim \frac{k^{r-2}}{r\cdot(r-2)!}$.

We will only use the case in which $q(t)-q_1(t)$ and $q(t)-p(t)$ are a constant, $|q(t)-q_1(t)| \le c+g_1+\cdots+g_c$ and $|q(t)-p(t)| \le c+g_1+\cdots+g_c$. Since $0 \le v_{r,a,x} \le x-1$, from (3) we get that if $k > 3(c+g_1+\cdots+g_c)$, then $|u_{r,p(k),k} - u_{r,p(k)+x,k}| \le 1$ for all x with $|x| \le c+g_1+\cdots+g_c$. Hence (for fixed r, $p(t)$, c and $g_1,\dots,g_c$) there is an integer k_1 such that for all integers $k \ge k_1$ and all such polynomials $q(t), q_1(t)$ we have $h^0(\mathcal{O}_Z(k)) = p(k)$, $u_{r,q(k),k} \sim \frac{k^{r-1}}{r!}$ and $u_{r,q(k),k} - u_{r,q_1(k-1),k-1} \sim \frac{k^{r-2}}{r\cdot(r-2)!}$.

4 Proofs of Theorems 1 and 2

If $r > 4$ we use induction on r to prove Theorems 1 and 2, except that in $\mathbb{P}^{r-1}$ we only require the case $g_i = 0$ for all i. Recall that we assumed that $\dim(Z) \le r-4$. Hence if $r=4$, then $Z \cap H = \emptyset$. Hence if $r=4$, then we may use Theorems 1 and 2 in $H = \mathbb{P}^3$ ([1]). For Theorem 2 we may take $k_0 = 5$ if $r=3$ and $Z=\emptyset$. Hence we introduce the following notation.

Notation 1 *If $r=4$, then set $\kappa := 5$. If $r>4$, then we assume that Theorems 1 and 2 are true in $H = \mathbb{P}^{r-1}$ for any c, but only for genera $g_i = 0$ and we call κ any integer which we may take as k_0 in the statement of Theorem 2 in H for $Z \cap H$, c, $g_i = 0$ for all i.*

The next lemma is Assertion B_t (and its proof as Claim 2) in part (i) of [2, §5].

Lemma 2 *There is an integer $k_2' \ge \kappa$ (depending only on the integer r and the Hilbert function $p(t)$ of Z) such that for all integers $k \ge k_2'$ a general union $Y \subset \mathbb{P}^r$ of a general $E \in Z(r, u_{r,p(k)+1,1} - v_{r,p(k)+1,1}, 0)$ and $v_{r,p(k)+1,1}$ lines satisfies $h^i(\mathcal{I}_{Z\cup Y}(k)) = 0$, $i=0,1$.*

Notation 2 *Fix an integer $k_2 \ge \max\{k_1, k_2', |\eta|+2\}$. For all $i=2,\dots,c$ set $e_i := \delta_r(g_i) - g_i$ if $g_i > 0$ and $e_i := 1$ if $g_i = 0$. Set $\alpha := u_{r,p(k_2+1)+c,k_2+1} - v_{r,p(k_2+1)+\eta,k_2+1} - \sum_{i=2}^{c} e_i$. We have $\alpha \ge u_{r,p(k_2),k_2} + \delta_r(g_1) + c$ and hence $\alpha \ge \delta_r(g_1)$.*

Lemma 3 *Let $Y \subset \mathbb{P}^r$ be a general union of a smooth curve of genus g_1 and degree α, $c-1$ smooth rational curves of degree $e_2,\dots,e_c$ and $v_{r,p(k_2+1)+c-g_1,k_2+1}$ lines. Then $h^i(\mathcal{I}_{Z\cup Y}(k_2+1)) = 0$, $i=0,1$.*

Proof. Since $k_2 > \max\{|\eta|, 1\}$, we have $|u_{r,p(x)+1,k} - u_{r,p(x)+\eta,x}| \le 1$ for $x = k_2$ and $x = k_2+1$ and $u_{r,p(k_2+1)+1,k_2+1} - u_{r,p(k_2)+1,k_2} \sim \frac{k_2^{r-2}}{r\cdot(r-2)!}$, we get $\alpha \ge u_{r,p(k_2)+1,k_2} + \delta_r(g_1) + 4k$. Fix $Y = Y_1 \sqcup E$ as in Lemma 2 with Y_1 a general element of $Z(r, u_{r,p(k_2)+1,k_2} - v_{r,p(k_2)+1,k_2}, g_1)$ and E a disjoint union of $v_{r,p(k_2)+1,k_2}$ lines. We have $h^i(\mathcal{I}_{Z\cup Y}(k_2)) = 0$, $i = 0,1$ (Lemma 2). Without losing generality we may also assume that Y is transversal to H and that $Y \cap H$ is formed by $\deg(Y)$ general points of H (Lemmma 1).

(a) First assume $v_{r,p(k_2+1)+\eta,k_2+1} \ge v_{r,p(k_2)+1,k_2}$. By the inductive assumption with only genera zero if $r > 4$ or [1] if $r = 4$, there are is curve $W \subset H$ with the following properties: $h^1(H, \mathcal{I}_{(Z\cap H)\cup W,H}(k_2+1)) = 0$, $W = W_1 \sqcup \cdots \sqcup W_{c+1}$, $W \cap Z = \emptyset$, W_{c+1} is a disjoint union of $v_{r,p(k_2+1)+\eta,k_2+1} - v_{r,p(k_2)+1,k_2}$ lines, $W_{c+1} \cap (Y \cap H) = \emptyset$, W_j, $2 \le j \le c$, is a smooth rational curve of degree e_j with $W_j \cap (Y \cap H) = \emptyset$, W_1 is a smooth rational curve of degree $u_{r,p(k_2+1)+c-g_1,k_2+1} - u_{r,p(k_2)+1,k_2} - \sum_{i=2}^{c} e_i - (v_{r,p(k_2+1)+\eta,k_2+1} - v_{r,p(k_2)+1,k_2})$, $W_1 \cap E = \emptyset$, and $\sharp(W_1 \cap Y_1) = g_1+1$. We need to check that we may achieve the last condition. It is sufficient to use that $Y_1 \cap H$ contains at least g_1+1 points of H and that a general rational curve $C \subset H$ with $\deg(C) = \deg(W_1)$ and C passes through g_1+1 general points of H. The first condition is satisfied by the case $g = 0$ of Remark 1 (we even have that $Y_1 \cap H$ is general in H and hence it is sufficient to notice that $\deg(Y_1) \ge g_1+1$ by our assumption on k_2. The second condition is satisfied by the case $g = 0$, $n = r-1$ of Remark 1, because our assumption on k_2 implies $\deg(W_1) \ge g_1+1$. Set $F := Y \cup W$. The curve F is a disjoint union of a nodal curve with arithmetic genus g_1, $c-1$ smooth rational curves of degree $e_2, \dots, e_c$ and $v_{r,p(k_2+1)+c-g_1}$ lines. From (3) we get $h^0(\mathcal{O}_{((Z\cup Y)\cap H)\cup W}(k_2+1)) = \binom{r+k_2}{r-1}$. Hence $h^0(\mathcal{O}_{(Z\cap H)\cup W}(k_2+1)) \le \binom{r+k_2}{r-1}$. If $r = 4$, then $Z \cap H = \emptyset$; since $\kappa \ge 5$, by [1] we have $h^1(H, \mathcal{I}_{(Z\cap H)\cup W,H}(k_2+1)) = 0$. If $r > 4$, then $h^1(H, \mathcal{I}_{(Z\cap H)\cup W,H}(k_2+1)) = 0$, because $k_2+1 \ge \kappa$ Since $Y \cap H \setminus W$ has cardinality $u_{r,p(k_2)+1,k_2} - g_1 - 1$ and it is general in H, $h^1(H, \mathcal{I}_{(Z\cap H)\cup W,H}(k_2+1)) = 0$ and $Z \cap H$ has $p(t) - p(t-1)$ as its Hilbert polynomial, (3) gives $h^i(H, \mathcal{I}_{((Z\cup Y)\cap H)\cup W}(k_2+1)) = 0$, $i = 0,1$. Castelnuovo's inequalities give $h^i(\mathcal{I}_{Z\cup Y\cup W}(k_2+1)) = 0$, $i = 0,1$. The semicontinuity theorem for cohomology gives the lemma when $v_{r,p(k_2+1)+\eta,k_2+1} \ge v_{r,p(k_2)+1,k_2}$.

(b) Now assume $v_{r,p(k_2+1)+c-g_1 k_2+1} < v_{r,p(k_2)+1,k_2}$. Write $E = E_1 \sqcup E_2$ with E_1 a disjoint union of $v_{r,p(k_2+1)+c-g_1,k_2+1}$ lines. By the inductive assumption with only genera zero if $r > 4$ or [1] if $r = 4$, there are a curve $W \subset H$ with $h^1(H, \mathcal{I}_{(Z\cap H)\cup W,H}(k_2+1)) = 0$, $W = W_1 \sqcup \cdots \sqcup W_c$ with W_j, $2 \le j \le c$, a smooth rational curve of degree e_j, $W \cap (Z \cap H) = \emptyset$, $W_j \cap Y = \emptyset$ for all $j \ne 1$, $W_1 \cap E_1 = \emptyset$, $\sharp(W_1 \cap Y_1) = g_1+1$ and $E_2 \cap H \subset W_1$. We need to check that W_1 contains $\beta := 1 + g_1 + v_{r,p(k_2)+1,k_2} - v_{r,p(k_2)+1,k_2}$ general points of H. This is true, because $\beta \le 2k_2$ and $\deg(W_1) \ge 2k_2$. We use $Y \cup W$ and the semicontinuity theorem.

Lemma 4 *Fix any integer $k \ge k_2 + 2$ and any integers $b_1, \dots, b_c$ such that $b_i \ge \delta_r(g_i)$ for all $i \ne 1$, $b_1 \ge \alpha$ (where α is defined in Lemma 3), $b_1 + \cdots + b_c = u_{r,p(k)+\eta,k} - v_{r,p(k)+\eta,k}$. Let $Y = Y_1 \sqcup \cdots \sqcup Y_c \sqcup Y_{c+1} \subset \mathbb{P}^r$ be a general union with $Y_i \in Z(r, b_i, g_i)$, $1 \le i \le c$, and Y_{c+1} a union of $v_{r,p(k)+\eta}$ disjoint lines. Then $h^i(\mathcal{I}_{Z\cup Y}(k)) = 0$, $i = 0, 1$.*

Proof. Fix k and the integers $b_i \ge \delta_r(g_i)$ for all $i \ne 1$, $b_1 \ge \alpha$, $b_1 + \cdots + b_c = u_{r,p(k)+\eta,k} - v_{r,p(k)+\eta}$.

(a) Assume $k = k_2 + 2$. Take the set-up of Lemma 3. Recall that $e_i = 1$ if $g_i = 0$ and $e_i = \delta_r(g_i) - 1$ if $g_i > 0$, $2 \le i \le c$. Hence $b_i - g_i \ge e_i \ge r$ if $g_i > 0$. Take $Y' = Y_1' \sqcup \cdots \sqcup Y_{c+1}'$ with $Y' \cap Z = \emptyset$, $h^i(\mathcal{I}_{Z\cup Y'}(k_2 + 1)) = 0$, $i = 0, 1$, $Y_i' \cap Y_j' = \emptyset$ for all $i \ne j$, $Y_1' \in Z(3, \alpha, g_1)$, $Y_i' \in Z(r, e_i, 0)$ for all $i = 2, \dots, c$ and Y_{c+1}' a disjoint union of $v_{r,p(k_2+2)+\eta,k_2+2}$ lines. Since $\alpha \ge 2rg_1$, we may assume that $Y' \cap H$ is a general subset of H with cardinality $\deg(Y')$.

(a1) First assume $v_{r,p(k_2+2)+\eta,k_2+2} \ge v_{r,p(k_2+1)+c-g_1,k_2+1}$. Take the curve $W = W_1 \sqcup \cdots \sqcup W_{c+1} \subset H$ in the following way. $W_i \cap W_j = \emptyset$ for all $i \ne j$; W_{c+1} is a disjoint union of $v_{r,p(k_2+2)+\eta,k_2+2} - v_{r,p(k_2+1)+c-g_1,k_2+1}$ lines; W_i, $2 \le i \le c$, is a smooth rational curve of degree $b_i - e_i$ containing containing exactly one point of Y_i' if $b_i > e_i$, while $W_i = \emptyset$ if $b_i = e_i$ (in the latter case we have $g_i = 1$ and $b_i = 1$); W_1 is a smooth rational curve contained exactly one point of Y_1'. From (3) we get $h^0(\mathcal{O}_{((Z\cup Y')\cap H\cup W}(k_2 + 2)) = \binom{r+k_2+1}{r-1}$. Since $Y' \cap H$ is general in H, it is sufficient to prove that $h^1(H, \mathcal{I}_{(Z\cap H)\cup W}(k_2 + 2)) = 0$. If $r = 4$, then $Z \cap H = \emptyset$ and hence we only need to prove that $h^1(H, \mathcal{I}_{W,H}(k)) = 0$ with $W \subset H$ a general union of at most c general rational curves with prescribed degrees. If $r = 4$, then $Z \cap H = \emptyset$; we use [1] and that $\kappa \ge 5$. If $r > 4$, then we use the definition of κ, i.e. inductive assumption in $H \cong \mathbb{P}^{r-1}$ for $H \cap Z$, c, and $g_i = 0$ for all i. Hence from (3) and the generality of the set $Y' \cap H$ we get $h^i(H, \mathcal{I}_{((Z\cup Y)\cap H)\cup W,H}(k_2 + 2)) = 0$, $i = 0, 1$. The Castelnuovo's inequalities give $h^i(\mathcal{I}_{Z\cup Y'\cup W}(k_2 + 2)) = 0$, $i = 0, 1$. Use the semicontinuity theorem for cohomology ([4, Theorem III.12.8]).

(a2) Now assume $v_{r,p(k_2+2)+\eta,k_2+2} < v_{r,p(k_2+1)+c-g_1,k_2+1}$. Write $Y_{c+1}' = E_1 \sqcup E_2$ with $\deg(E_1) = v_{r,p(k_2+1)+c-g_1,k_2+1} - v_{r,p(k_2+2)+\eta,k_2+2}$. We make the construction of step (a1) taking $W_{c+1} = \emptyset$, W_i as in step (a1) for $i = 1, \dots, c$ as in step (a1), except that $E_1 \cap H \subset W$. This is possible for the following reason. Since each component of E_1 is a line and $E_1 \subset \mathbb{P}^r$, $E_1 \cap H$ are $\deg(E_1)$ general points of H; we have $\deg(E_1) \le k_2$, W is a disjoint union of " general " rational curve and $\deg(W) \ge rk_2$ by (3) and our assumption on k_2. Use this new $Y' \cup W$ and the semicontinuity theorem for cohomology ([4, Theorem

III.12.8]).

(b) Now assume $k > k_2 + 3$ and that the lemma is true for the integer $k-1$. Fix $b_i \geq \delta_r(g_i)$ for all $i \neq 1$, $b_1 \geq \alpha$, $b_1 + \cdots + b_c = u_{r,p(k)+\eta,k} - v_{r,p(k)+\eta}$. By the definition of α we also have $b_1 \geq \delta_r(g_1)$.

Claim 1: There are integers $a_1 \geq \alpha$ and $a_i \geq \delta_r(g_i)$, $i = 2, \ldots, c$, such that $a_1 + \cdots + a_c = u_{r,p(k-1)+\eta,k-1} - v_{r,p(k-1)+\eta,k-1}$ and $a_i \leq b_i$ for all i.

Proof of Claim 1: We have $c - g_1 + k_2 u_{r,p(k_2)+c-g_1,k_2} + v_{r,p(k_2)+c-g_1,k_2} = \binom{t+k_2}{r}$ with $0 \leq v_{r,p(k_2)+c-g_1,k_2} \leq k_2 - 1$ and $u_{r,p(k_2)+c-g_1,k_2} - v_{r,p(k_2)+c-g_1,k_2} = \alpha + e_2 + \cdots + e_c = \alpha + \sum_{i=2}^{c} \delta_r(g_i) - (\sum_{i=2}^{c} g_i$. Since $k - 1 > k_2$, we get $(k-1)u_{r,p(k_2)+c-g_1,k_2} - v_{r,p(k_2)+c-g_1,k_2} \leq \binom{r+k-1}{r} - 2k - 2(k-1)(g_2 + \cdots + g_c)$. Hence $(k-1)(u_{r,p(k_2)+c-g_1,k_2} - v_{r,p(k_2)+c-g_1,k_2} + g_2 + \cdots + g_c$. We start with the c-ple $(\alpha, \delta_r(g_2), \ldots, \delta_r(g_c))$ and then increase in each step by one one of its entries, with the only restriction that after this step for all $i = 1, \ldots, c$ the i-th component is at most b_i. We make β steps, where $\beta := u_{r,p(k-1)+\eta,k-1} - v_{r,p(k-1)+\eta,k-1}$. After β steps we get an c-ple $(a_1, \ldots, a_c)$ with the properties of Claim 1.

By the inductive assumption there is a disjoint union $Y'' = Y_1'' \sqcup \cdots \sqcup Y_{c+1}''$ with $Y_i \in Z(r, a_i, g_i)$ if $i \leq c$, Y_{c+1}'' a disjoint union of $v_{r,p(k-1)+\eta,k-1}$ lines, $Y'' \cap Z = \emptyset$ and $h^i(\mathcal{I}_{Z \cup Y''}(k-1)) = 0$, $i = 0, 1$.

(b1) First assume $v_{r,p(k)+\eta,k} \geq v_{r,p(k-1)+\eta,k-1}$. We add $W_1 \sqcup \cdots \sqcup W_{c+1} \subset H$, with W_{c+1} a union of $v_{r,p(k)+\eta,k} v_{r,p(k-1)+\eta,k-1}$ disjoint lines, each W_i, $1 \leq i \leq c$, a smooth rational curve containing exactly one point of $Y_i'' \cap H$ (case $b_i > a_i$) or $W_i = \emptyset$ if $b_i = a_i$. As in step (a1) we get $h^i(\mathcal{I}_{Z \cup Y'' \cup W}(k)) = 0$, $i = 0, 1$.

(b2) Now assume $v_{r,p(k)+\eta,k} < v_{r,p(k-1)+\eta,k-1}$. Write $Y_{c+1}' = E_1 \sqcup E_2$ with $\deg(E_1) = v_{r,p(k)+c-g_1,k} - v_{r,p(k-1)+\eta,k-1}$ Now we add $W = W_1 \sqcup \cdots \sqcup W_c \subset H$, with each W_ia smooth rational curve containing exactly one point of $Y_i'' \cap H$ (case $b_i > a_i$) or $W_i = \emptyset$ if $b_i = a_i$; we also impose that each point of $E_1 \cap H$ is contained in W. As in step (a2) this is possible, because $\deg(W) \geq r\deg(E_1)$ (Remark 1).

Proof of Theorem 2: Fix an integer k_3 such that $k_3 \geq k_2$ and $u_{r,p(k_3-1)+\eta,k_3-1}$. We claim that we may take $k_0 := k_3 + 5$; we may also take $k_0 = k_2 + 5$ if we only look at integers $b_1, \ldots, b_c$ with the additional condition that $b_1 \geq \alpha$. Fix an integer $k \geq k_2 + 5$. Fix integers $b_i \geq \delta_r(g_i)$, $1 \leq i \leq c$, $b_1 \geq \alpha$, such that the union of Z and the disjoint union of c elements of $Z(r, b_i, g_i)$, $i = 1, \ldots, c$, has critical value k. Since $Z(r, d_1, g_1) \times \cdots \times Z(r, b_c, g_c)$ is irreducible, it is sufficient to find $A = A_1 \sqcup \cdots \sqcup A_c$ and $B = B_1 \sqcup \cdots \sqcup B_c$ with $A \cap Z = B \cap Z = \emptyset$, $A_i \in Z(r, b_i, g_i)$ for all i, $B_i \in Z(r, b_i, g_i)$ for all i, $h^1(\mathcal{I}_{Z\cup A}(k)) = 0$ and $h^0(\mathcal{I}_{Z\cup B}(k-1)) = 0$. Set $b := b_1 + \cdots + b_c$. By the definition of critical value and the inequalities $k - 1 \geq \kappa$ we have $h^0(\mathcal{O}_Z(k)) + kd + \eta \leq \binom{r+k}{r}$, $h^0(\mathcal{O}_Z(k-1) + (k-1)d + \eta > \binom{r+k-1}{r}$ and $u_{r,p(k-1)+\eta,k-1} < d \leq u_{r,p(k)+\eta,k}$.

(a) In this step we prove the existence of the curve A.

(a1) First assume $d \geq u_{r,p(k-1)+\eta,k-1} + rv_{r,p(k-1)+\eta,k-1}$. As in the proof of Claim 1 of the proof of Lemma 4 there are integers $a_1, \ldots, a_c$ such that $\delta_r(g_i) \leq a_i \leq b_i$ for all i, $a_1 \geq \alpha$ and $a_1 + \cdots + a_c = u_{r,p(k-1)+\eta,k-1} - v_{r,p(k-1)+\eta,k-1}$. By Lemma 4 applied to the integers $k-1$ and $a_1, \ldots, a_c$ there is $Y = Y_1 \sqcup \cdots \sqcup Y_{c+1}$ with $Y \cap Z = \emptyset$, $Y_i \in Z(r, a_i, g_i)$ for all $i \leq c$, Y_{c+1} a disjoint union of $v_{r,p(k-1)+\eta}$ lines and $h^i(\mathcal{I}_{Z\cup Y}(k-1)) = 0$, $i = 0, 1$. Let $W = W_1 \sqcup \cdots \sqcup W_c$ a general union of smooth rational curves of degree $b_i - a_i$ (the $\emptyset$ if $b_i = a_i$) with the only restriction that if $b_i > a_i$ ($i = 1, \ldots c$), then W_i contains exactly one point of Y_i and $Y_{c+1} \cap H \subset W$; we use Remark 1 or Lemma 1 to satisfy the last condition, because $\sum_i (b_i - a_i) \geq rv_{r,p(k-1)+\eta,k-1}$. The curve $Y \cup W$ has c connected components, $Y_1 \cup W_c, \ldots Y_c \cup W_c$, and $Y_i \cup W_i \in Z'(r, b_i, g_i)$ for all i. Hence by the semicontinuity theorem for cohomology it is sufficient to prove that $h^1(\mathcal{I}_{Z\cup Y\cup W}(k)) = 0$. Since $\mathrm{Res}_H(Z \cup Y \cup W) = Z \cup Y$ and $h^i(\mathcal{I}_{Z\cup Y}(k-1)) = 0$, $i = 0, 1$, it is sufficient to prove that $h^1(H, \mathcal{I}_{((Z\cup Y)\cap H)\cup W}(k)) = 0$. Since $d \leq u_{r,q(k)+\eta,k}$, (3) gives $h^0(\mathcal{O}_{((Z\cup Y)\cap H)\cup W}(k)) \leq \binom{r+k-1}{r-1} - v_{r,p(k)+\eta,k}$ and hence $h^0(\mathcal{O}_{((Z\cup Y)\cap H)\cap W}(k)) \leq \binom{r+k-1}{r-1}$. Since $H \cap Y$ is a general subset of H with cardinality $\deg(Y)$ (+++), it is sufficient to prove that $h^1(H, \mathcal{I}_{(Z\cap H)\cup W}(k)) = 0$. If $r = 4$, then $Z \cap H = \emptyset$ and hence we only need to prove that $h^1(H, \mathcal{I}_{W,H}(k)) = 0$ with $W \subset H$ a general union of at most c general rational curves with prescribed degrees. Since we use the definition of κ, i.e. we use the inductive assumption in $H\mathbb{P}^{r-1}$ for $H \cap Z$, c, and $g_i = 0$ for all i.

(a2) Assume $d < u_{r,p(k-1)+\eta,k-1} + rv_{r,p(k-1)+\eta,k-1}$. By (3) and the inequality $u_{r,p(k)+\eta,k} - u_{r,p(k-1)+\eta,k-1} \geq 2rk$ we have $h^0(\mathcal{O}_Z(k) + kd + \eta \leq \binom{r+k}{r} - k$ and hence $h^0(\mathcal{O}_Z(k) + kd + \eta \leq \binom{r+k}{r} - v_{r,p(k-1)+\eta,k-1}$. Since $d \geq u_{r,p(k-1)+\eta,k-1}$ there are integers $u_1, \ldots, u_c$ such that $b_i \geq u_i \geq c_i$ for all i, $u_1 \geq \alpha$, $u_i \geq \delta_r(g_i)$ for all i and $u_1 + \cdots + u_c = u_{r,p(k-1)+\eta,k-1}$. We modify step (a) of the proof of Lemma 4 (here from $k-2$ to $k-1$) in the following way. Since $k-2 \geq k_2 + 3$, there are integers $w_1, \ldots, w_c$ such that $u_1 \geq w_1 \geq \alpha$, and $u_i \geq w_i \geq \delta_r(g_i)$ for all $i \geq 2$. Take a solution $Y' = Y'_1 \sqcup \cdots \sqcup Y'_{c+1}$ with Y'_{c+1} a disjoint union of $v_{r,p(k-2)+\eta,k-2}$. Let $W' = W'_1 \sqcup \cdots \sqcup W'_c \subset H$ be a general of c rational curves of degree $u_i - w_i$ (or the $\emptyset$ if $u_i = w_i$), with W'_i containing exactly one point of Y''_i if $W_i \neq \emptyset$, so that $Y'' \cup W'$ has c connected components, $Y''_1 \cup W'_1, \ldots, Y''_c \cup W'_c$ with each $Y''_i \cup W'_i \in Z'(r, u_i, g_i)$. Using [1] (if $r = 4$) or the inductive assumption (if $r > 4$) we get $h^1(\mathcal{I}_{Z\cup Y''\cup W'}(k-1)) = 0$. Let

$T = T_1 \sqcup \cdots \sqcup T_c$ a general smoothing of $Y'' \cup W'$. By the semicontinuity theorem we have $h^1(\mathcal{I}_{Z\cup T}(k-1)) = 0$. Let $F = F_1 \sqcup \cdots \sqcup F_c \subset H$ be a general union of smooth rational curves of degree $b_1 - u_1, \dots, b_c - u_c$ (the $\emptyset$ if $b_i = u_i$) with the only restriction that if $b_i - a_i > 0$, then F_i contains exactly one point of T_i. The curve $T \cup F$ has c connected components, $T_1 \cup F_1, \dots, T_c \cup F_c$ with $F_i \in Z'(r, b_i, g_i)$ for all i.

(b) In this step we prove the existence of the curve B. We modify step (a2) in the following way. Since $b_1 \geq \alpha$, $b_i \geq \delta_r(g_i)$ for all i and $b_1 + \cdots + b_c > u_{r,p(k-1)+\eta,k-1} \geq u_{r,p(k-2)+\eta,k-2}$ as in the proof of Claim 1 in the proof of Lemma 4 we get the existence of integers z_i, $1 \leq i \leq c$, such that $z_1 \geq \alpha$, $b_i \geq z_i \geq \delta_r(g_i)$ for all i and $z_1 + \cdots + z_c = u_{r,p(k-2)+\eta,k-2} - v_{r,p(k-2)+\eta,k-2}$. By Lemma 4 there is $Y'' = Y''_1 \sqcup \cdots \sqcup Y''_{c+1}$ with $Y'' \cap Z = \emptyset$, Y''_{c+1} a disjoint union of $v_{r,p(k-2)+\eta,k-2}$ lines, $Y''_i \in Z(r, z_i, g_i)$ for all $i \leq c$ and $h^i(\mathcal{I}_{Z\cup Y''}(k-2)) = 0$, $i = 0, 1$. Let $W'' = W''_1 \sqcup \cdots \sqcup W''_c \subset H$ be a general of c rational curves of degree $b_i - z_i$ (or the $\emptyset$ if $b_i = z_i$), with W''_i containing exactly one point of Y''_i if $W_i \neq \emptyset$, so that $Y'' \cup W''$ has c connected components, $Y''_1 \cup W''_1, \dots, Y''_c \cup W'_c$ with each $Y''_i \cup W''_i \in Z'(r, b_i, g_i)$, and $Y''_{c+1} \cap H \subset W''$ (see step (a1) for the latter condition) . By the Castelnuovo's inequalities it is sufficient to prove that $h^0(H, \mathcal{I}_{((Z\cup Y'')\cap H)\cup W''}(k-1)) = 0$. Using [1] (if $r = 4$) or the inductive assumption (if $r > 4$) we get that either $h^1(H, \mathcal{I}_{(Z\cap H)\cup W''}(k-1)) = 0$ or $h^0(H, \mathcal{I}_{(Z\cap H)\cup W''}(k-1)) = 0$. If $h^0(\mathcal{I}_{(Z\cup H)\cup W'}(k-1)) = 0$, then $h^0(H, \mathcal{I}_{((Z\cup Y'')\cap H)\cup W''}(k-1)) = 0$. Hence we may assume $h^1(H, \mathcal{I}_{(Z\cap H)\cup W''}(k-1)) = 0$. Hence $h^0(H, \mathcal{I}_{(Z\cap H)\cup W''}(k-1)) = \binom{r+k-2}{r-1} - p(k-1) + p(k-2) - h^0(\mathcal{O}_{W''}(k-1))$. Call β the last integer. By (3) for the integer $k-1$ we have $\beta = -(k-1)(d - u_{r,p(k-1)+\eta,k-1}) + v_{r,p(k-1),k-1} + \sharp(Y'' \cap H) - \sharp(Y'' \cap W'')$. Since $Y'' \cap H \setminus Y'' \cap W''$ is general in H, we get $h^0(H, \mathcal{I}_{((Z\cup Y'')\cap H)\cup W''}(k-1)) = 0$.

(c) Now assume $k \geq k_3 + 5$. From steps (a) and (b) we conclude if we assume that $b_1, \dots, b_c$ satisfies $b_i \geq \delta_r(g_i)$ and $b_1 \geq \alpha$. Take arbitrary integers $w_i \geq \delta_r(g_i)$ such that $w_1 + \cdots + w_c \geq u_{r,p(k-1)+\eta,k-1}$. By the definition of the integer k_3 there is at list one integer $j \in \{1, \dots, c\}$ such that $b_j \geq \alpha$. We may rewrite the proof with the same k_1 and k_2 (we used bound on $g_1 + \cdots + g_c$, not on g_1).

Proof that Theorem 2 for r, Z, c, g_i implies Theorem 1 for the same data Fix integers $r \geq 4$, $c > 0$, $g_i \geq 0$ and an admissible polynomial $p(t)$ (i.e. the Hilbert polynomial of some closed subscheme of some projective space) such that $\deg(p(t)) \leq r - 4$. Let $Z \subset \mathbb{P}^r$ be a closed subscheme with $p(t)$ as its Hilbert polynomial. By Gotzmann's theorem there in an integer t_0 such that $h^i(\mathcal{O}_Z(t+1-i)) = 0$ for all $t \geq t_0$ and all $i > 0$ and t_0 depends only on the polynomial $p(t)$. We take $t_0 = 0$ if $Z = \emptyset$. Fix integers $b_i \geq \delta_r(g_i)$, $1 \leq i \leq c$, and set $d := b_1 + \cdots + b_c$. Set $\eta := c - g_1 - \cdots + g_c$. We assume that d is very large, e.g. $d > c\binom{r+t_0}{r} + g_1 + \cdots + g_c$. With these assumptions it is easy to check that a general disjoint union $Y = Y_1 \sqcup \cdots \sqcup Y_c$, $Y_i \in Z(r, b_i, g_i)$ satisfies $h^0(\mathcal{I}_Y(t_0+1)) = 0$. Hence to study the function $t \mapsto h^0(\mathcal{I}_{Z\cup Y}(t))$ it is sufficient to study it when $t > t_0$. We say that $(r, p(t), c, g_1, \ldots, g_c, b_1, \ldots, b_c)$ has *critical value* k if k is the minimal integer $> t_0$ such that $kd + \eta + p(k) \leq \binom{r+k}{r}$, i.e. $d \leq u_{r,p(k)+\eta,k}$. Since $u_{r,p(k)+\eta,k} > u_{r,p(k-1),k-1}$ (use (3)), we get that $(r, p(t), c, g_1, \ldots, g_c, b_1, \ldots, b_c)$ has critical value k if and only if $u_{r,p(k-1),k-1} < d \leq u_{r,p(k)+\eta,k}$. Assume that $(r, p(t), c, g_1, \ldots, g_c, b_1, \ldots, b_c)$ has critical value k. Fix $Y = Y_1 \sqcup \cdots \sqcup Y_c$, $Y_i \in Z(r, b_i, g_i)$, with $Z \cap Y = \emptyset$. If $Z \cup Y$ has maximal rank, then $h^0(\mathcal{I}_{Z\cup Y}(k-1)) = 0$ and $h^1(\mathcal{I}_Z(k)) = 0$. Now assume that $h^0(\mathcal{I}_{Z\cup Y}(k-1)) = 0$ and $h^1(\mathcal{I}_Z(k)) = 0$. Obviously $h^0(\mathcal{I}_{Z\cup Y}(t)) = 0$ for all $t \leq k-1$. Our assumptions on the integers b_i imply $h^1(\mathcal{O}_Y(1)) = 0$. Since $k \geq \max\{2, t_0\}$, Castelnuovo-Mumford's lemma implies $h^1(\mathcal{I}_{Z\cup Y}(t)) = 0$ for all $t > k$. Take k_3 as in the proof of Theorem 2 and take $\Delta := u_{r,p(k_3+1),k_3+1}$. Take $b_i \geq \delta_r(g_i)$, $i = 1, \ldots, c$, such that $d := b_1 + \cdots + b_c \geq \Delta$. Obviously $(r, p(t), c, g_1, \ldots, g_c, b_1, \ldots, b_c)$ has critical value $k > k_0$. Fix a general $Y = Y_1 \sqcup \cdots \sqcup Y_c$ be a general union with $Y_i \in Z(r, b_i, g_i)$. In particular $Z \cap Y = \emptyset$. Theorem 2 for the data $p(t), c, g_i$ gives that $h^1(\mathcal{I}_{Z\cup Y}(k)) = 0$. Since $k-1 \geq k_3$, it gives $h^0(\mathcal{I}_{Z\cup Y}(k-1)) = 0$. Hence $Z \cup Y$ has maximal maximal rank by the Castelnuovo-Mumford's lemma).

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