Gen. Math. Notes, Vol. 36, No. 1, September 2016, pp.22-33
ISSN 2219-7184;
www.i-csrs.org
Available free online at http://www.geman.in

(p,q)-Type of the Generalized Laplace-Stieltjes Transform

J.H. Ning[1], J.J. Chen[2] and W.P. Huang[3]

[1]College of Mathematics and Information Science
Jiangxi Normal University, Nanchang, Jiangxi, P.R. China
E-mail: ningjh2016@163.com
[2]Zhongshan Shiqi Central Primary School
Zhongshan, Guangzhou, P.R. China
E-mail: chenjj20070101@163.com
[3]College of Information System and Management
National University of Defense Technology
Changsha, Hunan, P.R. China
E-mail: jxnuhwp1@163.com

(Received: 6-8-16 / Accepted: 14-9-16)

Abstract

In this paper, we study relations among the maximum modulus, the maximum term, and the local maximum of entire functions represented by the generalized Laplace-Stieltjes transform, some theorems on estimating the (p,q)-type are obtained.

Keywords: *Generalized Laplace-Stieltjes transforms, (p,q)-type, entire functions, growth.*

1 Introduction

Let $g(z)$ be a nonconstant entire function and $M(r,g) = \max_{|z|=r} |g(z)|$, to estimate the growth of $g(z)$ precisely, in [4] the concept of order ρ is introduced by

$$\rho = \limsup_{r\to\infty} \frac{\ln\ln M(r,g)}{\ln r}.$$

If $0 < \rho < \infty$, the type τ of $g(z)$ is defined by

$$\tau = \limsup_{r\to\infty} \frac{\ln M(r,g)}{r^{\rho}}.$$

To compare the growth of entire functions having the same order, in [1] authors introduce the concept of (p,q)-order. To compare the growth of entire functions having the same (p,q)-order, in [2] authors introduce the concept of (p,q)-type.

The Laplace-Stieltjes transform

$$\int_0^{\infty} e^{-sy} d\alpha(y)(s = \sigma + it),$$

where $\alpha(y)$ is a function of bounded variation on every interval $[0, X](0 < X < \infty)$, has been investigated on many aspects[6], such as functional analysis, certain areas of theoretical and applied probability. In [3,5,7,8], the growth of analytic functions which represented by Laplace-Stieltjes transforms is mainly studied. In these studies, when $\{\lambda_n : 0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \uparrow +\infty\}$ is a real number sequences satisfying some density conditions, authors generally define its maximum modulus $M_u(\sigma, f)$, its local maximum A_n^* and its maximum term $\mu(\sigma, f)$ which is similar to the Dirichlet series as follows

$$M_u(\sigma, f) = \sup_{0<x<\infty, t\in R} \{\Big|\int_0^{x} e^{-(\sigma+it)y} d\alpha(y)\Big|\}, \sigma \in R;$$

$$A_n^* = \sup_{t\in R, \lambda_n < x \le \lambda_{n+1}} \{\Big|\int_{\lambda_n}^{x} e^{-ity} d\alpha(y)\Big|\}, n \in N; \mu(\sigma, f) = \max_{n\in N}\{A_n^* e^{-\lambda_n \sigma}\}, \sigma \in R,$$

and get some formulas about the relation among them.

If the Laplace-Stieltjes transform defines on a Jordan curve in the right half plane and represents an entire function, the (p,q)-type of this function with (p,q)-order is studied in the present paper. It seems this topic has never been treated before.

For convenience, some used symbols and definitions are introduced first. Let G_τ denote the closed angular domain $\{z : |argz| \le \tau < \frac{\pi}{2}(\tau \ge 0)\}$. Assume that L is a Jordan curve in G_τ starting from the origin and extending to infinity. For every $z \in L$, let L_z denote the part of L from the origin to z containing two endpoints origin and z, L_z is a rectifiable curve, $\alpha(z)$ be a complex valued function of a complex variable z of bounded variation in L_z.

The generalized Laplace-Stieltjes transform(GLST)

$$F(s) = \int_L e^{-sz} d\alpha(z), \ s = \sigma + it \tag{1.1}$$

and there exists a complex number sequence $\{\lambda_n\}_{n=1}^{\infty} \subset L$ satisfying the following conditions C:

(i) $Re\lambda_n = \omega_n, 0 = \omega_0 < \omega_1 < \omega_2 < \cdots < \omega_n < \cdots, \lim_{n\to\infty} \omega_n = +\infty$;
(ii) $\overline{\lim_{n\to\infty}} \frac{\ln n}{\omega_n} = D < +\infty$;
(iii) Let $L_{\lambda_{n+1}} - L_{\lambda_n}$ denote the part of $L_{\lambda_{n+1}}$ deleting L_{λ_n}. There exists a positive constant A_0 such that when n is sufficiently large and $\lambda \in L_{\lambda_{n+1}} - L_{\lambda_n}$, then $|I_\lambda - I_{\lambda_n}| \le A_0$, where I_λ is the length of L_λ.

To study the (p,q)-type of $F(s)$ represented by (1.1), we need following definitions:

Definition 1.1 *The definition of the maximum modulus $M(r,F)$, the maximum term $m(r,F)$ and the local maximum A_n of $F(s)$ represented by (1.1) is given as follows*

$$M(r,F) = \sup_{\lambda\in L, |s|=r} \left| \int_{L_\lambda} e^{-sz} d\alpha(z) \right|, \quad m(r,F) = \max_{1\le n<+\infty} A_n e^{hr\omega_n},$$

where $h = 1 + tan\tau (0 \le \tau < \frac{\pi}{2})$ and

$$A_n = \sup_{\lambda\in L_{\lambda_{n+1}} - L_{\lambda_n}, |s|=r\neq 0} \left| \int_{L_\lambda - L_{\lambda_n}} e^{(-s-hr)z} d\alpha(z) \right|.$$

Similar to [1, 2], we define the (p,q)-order and (p,q)-type of $F(s)$ represented by (1.1) as follows

Definition 1.2 *For the entire function $F(s)$ represented by (1.1), let*

$$\rho(p,q) = \overline{\lim_{r\to\infty}} \frac{\ln^{[p+1]} M(r,F)}{\ln^{[q]} r},$$

where p, q $(p \ge q \ge 0)$ are integers, $M(r,F)$ is defined by the definition 1.1, $\exp^{[0]} x = \ln^{[0]} x = x, \exp^{[n+1]} x = \exp\{\exp^{[n]} x\}, \ln^{[n+1]} x = \ln\{\ln^{[n]} x\}, n = 1, 2, \cdots$.

Definition 1.3 *If $\rho(p-1, q-1) = 0$ or $+\infty$ $(p \ge q \ge 1)$, and $l < \rho(p,q) < +\infty$(when p=q, l=1; when p>q, l=0), then the entire function $F(s)$ is said to have index-pair (p,q). If $F(s)$ has the index-pair (p,q), then $\rho = \rho(p,q)$ is called its (p,q)-order.*

Definition 1.4 *The entire function $F(s)$ of (p,q)-order $\rho(l < \rho < +\infty, p = q, l = 1; p > q, l = 0)$ represented by (1.1) is said to be of (p,q)-type T if*

$$T = \overline{\lim_{r\to\infty}} \frac{\ln^{[p]} M(r,F)}{(\ln^{[q-1]} r)^\rho} (0 \le T \le +\infty),$$

where p, q are integers and $p \ge 1$, $q \ge 0$, $p \ge q$.

Remark 1.5 *In the definition 1.2 and 1.4, if let $p = q = 1$, then we get the ordinary order ρ and type τ respectively.*

Remark 1.6 *Throughout the paper, we denote A by a positive constant, not necessarily the same at each occurrence.*

2 Main Results

In this section, we will introduce our main conclusions.

Theorem 2.1 *If the sequence* $\{\lambda_n\}_{n=1}^{\infty} \subset L$ *satisfies conditions* C*, the entire function* $F(s)$ *represented by* (1.1) *has (p,q)-order* $\rho(l < \rho < +\infty, p = q, l = 1; p > q, l = 0)$*, then*

$$\overline{\lim_{r\to+\infty}}\frac{\ln^{[p]} M(r,F)}{(\ln^{[q-1]} r)^{\rho}} = \overline{\lim_{r\to\infty}}\frac{\ln^{[p]} m(r,F)}{(\ln^{[q-1]} r)^{\rho}},$$

where p*,* q *are integers and* $p \geq 1,\ q \geq 0,\ p \geq q$*.*

Proof: Obviously, for every $\lambda \in L$, there exists an n such that $\omega_n \leq Re\lambda < \omega_{n+1}$, then we have

$$\int_{L_\lambda} e^{-sz} d\alpha(z) = \sum_{k=1}^{n-1}\int_{L_{\lambda_{k+1}}-L_{\lambda_k}} e^{-sz} d\alpha(z) + \int_{L_\lambda - L_{\lambda_n}} e^{-sz} d\alpha(z).$$

Let $I_k(\lambda,s) = \int_{L_\lambda - L_{\lambda_k}} e^{(-s-hr)z} d\alpha(z) (\omega_k \leq Re\lambda < \omega_{k+1})$, it holds $|I_k(\lambda,s)| \leq A_k \leq m(r,F)e^{-h\omega_k r}$. By (ii), there exists a constant $b > 0$ such that $\omega_n > b\ln n (n = 1, 2, \cdots)$. By (iii), for any $\epsilon > 0$, there exists an $N \in N^+$, and $n > N$, such that $\omega_{n+1} < (1+\epsilon)\omega_n$. If $r \geq \frac{2}{\epsilon b}$, we get

$$\begin{aligned}
&\left|\int_{L_\lambda} e^{-sz} d\alpha(z)\right| \\
&= \left|\sum_{k=1}^{n-1}\int_{L_{\lambda_{k+1}}-L_{\lambda_k}} e^{hrz}e^{(-s-hr)z} d\alpha(z) + \int_{L_\lambda - L_{\lambda_n}} e^{hrz}e^{(-s-hr)z} d\alpha(z)\right| \\
&= \left|\sum_{k=1}^{n-1}\int_{L_{\lambda_{k+1}}-L_{\lambda_k}} e^{hrz} dI_k(z;s) + \int_{L_\lambda - L_{\lambda_n}} e^{hrz} dI_n(z;s)\right| \\
&\leq \left|\sum_{k=1}^{n-1}\left[e^{hr\lambda_{k+1}}I_k(\lambda_{k+1},s) - e^{hr\lambda_k}I_k(\lambda_k,s) - hr\int_{L_{\lambda_{k+1}}-L_{\lambda_k}} e^{hrz}I_k(z;s)dz\right]\right| \\
&+ \left|e^{hr\lambda}I_n(\lambda,s) - e^{hr\lambda_n}I_n(\lambda_n,s) - hr\int_{L_\lambda - L_{\lambda_n}} e^{hrz}I_n(z;s)dz\right| \\
&\leq \sum_{k=1}^{n-1}\left[e^{hr\omega_{k+1}}A_k + e^{hr\omega_k}A_k + hrA_k\int_{L_{\lambda_{k+1}}-L_{\lambda_k}} |e^{hrz}|\cdot|dz|\right] + e^{hrRe\lambda}A_n \\
&+ e^{hr\omega_n}A_n + hrA_n\int_{L_\lambda - L_{\lambda_n}} |e^{hrz}|\cdot|dz| \\
&\leq \sum_{k=1}^{n-1}\left[2e^{hr\omega_{k+1}}A_k + hrA_k e^{hrRe\lambda'}(I_{\lambda_{k+1}} - I_{\lambda_k})\right] \\
&+ 2e^{hrRe\lambda}A_n + hrA_n e^{hrRe\lambda''}(I_\lambda - I_{\lambda_n}),
\end{aligned}$$

where $\lambda' \in L_{\lambda_{k+1}} - L_{\lambda_k}, \lambda'' \in L_\lambda - L_{\lambda_n}$, it is easy to prove that, when $r \geq 1$, then there exist a positive constant A, such that

$$\begin{aligned} M(r,F) &\leq Ar \sum_{n=1}^{\infty} A_n e^{hr\omega_{n+1}} \\ &\leq Ar \left[\sum_{n=1}^{N} A_n e^{hr\omega_{n+1}} + \sum_{n=N+1}^{\infty} m((1+2\epsilon)r, F) e^{-(1+2\epsilon)hr\omega_n} e^{hr\omega_{n+1}} \right], \end{aligned}$$

where

$$\sum_{n=1}^{N} A_n e^{hr\omega_{n+1}} = \sum_{n=1}^{N} A_n e^{hr\omega_n} e^{hr(\omega_{n+1}-\omega_n)} \leq m(r,F) \sum_{n=1}^{N} e^{hr(\omega_{n+1}-\omega_n)},$$

If $C_N = \max_{1 \leq n \leq N} (\omega_{n+1} - \omega_n)$, then

$$\sum_{n=1}^{N} A_n e^{hr\omega_{n+1}} \leq m(r,F) \sum_{n=1}^{N} e^{hr(\omega_{n+1}-\omega_n)} \leq m(r,F) N e^{hrC_N}.$$

Since

$$\begin{aligned} &\sum_{n=N+1}^{\infty} m((1+2\epsilon)r, F) e^{-(1+2\epsilon)hr\omega_n} e^{hr\omega_{n+1}} \\ &= \sum_{n=N+1}^{\infty} m((1+2\epsilon)r, F) e^{(\omega_{n+1}-(1+2\epsilon)\omega_n)hr} \\ &\leq m((1+2\epsilon)r, F) \sum_{n=N+1}^{\infty} e^{-\epsilon hr\omega_n} \leq m((1+2\epsilon)r, F) \sum_{n=N+1}^{\infty} \frac{1}{n^{2h}} (h \geq 1), \end{aligned}$$

we have $M(r,F) \leq Ar[m(r,F)Ne^{hrC_N} + Am((1+2\epsilon)r, F)]$, then

$$M(r,F) \leq Ae^{Ar} m((1+2\epsilon)r, F). \tag{2.1}$$

On the other hand, let $J_n(\lambda, s) = \int_{L_\lambda - L_{\lambda_n}} e^{-sz} d\alpha(z)$, then $|J_n(\lambda, s)| \leq 2M(r,F)$. For any n, when $\lambda \in L_{\lambda_{n+1}} - L_{\lambda_n}$, we have

$$\begin{aligned} &\left| \int_{L_\lambda - L_{\lambda_n}} e^{(-s-hr)z} d\alpha(z) \right| = \left| \int_{L_\lambda - L_{\lambda_n}} e^{-hrz} dJ_n(z,s) \right| \\ &= \left| e^{-hr\lambda} J_n(\lambda, s) - e^{-hr\lambda_n} J_n(\lambda_n, s) + hr \int_{L_\lambda - L_{\lambda_n}} e^{-hrz} J_n(z,s) dz \right| \\ &\leq e^{-rh\omega_n} 4M(r,F) + rAM(r,F) \int_{L_\lambda - L_{\lambda_n}} |e^{-hrz}| \cdot |dz|. \end{aligned}$$

For all sufficiently large values of n, then $A_n \leq ArM(r,F)e^{-hr\omega_n}$, we have

$$m(r,F) \leq ArM(r,F). \tag{2.2}$$

Since, by (2.1), for r sufficiently large, we get

$$\ln M(r,F) \le \ln A + Ar + \ln m((1+2\epsilon)r,F),$$

$$\ln^{[2]} M(r,F) \le \ln^{[2]} A + \ln(Ar) + \ln^{[2]} m((1+2\epsilon)r,F),$$

$$\cdots\cdots$$

$$\ln^{[p]} M(r,F) \le \ln^{[p]} A + \ln^{[p-1]} A + \ln^{[p-1]} r + \ln^{[p]} m((1+2\epsilon)r,F),$$

Proceeding to limits we get

$$\begin{aligned} \overline{\lim_{r\to+\infty}} \frac{\ln^{[p]} M(r,F)}{(\ln^{[q-1]} r)^{\rho}} &\le \overline{\lim_{r\to+\infty}} \frac{\ln^{[p-1]} r}{(\ln^{[q-1]} r)^{\rho}} \\ &+ \overline{\lim_{r\to+\infty}} \frac{\ln^{[p]} m((1+2\epsilon)r,F)}{(\ln^{[q-1]}((1+2\epsilon)r))^{\rho}} \frac{(\ln^{[q-1]}((1+2\epsilon)r))^{\rho}}{(\ln^{[q-1]} r)^{\rho}} \\ &= \overline{\lim_{r\to+\infty}} \frac{\ln^{[p]} m(r,F)}{(\ln^{[q-1]} r)^{\rho}}. \end{aligned} \tag{2.3}$$

By (2.2), we have $\ln^{[p]} m(r,F) \le \ln^{[p]} A + \ln^{[p]} r + \ln^{[p]} M(r,F)$, thus

$$\begin{aligned} \overline{\lim_{r\to+\infty}} \frac{\ln^{[p]} m(r,F)}{(\ln^{[q-1]} r)^{\rho}} &\le \overline{\lim_{r\to+\infty}} \frac{\ln^{[p]} r}{(\ln^{[q-1]} r)^{\rho}} + \overline{\lim_{r\to+\infty}} \frac{\ln^{[p]} M(r,F)}{(\ln^{[q-1]} r)^{\rho}}, \\ &= \overline{\lim_{r\to+\infty}} \frac{\ln^{[p]} M(r,F)}{(\ln^{[q-1]} r)^{\rho}}. \end{aligned} \tag{2.4}$$

The inequalities (2.3) and (2.4) together give

$$\overline{\lim_{r\to+\infty}} \frac{\ln^{[p]} M(r,F)}{(\ln^{[q-1]} r)^{\rho}} = \overline{\lim_{r\to\infty}} \frac{\ln^{[p]} m(r,F)}{(\ln^{[q-1]} r)^{\rho}}.$$

Remark 2.2 *From the proof of theorem 2.1 we can get*

$$\left| \int_{L_\lambda} e^{-sz} d\alpha(z) \right| \le Ar \sum_{n=1}^{\infty} A_n e^{hr\omega_{n+1}},$$

if let $\limsup_{n\to\infty} \frac{\ln A_n}{\omega_n} = -\infty$, *for any compact set* $|s| \le r \le r_0 (r_0 > 0)$ *of the complex plane we can get the uniform convergence of the series in the right of the above inequality, so the GLST can represent an entire function in the complex plane.*

Theorem 2.3 *If the sequence $\{\lambda_n\}_{n=1}^{\infty} \subset L$ satisfies conditions C, the entire function $F(s)$ with (p,q)-order $\rho(l < \rho < +\infty, p = q, l = 1; p > q, l = 0)$ represented by (1.1) has (p,q)-type T if and only if $T{=}MV$, where $V = \overline{\lim\limits_{n\to\infty}} \frac{\ln^{[p-1]}(h\omega_n)}{(\ln^{[q]} A_n^{-\frac{1}{(h\omega_n)}})^{\rho-a}}$, we have $a = 1$ if $(p,q) = (1,1)$ and $a = 0$ if $(p,q) \neq (1,1)$. And $M = \frac{1}{e\rho}$ if $p = 1, q = 0$; $M = \frac{(\rho-1)^{\rho-1}}{(\rho)^{\rho}}$ if $p = q = 1$ and $M = 1$ if $p \geq q \geq 2$.*

Proof: (a) If $p = 1, q = 0$, and $T < +\infty$, by the definition 1.4 and theorem 2.1 we have

$$T = \overline{\lim_{r\to+\infty}} \frac{\ln M(r,F)}{(\ln^{-1} r)^{\rho}} = \overline{\lim_{r\to+\infty}} \frac{\ln m(r,F)}{e^{r\rho}},$$

for any given $\epsilon > 0$, and for sufficiently large values of r and the definition of $m(r,F)$ we have $\ln m(r,F) < (T+\epsilon)e^{r\rho}$, then $\ln A_n < (T+\epsilon)e^{r\rho} - h\omega_n r$, let $r = \ln(\frac{h\omega_n}{\rho(T+\epsilon)})^{\frac{1}{\rho}}$, then

$$\ln A_n < \frac{h\omega_n}{\rho} - \frac{h\omega_n}{\rho} \ln \frac{h\omega_n}{\rho(T+\epsilon)},$$

and $\frac{h\omega_n}{(A_n^{-\frac{1}{h\omega_n}})^{\rho}} < e\rho(T+\epsilon)$. For any ϵ, we have

$$\overline{\lim_{n\to\infty}} \frac{h\omega_n}{(A_n^{-\frac{1}{h\omega_n}})^{\rho}} \leq e\rho T,$$

hence $MV \leq T$.

Let $V < \frac{T}{M}$, then there exists a $\delta \in (0,T)$, such that $V < \frac{T-\delta}{M}$, we have

$$\overline{\lim_{n\to\infty}} \frac{h\omega_n}{(A_n^{-\frac{1}{h\omega_n}})^{\rho}} < \frac{T-\delta}{M},$$

therefore

$$\frac{h\omega_n}{(A_n^{-\frac{1}{h\omega_n}})^{\rho}} < e\rho(T-\delta),$$

and $\ln A_n e^{h\omega_n r} < -\frac{h\omega_n}{\rho} \ln \frac{h\omega_n}{e\rho(T-\delta)} + h\omega_n r$. Let $\varphi(x) = -\frac{x}{b}\ln x + cx$, where b and c are positive constants and $x > 0$, when $x = e^{bc-1}$, we get maximum value $\frac{1}{b}e^{bc-1}$. Therefore

$$\frac{\ln m(r,F)}{e^{r\rho}} < \frac{-(T-\delta)e^{r\rho}(r\rho-1) + (T-\delta)e^{r\rho}r\rho}{e^{r\rho}},$$

then

$$\overline{\lim_{r\to+\infty}} \frac{\ln m(r,F)}{e^{r\rho}} \leq (T-\delta) < T.$$

It is a contradict of theorem 2.1, so we have $T = MV$.

(b) When $p = q = 1$, then let $T < +\infty$, by the definition of T, for any given $\epsilon > 0$, and by theorem 2.1 for sufficiently large r we have $\ln m(r,F) < (T+\epsilon)r^{\rho}$, hence $\ln A_n \le \ln m(r,F) - h\omega_n r \le (T+\epsilon)r^{\rho} - h\omega_n r$. let $r = (\frac{h\omega_n}{\rho(T+\epsilon)})^{\frac{1}{\rho-1}}$, we get

$$-\frac{1}{h\omega_n}\ln A_n \ge -(T+\epsilon)\frac{(h\omega_n)^{\frac{1}{\rho-1}}}{[\rho(T+\epsilon)]^{\frac{\rho}{\rho-1}}} + \frac{(h\omega_n)^{\frac{1}{\rho-1}}}{[\rho(T+\epsilon)]^{\frac{1}{\rho-1}}},$$

then

$$(\ln A_n^{-\frac{1}{h\omega_n}})^{\rho-1} \ge \frac{(\rho-1)^{\rho-1}}{(\rho)^{\rho}}\frac{h\omega_n}{T+\epsilon} = \frac{Mh\omega_n}{T+\epsilon},$$

we obtain $T \ge MV$.

On the other hand, $T \le MV$. In fact, by the definition of V, for sufficiently large $N_1(N_1 > N)$ we have $A_n < \exp[\frac{-(h\omega_n)^{\frac{\rho}{\rho-1}}}{(V+\epsilon)^{\frac{1}{\rho-1}}}]$ and by the proof of theorem 2.1, we get

$$\begin{aligned}\left|\int_{L_\lambda} e^{-sz}d\alpha(z)\right| &\le Ar\sum_{k=1}^{\infty}A_k e^{hr\omega_{k+1}} \le Ar\sum_{k=1}^{\infty}A_k e^{hr(\omega_k+A)}\\ &\le Ae^{Ar}\sum_{k=1}^{N_1}A_k e^{hr\omega_k} + Ae^{Ar}\sum_{k=N_1+1}^{\infty}A_k e^{hr\omega_k}\\ &= J_1 + J_2,\end{aligned}$$

where

$$J_1 = Ae^{Ar}\sum_{k=1}^{N_1}A_k e^{hr\omega_k} \le AN_1 e^{Ar}e^{h\omega_{N_1}r}\max_{1\le k\le N_1}A_k,$$

$$\begin{aligned}J_2 &= Ae^{Ar}\sum_{k=N_1+1}^{\infty}e^{h\omega_k r}A_k\\ &\le Ae^{Ar}\sum_{k=N_1+1}^{\infty}\exp\{h\omega_k r - \frac{(h\omega_k)^{\frac{\rho}{\rho-1}}}{(V+\epsilon)^{\frac{1}{\rho-1}}} + \frac{2}{b}h\omega_k - \frac{2}{b}h\omega_k\}\end{aligned}$$

let $\varphi(x) = -x^{\frac{\rho}{\rho-1}} + cx$, where ρ and c are positive constants and $x > 0$, when $x = c^{\rho-1}(\frac{\rho-1}{\rho})^{\rho-1}$, we can get the maximum value $c^{\rho}\frac{(\rho-1)^{\rho-1}}{\rho^{\rho}}$, so we have

$$J_2 \le Ae^{Ar}\exp\{\frac{(\rho-1)^{\rho-1}}{\rho^{\rho}}(r+\frac{2}{b})^{\rho}(V+\epsilon)\}\sum_{k=N+1}^{\infty}e^{-\frac{2}{b}h\omega_k}$$

$$\le \exp\{\frac{(\rho-1)^{\rho-1}}{\rho^{\rho}}(r+\frac{2}{b})^{\rho}(V+\epsilon)\}e^{Ar}\sum_{k=N+1}^{\infty}\frac{1}{n^{2h}},$$

then

$$\ln M(r,F) \le \ln[AN_1 e^{Ar} e^{h\omega_{N_1} r} \max_{1\le k\le N_1} A_k$$
$$+\exp\{\frac{(\rho-1)^{\rho-1}}{\rho^\rho}(r+\frac{2}{b})^\rho(V+\epsilon)\}e^{Ar}\sum_{k=N+1}^{\infty}\frac{1}{n^{2h}}]$$
$$\le \ln(AN_1 e^{Ar} e^{h\omega_{N_1} r} \max_{1\le k\le N_1} A_k) + \frac{(\rho-1)^{\rho-1}}{\rho^\rho}(r+\frac{2}{b})^\rho(V+\epsilon)+Ar,$$

$$\frac{\ln M(r,F)}{r^\rho} \le \frac{(A+h\omega_{N_1})r+A}{r^\rho} + \frac{\frac{(\rho-1)^{\rho-1}}{\rho^\rho}(r+\frac{2}{b})^\rho(V+\epsilon)}{r^\rho} + \frac{Ar}{r^\rho}.$$

Hence $\overline{\lim\limits_{r\to+\infty}} \frac{\ln M(r,F)}{r^\rho} \le \frac{(\rho-1)^{\rho-1}}{\rho^\rho}V = MV$. We have $T \le MV$.

The inequalities $T \ge MV$ and $T \le MV$ together imply $T = MV$.

(c) If $p \ge q \ge 2$ and $T < +\infty$, by the definition of T, for any given $\epsilon > 0$, and for sufficiently large r we have

$$\ln m(r,F) < \exp^{[p-1]}\{(T+\epsilon)(\ln^{[q-1]} r)^\rho\},$$

then $\ln A_n \le \exp^{[p-1]}\{(T+\epsilon)(\ln^{[q-1]} r)^\rho\} - h\omega_n r$.

Let $r = \exp^{[q-1]}\{(\frac{1}{T+\epsilon}\ln^{[p-1]}(h\omega_n))^{\frac{1}{\rho}}\}$, we have

$$\begin{aligned}\ln A_n \quad & < \exp^{[p-1]}\{(T+\epsilon)(\ln^{[q-1]}\exp^{[q-1]}\{(\frac{1}{T+\epsilon}\ln^{[p-1]}(h\omega_n))^{\frac{1}{\rho}}\})^\rho\} \\ & -(h\omega_n)\exp^{[q-1]}\{(\frac{1}{T+\epsilon}\ln^{[p-1]}(h\omega_n))^{\frac{1}{\rho}}\} \\ & < h\omega_n - (h\omega_n)\exp^{[q-1]}\{(\frac{1}{T+\epsilon}\ln^{[p-1]}(h\omega_n))^{\frac{1}{\rho}}\}.\end{aligned}$$

and

$$(\ln^{[q]} A_n^{-\frac{1}{h\omega_n}})^\rho > \frac{1}{T+\epsilon}\ln^{[p-1]}(h\omega_n) + o(1),$$

then

$$\overline{\lim_{n\to\infty}}\frac{\ln^{[p-1]}(h\omega_n)}{(\ln^{[q]} A_n^{-\frac{1}{h\omega_n}})^\rho} < (T+\epsilon).$$

So $V \le T$.

On the other hand $V \ge T$. In fact, let $V < +\infty$, by the definition of V, for any given $\epsilon > 0$, there exists an integer $N_2 > 0$, when $n > N_2$,

$$A_n < \exp\{-(h\omega_n)\exp^{[q-1]}\{\frac{1}{V+\epsilon}\ln^{[p-1]}(h\omega_n)\}^{\frac{1}{\rho}}\}.$$

For sufficiently large r, there exists an integer $s > N_2$ such that

$$h\omega_s \le \exp^{[p-1]}\{(V+\epsilon)(\ln^{[q-1]}(r+\frac{2}{b}))^\rho\} < h\omega_{s+1}.$$

By (iii) , there exists a constant η, $0 < \eta < +\infty$ such that $\omega_{n+1} < \omega_n + \eta (n = 1, 2, \cdots)$. Then

$$\left|\int_{L_\lambda} e^{-sz} d\alpha(z)\right| \leq Ar \sum_{k=1}^{\infty} A_k e^{hr\omega_{k+1}} \leq Ar \sum_{k=1}^{\infty} A_k e^{hr(\omega_k+\eta)} \leq Ae^{Ar} \sum_{k=1}^{\infty} A_k e^{rh\omega_k}$$
$$\leq Ae^{Ar} \sum_{k=1}^{N_2} A_k e^{rh\omega_k} + Ae^{Ar} \sum_{k=N_2+1}^{s} A_k e^{rh\omega_k} + Ae^{Ar} \sum_{k=s+1}^{\infty} A_k e^{rh\omega_k}$$
$$= B_0 + B_1 + B_2,$$

where $B_0 = Ae^{Ar} \sum_{k=1}^{N_2} A_k e^{rh\omega_k} \leq ANe^{Ar} e^{h\omega_{N_2} r} \max_{1 \leq k \leq N_2} A_k \leq e^{Ar} e^{h\omega_{N_2} r}$.

$$B_1 = Ae^{Ar} \sum_{k=N_2+1}^{s} A_k e^{rh\omega_k} \leq Ae^{Ar} e^{h\omega_s r} \sum_{k=N_2+1}^{s} A_k$$
$$\leq A \exp\{Ar + r \exp^{[p-1]}\{(V+\epsilon)(\ln^{[q-1]}(r + \frac{2}{b}))^{\rho}\}\} \sum_{k=N_2+1}^{s} A_k$$
$$\leq A \exp\{Ar + r \exp^{[p-1]}\{(V+\epsilon)(\ln^{[q-1]}(r + \frac{2}{b}))^{\rho}\}\}$$
$$\sum_{k=N_2+1}^{\infty} \exp\{-(h\omega_k) \exp^{[q-1]}\{\frac{1}{V+\epsilon} \ln^{[p-1]}(h\omega_k)\}^{\frac{1}{\rho}}\}$$
$$\leq A \exp\{Ar + r \exp^{[p-1]}\{(V+\epsilon)(\ln^{[q-1]}(r + \frac{2}{b}))^{\rho}\}\}$$
$$\sum_{k=N_2+1}^{\infty} k^{-(bh \exp^{[q-1]}\{(\frac{1}{V+\epsilon} \ln^{[p-1]}(h\omega_k))^{\frac{1}{\rho}}\})}$$
$$\leq Ae^{Ar} \exp\{r \exp^{[p-1]}\{(V+\epsilon)(\ln^{[q-1]}(r + \frac{2}{b}))^{\rho}\}\}.$$

In B_2, by $h\omega_k > \exp^{[p-1]}\{(V+\epsilon)(\ln^{[q-1]}(r + \frac{2}{b}))^{\rho}\}$, then

$$r < \exp^{[q-1]}\{(\frac{1}{V+\epsilon} \ln^{[p-1]}(h\omega_k))^{\frac{1}{\rho}}\} - \frac{2}{b}.$$

We have

$$B_2 = Ae^{Ar} \sum_{k=s+1}^{\infty} A_k e^{rh\omega_k}$$
$$\leq Ae^{Ar} \sum_{k=s+1}^{\infty} \exp\{-(h\omega_k) \exp^{[q-1]}\{\frac{1}{V+\epsilon} \ln^{[p-1]}(h\omega_k)\}^{\frac{1}{\rho}}\}$$
$$\exp\{(h\omega_k)(\exp^{[q-1]}\{(\frac{1}{V+\epsilon} \ln^{[p-1]}(h\omega_k))^{\frac{1}{\rho}}\} - \frac{2}{b})\}$$
$$\leq Ae^{Ar} \sum_{k=s+1}^{\infty} e^{-\frac{2h\omega_k}{b}} \leq Ae^{Ar} \sum_{k=s+1}^{\infty} \frac{1}{n^{2h}} \leq Ae^{Ar} (h \geq 1).$$

Therefore

$$\begin{aligned} M(r,F) &\le e^{Ar}e^{h\omega_{N_2}r} \\ &+Ae^{Ar}\exp\{r\exp^{[p-1]}\{(V+\epsilon)(\ln^{[q-1]}(r+\frac{2}{b}))^{\rho}\}\}+Ae^{Ar}, \end{aligned}$$

so

$$\ln^{[p]}M(r,F)\le (V+\epsilon)(\ln^{[q-1]}(r+\frac{2}{b}))^{\rho}+A\ln^{[p-1]}r+o(1)(r\to+\infty).$$

then $T=\overline{\lim_{r\to+\infty}}\frac{\ln^{[p]}M(r,F)}{(\ln^{[q-1]}r)^{\rho}}\le V$.

For $T=+\infty$, we can use the similar method to prove. Theorem 2.3 is proved.

The theorems given above carry over directly to the case of a Laplace-stieltjes transform. Obviously, if $\tau=0$, then $G_{\tau}=\{x\ge 0, y=0: z=x+iy\}$ and $h=1+tan\tau=1$, L is the positive real half axis starting from the origin, the sequence $\{\lambda_n\}_{n=1}^{\infty}\subset L$ is a real sequence, then we have

$$F(s)=\int_0^{\infty}e^{-sx}d\alpha(x). \tag{2.5}$$

Let

$$M(r,F)=\sup_{\lambda\in L,|s|=r}\left|\int_0^{\lambda}e^{-sx}d\alpha(x)\right|,\ m(r,F)=\max_{1\le n<+\infty}A_ne^{r\lambda_n}, \tag{2.6}$$

where $A_n=\sup_{\lambda_n\le\lambda<\lambda_{n+1},|s|=r\ne 0}\left|\int_{\lambda_n}^{\lambda}e^{(-s-r)x}d\alpha(x)\right|$.

Corollary 2.4 *For $\tau=0$, if the sequence $\{\lambda_n\}_{n=1}^{\infty}\subset L$ satisfies conditions C, the entire function $F(s)$ represented by* (2.5) *has (p,q)-order $\rho(l<\rho<+\infty, p=q, l=1; p>q, l=0)$, then*

$$\overline{\lim_{r\to+\infty}}\frac{\ln^{[p]}M(r,F)}{(\ln^{[q-1]}r)^{\rho}}=\overline{\lim_{r\to\infty}}\frac{\ln^{[p]}m(r,F)}{(\ln^{[q-1]}r)^{\rho}},$$

where p and q are integers such that $p\ge 1, q\ge 0, p\ge q, M(r,F), m(r,F)$ is defined by (2.6).

Corollary 2.5 *For $\tau=0$, if the sequence $\{\lambda_n\}_{n=1}^{\infty}\subset L$ satisfies conditions C, the entire function $F(s)$ with (p,q)-order $\rho(l<\rho<+\infty, p=q, l=1; p>q, l=0)$ represented by* (2.5) *has (p,q)-type T if and only if T=MV, where $V=\overline{\lim_{n\to\infty}}\frac{\ln^{[p-1]}(h\lambda_n)}{(\ln^{[q]}A_n^{-\frac{1}{(h\lambda_n)}})^{\rho-a}}$, we have $a=1$ if $(p,q)=(1,1)$ and $a=0$ if $(p,q)\ne(1,1)$. And $M=\frac{1}{e\rho}$ if $p=1, q=0$; $M=\frac{(\rho-1)^{\rho-1}}{(\rho)^{\rho}}$ if $p=q=1$ and $M=1$ if $p\ge q\ge 2$.*

Acknowledgements: The authors would like to thank the anonymous reviewers and editor for their valuable comments and suggestions to improve the quality of the paper. This project is supported by the National Natural Science Foundation of China (11661044).

References

[1] O.P. Juneja, G.P. Kapoor and S.K. Bajpai, On the (p,q)-order and lower (p,q)-order of an entire function, *Journal für die reine und angewandte Mathematik*, 282(1976), 53-67.

[2] O.P. Juneja, G.P. Kapoor and S.K. Bajpai, On the (p,q)-type and lower (p,q)-type of an entire function, *Journal für die reine und angewandte Mathematik*, 290(1977), 180-190.

[3] Y.Y. Kong and Y. Yang, On the growth properties of the Laplace-Stieltjes transform, *Complex Variables and Elliptic Equations*, 59(4) (2014), 553-563.

[4] B. Ja. Levin, Lectures on entire functions, *American Mathematical Society*, 150(1996), 3-5.

[5] L.N. Shang and Z.S. Gao, The growth of entire functions of infinite order represented by Laplace-Stieltjes transformation, *Mathematical Acta Scientia*, 27(6) (2007), 1035-1043.

[6] D.V. Widder, *Laplace Transform (PMS-6)*, Princeton University Press, (2015).

[7] J.R. Yu, The Borel line of entire functions defined by Laplace-Stieltjes transform, *Acta Mathematica Sinica (Chinese Series)*, 13(3) (1963), 471-484.

[8] H.S. Zhang and D.C. Sun, The value distribution of Laplace-Stieltjes transforms in the right half plane, *Acta Mathematica Sinica (Chinese Series)*, 55(3) (2012), 535-542.