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Meromorphic Continuation of the Spectral Zeta Kernel

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Abstract

The meromorphic extension of spectral zeta kernel, $\zeta_g(s, x, x)$, of the Laplacian on a Riemannian manifold, (M, g), is constructed using a relationship between the zeta kernel and the heat kernel. This construction is facilitated by the analytic properties of the Mellin transform of measurable functions on Riemannian manifolds. The pole structure of the spectral zeta function is highlighted from the relationship between it and the zeta kernel.

Keywords: *Laplacian, Zeta function, Zeta kernel, Heat kernel, Mellin Transform, Meromorphic Extension.*

1 Introduction

The systematic use of the spectral zeta function to give sense to infinite series (regularization) is believed to be begun by Godfrey H. Hardy and John E. Littlewood from the second decade of the last century. Elizalde [9] among other literature posits that they actually established the convergence and equivalence of series regularized with the heat and zeta functions methods. The most significant improvement, published in specialized literature in 1949, was due to Subharamiah Minakshisundaram and Åke Pleijel [17] who showed that for the Laplacian of a compact Riemannian manifold, the corresponding zeta function has a meromorphic continuation to the whole complex plane. This is what has come to be known and called Minakshisundaram-Pleijel spectral zeta function.

Since then various generalisations of the spectral zeta functions have appeared; see e.g [22, 18, 23, 19, 20] and [21] among many literature. In this paper, the meromorphic continuation of the spectral kernel is constructed. The relationships between the spectral zeta function, zeta kernel and heat kernel are used to aid the construction and discuss other spectral quantities e.g the poles and residues of the spectral zeta functions.

The paper is organised in sections. This section is the introduction. The concept of analytic continuation is briefly highlighted in the second section. The spectral functions of interest, namely, spectral zeta and heat kernels are introduced in the third and fourth sections. The fifth section contains the main result of the paper which is the meromorphic continuation of the zeta kernel. The last section contains the concluding remarks and some information on the pole structure of the zeta function and the zeta kernel.

2 Analytic Continuation

Analytic continuation of a given function is a method to extend the domain of the function, i.e find other values of the function which are initially undefined, [25]. Given a function f with domain Ω_1, find a function g that matches f on Ω_1 but now defined in a larger region Ω_2. Alternatively, on a complex plane, take a non-singular point x_0 of f with the associated circle of convergence of Taylor series. This circle of convergence will pass through the nearest singularity of f at x_0. Then take another non-singular point x_1 on this circle of convergence of its Taylor series. Repeat this process. The union of sets of all these circles form a larger region/domain Ω_3 say, on which f is defined thus $g = (f, \Omega_3)$ forms the analytic continuation of f. Note, if the analytic continuation of f exits, it is unique. This process is called "meromorphic continuation" when the function is continued to be analytic in the given domain except for simple poles at given points of the domain. Well known examples of meromorphic continuation are those of the gamma and Riemann zeta functions, see for example [13, 14, 6, 12, 15, 25] and [26].

To compute the analytic or meromorphic continuations of functions such as the zeta or gamma functions, sums and integrals are often interchanged, [12, 13]. Sums and integrals are switched throughout this work using the Fubini - Tonelli theorem stated below.

Theorem 2.1 *[4] (Fubini - Tonelli theorem). Let $\{\psi_k\}$ be a sequence of measurable functions. Sum and integral such as $\sum_k \int \psi_k(x)dx$ can be interchanged in either of the following cases: $\psi_k \geq 0, \forall k \in \mathbb{N}$* or $\sum_k \int |\psi_k(x)|dx < \infty$.

3 The Spectral Zeta Kernel

Let (M, g) be a closed connected smooth Riemannian manifold. The Laplacian on $C^\infty(M)$ is the operator

$$\Delta_g : C^\infty(M) \to C^\infty(M) \tag{1}$$

defined in local coordinates by

$$\Delta_g = -\ \mathrm{div}(\mathrm{grad}) = -\frac{1}{\sqrt{|g|}} \sum_{i,j} \frac{\partial}{\partial x^i}(\sqrt{|g|} g^{ij} \frac{\partial}{\partial x^j}). \tag{2}$$

The operator Δ_g extends to a self-adjoint operator on $L^2(M) \supset H^2(M) \to L^2(M)$ with compact resolvent. This implies that there exists orthonormal basis $\{\psi_k\} \subset L^2(M)$ consisting of eigenfunctions such that

$$\Delta_g \psi_k = \lambda_k \psi_k \tag{3}$$

where the eigenvalues $\{\lambda_k\}_{k=1}^\infty$ are listed with multiplicities; see e.g [7, 24, 1, 2], etc.

The spectral zeta function is a generalisation of the popular Riemann zeta function, ζ_R, usually defined as the function

$$\zeta_R : \{s \in \mathbb{C} : \Re(s) > 1\} \to \mathbb{C}$$

with

$$\zeta_R(s) = \sum_{k=1}^\infty \frac{1}{k^s}; \quad \Re(s) > 1. \tag{4}$$

From the Riemann zeta function, notice that since

$$\sum_{k=1}^\infty |\frac{1}{k^s}| = \sum_{k=1}^\infty \frac{1}{k^{\Re(s)}}, \tag{5}$$

the series on the right-hand-side of (5) converges absolutely if and only if $\Re(s) > 1$; c.f: [24, 9, 10, 17] and [22]. The Riemann zeta function defined by (4) above is holomorphic in the region indicated. It, however, admits a meromorphic continuation to the whole s-complex plane with only simple pole at $s = 1$ and has residue 1; see e.g. Titchmarsh [24].

The Laplacian defined in (2) acting on functions on the closed and connected Riemannian n-manifold is a non-negative operator and has the discrete spectrum $\{\lambda_k\}_{k=1}^\infty$ listed with multiplicities. The spectral zeta function is defined to be

$$\zeta_g(s) = \sum_{k=1}^\infty \frac{1}{\lambda_k^s}; \qquad \Re(s) > \frac{n}{2}. \tag{6}$$

The kernel, $\zeta_g(s,x,y)$, of the spectral zeta function, $\zeta_g(s)$, is the integral kernel of the operator Δ_g^{-s}. The operator Δ_g^{-s} is uniquely defined by the following properties (see e.g [21] and [8]):

(1.) it is linear on $L^2(M)$ with 1-dimensional null space consisting of constant functions. This ensures that the smallest eigenvalue of Δ_g^{-s} is 0 of multiplicity 1 with corresponding eigenfunction $\frac{1}{\sqrt{V}}$ where V is the volume of M;

(2.) the image of Δ_g^{-s} is contained in the orthogonal complement of constant functions in $L^2(M)$ i.e.

$$\int_M \Delta_g^{-s}\psi dV_g = 0 \;\; \forall\, \psi \in L^2(M) \;\; \text{constant;} \quad \text{and}$$

(3.) $\Delta_g^{-s}\psi_k(x) = \lambda_k^{-s}\psi_k(x)$ for all $\psi_k;\; k > 0$ an orthonormal basis of eigenfunction of Δ_g.

Then for $\Re(s) > \frac{n}{2}$, we see by property (3.) that Δ_g^{-s} is trace class, with trace given by the spectral zeta function, namely

$$\zeta_g(s) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k^s} = \mathrm{Tr}(\Delta_g^{-s}) = \int_M \zeta_g(s,x,x)dV \;; \qquad \Re(s) > \frac{n}{2}. \tag{7}$$

Theorem 3.1 *[16] Let $\{\psi_k\}_{k=1}^{\infty}$ be an orthonormal eigenbasis for Δ_g corresponding to the eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ listed with multiplicities. Then the zeta kernel, $\zeta_g(s,x,y)$, (also called the point-wise zeta function), is equal to*

$$\zeta_g(s,x,y) = \sum_{k=1}^{\infty} \frac{\psi_k(x)\bar{\psi}_k(y)}{\lambda_k^s}; \quad \Re(s) > \frac{n}{2}. \tag{8}$$

From here on, we suppress the subscript g in $\zeta_g(s)$ and Δ_g. We simply write $\zeta(s)$ and Δ for $\zeta_g(s)$ and Δ_g respectively, unless for purpose of emphasis.

4 The Heat Kernel

The meromorphic extension of $\zeta_g(s)$ is proved in this paper by showing a relationship between the zeta kernel and the heat kernel. The heat kernel, $K(t,x,y) : (0,\infty) \times M \times M \to \mathbb{R}$, is a continuous function on $(0,\infty) \times M \times M$. It is the fundamental solution to the heat equation [7], i.e, it is the unique solution to the following system of equations:

$$\left.\begin{array}{rcl} (\frac{\partial}{\partial t} + \Delta_x)K(t,x,y) &=& 0 \\ \lim_{t\to 0}\int_M K(t,x,y)\psi(y)dV_y &=& \psi(x) \end{array}\right\} \tag{9}$$

for $t > 0$; $x, y \in M$ and Δ_x is the Laplacian acting on any $\psi \in L^2(M)$, where the limit in the second equation of (9) is uniform for every $\psi \in C^\infty(M)$; [5, 7, 22], etc.

The heat operator $e^{-t\Delta} : L^2(M) \to L^2(M)$ is the operator defined by the integral kernel $K(t, x, y)$ as

$$(e^{-t\Delta}\psi)(y) := \int_M K(t, x, y)\psi(x)dV_x$$

for $\psi \in L^2(M)$. The heat kernel is symmetric in the space variables, that is $K(t, x, y) = K(t, y, x) \;\; \forall \, x, y \in M$.

In terms of these eigenfunctions, the heat operator, $e^{-t\Delta}$, is trace-class for all $t > 0$ and one can write the heat kernel as

$$K(t, x, y) = \sum_{k=0}^{\infty} e^{-\lambda_k t}\psi_k(x)\bar{\psi}_k(y). \tag{10}$$

The convergence for all $t > 0$ is uniform on $M \times M$; [5]. In particular, the trace of the heat operator

$$\mathrm{Tr}(e^{-\Delta_g t}) = \sum_{k=0}^{\infty} e^{-\lambda_k t}|\psi_k(x)|^2 = \sum_{k=0}^{\infty} e^{-\lambda_k t} = \int_M K(t, x, x)dV_g(x) < \infty. \tag{11}$$

The heat kernel admits an expansion along the diagonal. This is the so-called Minakshisundaram-Pleijel expansion of the trace of the heat kernel:

$$\mathrm{Tr}(e^{-\Delta_g t}) = \frac{1}{(4\pi t)^{n/2}}\Big\{a_0 + a_1 t + a_2 t^2 + \cdots + a_N t^N + O\big(t^{N+1}\big)\Big\}; \tag{12}$$

as $t \to 0^+$; where $a_j = \int_M \mathcal{O}_j(x)dV_g(x)$ with $\mathcal{O}_j(x)$ being some smooth functions on M which depend only on the geometric data at the point $x \in M$; see e.g. [22, 17] and [21]. For example, $a_0 = \text{volume}(M)$ and $a_1(x) = \frac{(4\pi)^{-n/2}c(x)}{6}$ with $c(x)$ being the scalar curvature of M at the point x; see e.g ([17]).

Recall the Mellin transform of a measurable function ψ is defined by

$$(M\psi)(s) := \int_0^\infty \psi(t)t^{s-1}dt. \tag{13}$$

Furthermore, the Mellin transform as a function of s is meromorphic on a region of $\mathbb{C}$ that depends upon the decay properties of ψ; [3]. Specifically, if

$$\psi(t) = \begin{cases} \mathcal{O}(t^\alpha) & t \to 0 \\ \mathcal{O}(t^\beta) & t \to \infty, \end{cases} \tag{14}$$

then $(M\psi)(s)$ is meromorphic for $\Re(s) \in (-\alpha, -\beta)$, c.f: [11]. We use this to prove the meromorphic continuation of the zeta kernel using the following relationship between the two kernels.

Lemma 4.1 *The zeta kernel and the heat kernel are related by*

$$\zeta_g(s,x,y) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}(K(t,x,y) - \frac{1}{V})dt; \qquad \Re(s) > \frac{n}{2}. \tag{15}$$

Proof: Observe that for any $x > 0$ and $\Re(s) > 0$,

$$x^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-xt} t^{s-1} dt. \tag{16}$$

Since a change of variable from, say, xt to τ gives x^{-s} and since $\Gamma(s)$ is holomorphic for $\Re(s) > 0$.

Consequently,

$$\lambda_k^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-\lambda_k t} t^{s-1} dt. \tag{17}$$

Thus,

$$\zeta_g(s,x,y) = \sum_{k=1}^\infty \left[\psi_k(x)\overline{\psi}_k(y) \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-\lambda_k t} dt \right]; \quad \Re(s) > \frac{n}{2}. \tag{18}$$

Therefore, using Fubini-Tonelli theorem, (2.1), to switch the order of the sum and the integral, we have

$$\begin{aligned} \zeta_g(s,x,y) &= \frac{1}{\Gamma(s)} \int_0^\infty \left(\sum_{k=1}^\infty e^{-\lambda_k t} \psi_k(x)\overline{\psi}_k(y) \right) t^{s-1} dt \\ &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} (\int_M K(t,x,y) dV_g - \frac{1}{V}) dt. \end{aligned} \tag{19}$$

Thus,

$$\zeta_g(s,x,y) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} (K(t,x,y) - \frac{1}{V}) dt. \tag{20}$$

5 Meromorphic Continuation of the Zeta Kernel

Now let's consider how to meromorphically continue the zeta kernel. Lemma (4.1) says that up to the Gamma function, $\zeta_g(s)$ is the Mellin transform of the heat kernel minus $\frac{1}{V}$. We want to use this relationship for $\Re(s) > \frac{n}{2}$ to extend the definition of the zeta kernel meromorphically to the rest of $\mathbb{C}$. We do this by breaking up the integrand into an infinite sum that has the appropriate decay and a finite sum that we can examine directly.

The following theorem about the decay of the heat kernel as $t \to \infty$, will be used. It states thus:

Theorem 5.1 *There is a constant $\beta = \beta(M) > 0$, depending only on M, such that for all $x, y \in M$ and $t \geq 1$, the following inequality holds:*

$$|K(t,x,y) - \frac{1}{V}| \leq \beta e^{-\lambda_1 t}. \tag{21}$$

Proof: Since M is compact, the claim is a consequence of the fact that

$$\begin{aligned} |K(t,x,y) - \frac{1}{V}| &= |\sum_{k=1}^{\infty} e^{-\lambda_k t}\psi_k(x)\bar{\psi}_k(y)| \\ &= e^{-\lambda_1 t}|\sum_{k=1}^{\infty} e^{-(\lambda_k-\lambda_1)t}\psi_k(x)\bar{\psi}_k(y)| \\ &\leq e^{-\lambda_1 t}(\sum_{k=1}^{N} e^{-(\lambda_k-\lambda_1)t}|\psi_k(x)\bar{\psi}_k(y)| \\ &+ |\sum_{k=N+1}^{\infty} e^{-(\lambda_k-\lambda_1)t}\psi_k(x)\bar{\psi}_k(y)|). \end{aligned}$$

As $K(t,x,y)$ is uniformly convergent on $[1,\infty)\times M \times M$, one can find a positive integer N such that

$$e^{-\lambda_1 t}(|\sum_{k=N+1}^{\infty} e^{-(\lambda_k-\lambda_1)t}\psi_k(x)\bar{\psi}_k(y)|) = 1. \tag{22}$$

Again, using that M is compact, we have

$$e^{-\lambda_1 t}(\sum_{k=1}^{N} e^{-(\lambda_k-\lambda_1)t}|\psi_k(x)\bar{\psi}_k(y)|) \leq \beta e^{-\lambda_1 t}; \tag{23}$$

thus, the claim follows

On the diagonal, one can write a modified version of this result as corollary (5.2) below.

Corollary 5.2 *Let (M, g) be smooth, compact and connected Riemannian manifold with volume V and $K(t,x,y)$ the heat kernel. Then for any fixed $t_0 > 0$ given,*

$$(K(t,x,x) - \frac{1}{V}) \leq (K(t_0,x,x) - \frac{1}{V}) \cdot e^{-\lambda_1(t-t_0)} \tag{24}$$

provided $t \geq t_0$ and λ_1 is the first positive eigenvalue of the Laplacian on M.

Since $\lambda_1 > 0$, these results imply that as $t \to \infty$, we have exponential decay, which means $-\beta = \infty$ in Equation (14) above. Thus we only need to split up the heat kernel to control the decay as $t \to 0$.

Now we will complete the proof of meromorphic continuation of the zeta kernel which is the main result of this paper.

Theorem 5.3 *The zeta kernel, $\zeta_g(s,x,y)$, admits a meromorphic continuation to the whole of the complex s-plane.*

Proof: For $x \neq y$, we have that $K(t,x,y) - \frac{1}{V}$ decays exponentially by theorem (5.1) above. So, we are left with the case $x = y$. Using the asymptotic expansion of the heat kernel (12), we have

$$\begin{aligned}
\Gamma(s)\zeta_g(s,x,x) &= \int_0^\infty (K(t,x,x) - \frac{1}{V})t^{s-1}dt \\
&= \int_0^\infty \left(t^{-n/2}\left\{a_1(x)t + \cdots + a_N(x)t^N\right\} + R_N(x,t) - \frac{1}{V}\right) t^{s-1}\, dt \\
&= \int_0^\infty (t^{-n/2}\left\{a_1(x)t + \cdots + a_N(x)t^N\right\} - \frac{1}{V})t^{s-1}\, dt \\
&+ \ M(R_N)(s),
\end{aligned}$$

where $R_N(x,t) = O\big(t^{N+1-(n/2)}\big)$. Thus $M(R_N)(s)$ is meromorphic for $\Re(s) \in ((n/2) - N - 1, \infty)$. Next, we need to show the first integral can be meromorphically continued.

To do that, we have to split up the whole heat kernel integral since its asymptotic expansion is only good near $t = 0$. To do that, we fix the following continuous cut-off function.

$$\chi(t) = \begin{cases} 1 & \text{if } t < \frac{1}{2} \\ 2 - 2t & \text{if } \frac{1}{2} \leq t \leq 1 \\ 0 & \text{if } t > 0. \end{cases} \tag{25}$$

Then we split up the integral into

$$\int_0^\infty \chi(t)(K(t,x,x) - \frac{1}{V})t^{s-1}dt + \int_0^\infty (1 - \chi(t))(K(t,x,x) - \frac{1}{V})t^{s-1}dt. \tag{26}$$

Now the second of these integrals is the Mellin transform of a function that decays exponentially both as $t \to 0$ and as $t \to \infty$. So, the critical strip in the Mellin transform is the whole complex plane. Thus, this second integral is holomorphic in s.

In the first integral, we can use the asymptotic expansion to get that it is equal to

$$\int_0^1 \chi(t)R_N(x,t)t^{s-1}dt + \sum_{j=1}^{N} a_j(x)\int_0^1 \chi(t)t^{j-(\frac{n}{2}+s-1)}dt - \frac{1}{sV} \qquad (27)$$

where $R_N(x,t) = O\big(t^{N+1-\frac{n}{2}}\big)$ as $t \to 0$. We can therefore rewrite the first integral as

$$\int_0^\infty \chi(t)R_N(x,t)t^{s-1}dt, \qquad (28)$$

which is now the Mellin transform of a continuous function that vanishes to infinite order as $t \to \infty$ because of the cutoff function $\chi(t)$ and vanishes like $t^{N+1-\frac{n}{2}}$ as $t \to 0$. Thus the critical strip is $(\frac{n}{2} - N - 1, \infty)$. So, taking N sufficiently large, one can also make this piece extend holomorphically as far as one likes, on the right side of the complex plane.

We are left with

$$\sum_{j=1}^{N} a_j(x)\int_0^1 \chi(t)t^{j-(\frac{n}{2}+s-1)}dt - \frac{1}{sV} \qquad (29)$$

which can be directly integrated for $s > j - \frac{n}{2}$, although we have to do it in two pieces. We can just look at the integral, but when one wishes to calculate the residues, he/she has to keep track of the coefficients $a_j(x)$. The integral

$$\begin{aligned}\int_0^1 \chi(t)t^{j-(\frac{n}{2}+s-1)}dt &= \int_0^{\frac{1}{2}} t^{j-(\frac{n}{2}+s-1)}dt + \int_{\frac{1}{2}}^1 (2-2t)t^{j-(\frac{n}{2}+s-1)}dt \\ &= \int_0^1 t^{j-(\frac{n}{2}+s-1)}dt + 2\int_{\frac{1}{2}}^1 t^{j-(\frac{n}{2}+s-1)}dt \\ &- 2\int_{\frac{1}{2}}^1 t^{j-(\frac{n}{2}+s)}dt.\end{aligned}$$

The last two pieces of the integral are holomorphic and the first for $\Re(s) > \frac{n}{2} - j$ yields

$$\sum_{j=0}^{N} \frac{a_j(x)}{s - n/2 + j}.$$

Therefore,

$$\zeta_g(s,x,x) = \frac{1}{\Gamma(s)}\left\{\sum_{j=0}^{N} \frac{a_j(x)}{s - n/2 + j} - \frac{1}{sV}\right\} + G_k(s); \qquad (30)$$

where $G_k(s)$ is a holomorphic function for $\Re(s) > n/2 - N - 1$.

6 Conclusion

In this paper, the meromorphic continuation of the spectral zeta kernel (8) of the Laplacian along the diagonal of a Riemannian manifold (M, g) has been computed to be (30). With this formula, one can easily evaluate the value of the kernel at any given point. To conclude, we also mention that the poles structure of $\zeta_g(s)$, comes from the relationship between the spectral zeta function and the zeta kernel (7). So, since

$$\lim_{s\to 0}\frac{1}{s\Gamma(s)} = \lim_{s\to 0}\frac{1}{\Gamma(s+1)} = 1, \tag{31}$$

$\zeta_g(s)$ is holomorphic at $s = 0$. Furthermore, since the poles of $\Gamma(s)$ are at $s = -j,\ j \in \mathbb{N}$, the poles of $\zeta_g(s)$ which are all simple are located at

$$s = \begin{cases} \frac{n}{2}, \frac{n}{2}-1, \frac{n}{2}-2, \frac{n}{2}-3, \frac{n}{2}-4, \cdots, 2, 1 & \text{n even, and} \\ \frac{n}{2}, \frac{n}{2}-1, \cdots, \frac{1}{2}, -\frac{1}{2}, \cdots, [\frac{n-1}{2}] - \frac{n}{2} & \text{for } n \text{ odd.} \end{cases} \tag{32}$$

The residue of $\zeta_g(s)$ at $s = k$ is given by

$$\mathrm{Res}_{s=k}\zeta_g(s) = \frac{a_{\frac{n}{2}-k}(x)}{\Gamma(k)} \tag{33}$$

which agrees with known results such as those of [8, 17] and [21].

Observe also that from (30), that at $s = 0$, the pole in $\Gamma(s)$ forces $G_k(s)$ to vanish and so,

$$\zeta_g(0, x, x) = a_{\frac{n}{2}}(x) - \frac{1}{V}.$$

Thus, a knowledge of $a_{\frac{n}{2}}$ is sufficient to determine $\zeta_g(0, x, x)$ for example,

$$\zeta_g(0, x, x) = \begin{cases} -\frac{1}{V} & \text{for } n \text{ odd;} \\ a_{\frac{n}{2}}(x) - \frac{1}{V} & \text{for } n \text{ even} \end{cases} \tag{34}$$

where V is the volume of M.

Another example is to evaluate the spectral zeta kernel, $\zeta_g(s, x, x)$ at $s = -\frac{1}{2}$, which is usually called the Casimir energy; [9] and [10]. This has been computed for different Riemannian manifolds of different dimensions; see e.g [10] and [8].

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