Gen. Math. Notes, Vol. 35, No. 2, August 2016, pp.29-39
ISSN 2219-7184; Copyright ©ICSRS Publication, 2016
www.i-csrs.org
Available free online at http://www.geman.in

Operation on Functions with a New Type of Closed Graph

Baravan A. Asaad

Deptartment of Mathematics, Faculty of Science
University of Zakho, Kurdistan-Region, Iraq
E-mail: baravan.asaad@gmail.com

(Received: 9-4-16 / Accepted: 14-7-16)

Abstract

The aim of this paper is to introduce some new classes of functions by using the operation γ on $P_SO(X)$ and the set P_S^{γ}-open. In addition, some basic properties of functions with a $P_S^{(\gamma,\beta)}$-closed graph have been investigated.

Keywords: *P_S^{γ}-open sets, γ operation on $P_SO(X)$, $P_S^{(\gamma,\beta)}$-continuous function, $P_S^{(\gamma,\beta)}$-closed graph.*

1 Introduction

Throughout this paper, the spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. The concept of preopen sets and semiopen sets was introduced respectively by Mashhour et al. [6] and Levine [5]. While the concept of P_S-open set was defined by Khalaf and Asaad [4]. Recently, Asaad [1] introduced and investigated the concept of the mapping γ on the collection of all P_S-open ($P_SO(X)$) subsets of (X, τ), and defined the notion of P_S^{γ}-open sets and studied some of their topological properties. He also defined some separation axioms by utilizing the operation γ on $P_SO(X)$ and the set P_S^{γ}-open.

The aim of this paper is to introduce some new classes of functions by using the operation γ on $P_SO(X)$ and the set P_S^{γ}-open. In Section 4, we investigate some basic properties of functions with a $P_S^{(\gamma,\beta)}$-closed graph by using P_S^{γ}-open and P_S^{β}-open sets.

2 Preliminaries

For a subset A of a space X, the closure and the interior of A are denoted by $Cl(A)$ and $Int(A)$, respectively. A subset A of a space X is said to be preopen [6] (respectively, semiopen [5]) if $A \subseteq Int(Cl(A))$ (respectively, $A \subseteq Cl(Int(A))$). The complement of semiopen set is said to be semiclosed [2]. A preopen subset A of a topological space (X,τ) is called P_S-open [4] if for each $x \in A$, there exists a semiclosed set F such that $x \in F \subseteq A$. An operation γ on $P_SO(X)$ is a mapping $\gamma\colon P_SO(X) \to P(X)$ such that $G \subseteq G^{\gamma}$ for every $G \in P_SO(X)$, where $P(X)$ is the power set of X and G^{γ} is the value of γ at G [1]. It is clear that $X^{\gamma} = X$ for any operation $\gamma\colon P_SO(X) \to P(X)$. A nonempty subset A of a topological space X with an operation $\gamma\colon P_SO(X) \to P(X)$ on $P_SO(X)$ is said to be P_S^{γ}-open [1] if for each $x \in A$, there exists a P_S-open set G such that $x \in G$ and $G^{\gamma} \subseteq A$. We assume that the empty set ϕ is a P_S^{γ}-open set for any operation $\gamma\colon P_SO(X) \to P(X)$. The class of all P_S^{γ}-open subsets of a space (X,τ) is denoted by $P_S^{\gamma}O(X)$. For each $x \in X$, the class of all P_S^{γ}-open sets of (X,τ) containing a point x is denoted by $P_S^{\gamma}O(X,x)$. The complement of a P_S^{γ}-open set of X is P_S^{γ}-closed [1]. The P_S^{γ}-interior of a subset A of X is defined as the union of all P_S^{γ}-open sets contained in A and it is denoted by $P_S^{\gamma}Int(A)$ [1] , and the P_S^{γ}-closure of a subset A of X is defined as the intersection of all P_S^{γ}-closed sets containing A and it is denoted by $P_S^{\gamma}Cl(A)$ [1]. The set A is P_S^{γ}-closed if and only if $P_S^{\gamma}Cl(A) = A$ [1], and the set A is P_S^{γ}-open if and only if $P_S^{\gamma}Int(A) = A$ [1].

The following definitions and theorem can be found in [1] .

Definition 2.1. *Let A be any subset of a topological space (X,τ) and γ be an operation on $P_SO(X)$. Then*

1. *The P_S^{γ}-boundary of A is denoted by $P_S^{\gamma}Bd(A)$ and is defined by $P_S^{\gamma}Bd(A) = P_S^{\gamma}Cl(A)\backslash P_S^{\gamma}Int(A)$ equivalently $P_S^{\gamma}Bd(A) = P_S^{\gamma}Cl(A) \cap P_S^{\gamma}Cl(X\backslash A)$.*

2. *A point $x \in X$ is said to be P_S^{γ}-limit point of A if for evey P_S^{γ}-open set G containing x, $G \cap (A\backslash\{x\}) \neq \phi$. The set of all P_S^{γ}-limit points of A is called a P_S^{γ}-derived set of A and it is denoted by $P_S^{\gamma}D(A)$.*

Theorem 2.2. *Let A be any subset of a topological space (X,τ) and γ be an operation on $P_SO(X)$. Then $x \in P_S^{\gamma}Cl(A)$ if and only if $A \cap G \neq \phi$ for every P_S^{γ}-open set G of X containing x.*

Definition 2.3. *A topological space (X, τ) with an operation γ on $P_SO(X)$ is called P_S^γ-T_0 if for each pair of distinct points x, y in X, there exists a P_S^γ-open set G containing one of the points but not the other.*

Definition 2.4. *A topological space (X, τ) with an operation γ on $P_SO(X)$ is called P_S^γ-T_1 if for each pair of distinct points x, y in X, there exist two P_S^γ-open sets G and H such that $x \in G$ but $y \notin G$ and $y \in H$ but $x \notin H$.*

Definition 2.5. *A topological space (X, τ) with an operation γ on $P_SO(X)$ is called P_S^γ-T_2 if for each pair of distinct points x, y in X, there exist disjoint P_S^γ-open sets G and H containing x and y respectively.*

3 $P_S^{(\gamma,\beta)}$-Continuous Functions in Terms of P_S^γ-Open Sets

Throughout Section 3 and Section 4, let $\gamma\colon P_SO(X) \to P(X)$ and $\beta\colon P_SO(Y) \to P(Y)$ be operations on $P_SO(X)$ and $P_SO(Y)$ respectively. In this section, we introduce a new class of functions called $P_S^{(\gamma,\beta)}$-continuous by using P_S^γ-open set. Some characterizations and properties of this function are investigated.

Definition 3.1. *Let (X, τ) and (Y, σ) be two topological spaces. A function $f\colon (X, \tau) \to (Y, \sigma)$ is called $P_S^{(\gamma,\beta)}$-continuous if the inverse image of every P_S^β-open set of Y is P_S^γ-open in X, or equivalently, if the inverse image of every P_S^β-closed set of Y is P_S^γ-closed in X.*

Theorem 3.2. *If $f\colon (X, \tau) \to (Y, \sigma)$ is $P_S^{(\gamma,\beta)}$-continuous and $g\colon (Y, \sigma) \to (Z, \rho)$ is $P_S^{(\beta,\alpha)}$-continuous, then $g \circ f\colon (X, \tau) \to (Z, \rho)$ is $P_S^{(\gamma,\alpha)}$-continuous for any operation $\alpha\colon P_SO(Z) \to P(Z)$ on $P_SO(Z)$.*

Proof: Let $U \in P_S^\alpha O(Z, \rho)$. Since g is $P_S^{(\beta,\alpha)}$-continuous, then $g^{-1}(U) \in P_S^\beta O(Y, \sigma)$. But f is $P_S^{(\gamma,\beta)}$-continuous, then

$$(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) \in P_S^\gamma O(X, \tau).$$

Hence, $g \circ f$ is $P_S^{(\gamma,\alpha)}$-continuous.

Some characterizations of $P_S^{(\gamma,\beta)}$-continuous function are mentioned in the following results.

Theorem 3.3. *For a function $f\colon (X, \tau) \to (Y, \sigma)$. The following statements are equivalent:*

1. *f is $P_S^{(\gamma,\beta)}$-continuous.*

2. *For every* $x \in X$ *and for every* $V \in P_S^{\beta}O(Y, f(x))$*, there exists* $U \in P_S^{\gamma}O(X, x)$ *such that* $f(U) \subseteq V$*.*

3. *For every* $x \in X$ *and for every* P_S^{β}*-neighbourhood* $N \subseteq Y$ *of* $f(x)$*, there exists a* P_S^{γ}*-neighbourhood* $M \subseteq X$ *of* x *such that* $f(M) \subseteq N$*.*

4. *The inverse image of every* P_S^{β}*-neighbourhood of* $f(x) \in Y$ *is* P_S^{γ}*-neighbourhood of* $x \in X$*.*

Proof: The proof is obvious and hence it is omitted.

Theorem 3.4. *The following properties are equivalent for any function* $f\colon (X, \tau) \to (Y, \sigma)$*:*

1. f *is* $P_S^{(\gamma,\beta)}$*-continuous.*

2. $P_S^{\beta}Int(f(A)) \subseteq f(P_S^{\gamma}Int(A))$*, for every* $A \subseteq X$*.*

3. $f^{-1}(P_S^{\beta}Int(B)) \subseteq P_S^{\gamma}Int(f^{-1}(B))$*, for every* $B \subseteq Y$*.*

4. $P_S^{\gamma}Cl(f^{-1}(B)) \subseteq f^{-1}(P_S^{\beta}Cl(B))$*, for every* $B \subseteq Y$*.*

5. $f(P_S^{\gamma}Cl(A)) \subseteq P_S^{\beta}Cl(f(A))$*, for every* $A \subseteq X$*.*

6. $f(P_S^{\gamma}D(A)) \subseteq P_S^{\beta}Cl(f(A))$*, for every* $A \subseteq X$*.*

7. $P_S^{\gamma}D(f^{-1}(B)) \subseteq f^{-1}(P_S^{\beta}Cl(B))$*, for every* $B \subseteq Y$*.*

Proof: Straightforward.

Lemma 3.5. *For any subset* A *of a space* X *with an operation* γ *on* $P_SO(X)$*. Then* A *is* P_S^{γ}*-closed if and only if* $P_S^{\gamma}Bd(A) \subseteq A$*.*

Proof: Let A be any P_S^{γ}-closed set in X. Then

$$P_S^{\gamma}Bd(A) = P_S^{\gamma}Cl(A) \backslash P_S^{\gamma}Int(A) = A \backslash P_S^{\gamma}Int(A) \subseteq A.$$

Conversely, assume that $P_S^{\gamma}Bd(A) \subseteq A$. Then $P_S^{\gamma}Bd(A) \cap X \backslash A = \phi$ which implies that $P_S^{\gamma}Cl(A) \cap P_S^{\gamma}Cl(X \backslash A) \cap X \backslash A = \phi$ and hence $P_S^{\gamma}Cl(A) \cap X \backslash A = \phi$. Therefore, $P_S^{\gamma}Cl(A) \subseteq A$. Since in general $A \subseteq P_S^{\gamma}Cl(A)$. So $A = P_S^{\gamma}Cl(A)$. This means that A is P_S^{γ}-closed set in X.

Theorem 3.6. *Let* $f\colon (X, \tau) \to (Y, \sigma)$ *be any function. Then the following properties are equivalent:*

1. f *is* $P_S^{(\gamma,\beta)}$*-continuous.*

2. $P_S^{\gamma}Bd(f^{-1}(B)) \subseteq f^{-1}(P_S^{\beta}Bd(B))$*, for each* $B \subseteq (Y, \sigma)$*.*

3. $f(P_S^\gamma Bd(A)) \subseteq P_S^\beta Bd(f(A))$, *for each* $A \subseteq (X,\tau)$.

Proof: (1) $\Rightarrow$ (2): Let f be a $P_S^{(\gamma,\beta)}$-continuous function and let $B \subseteq (Y,\sigma)$. Then by Theorem 3.4 (4), we have

$$\begin{aligned} P_S^\gamma Bd(f^{-1}(B)) &= P_S^\gamma Cl(f^{-1}(B)) \backslash P_S^\gamma Int(f^{-1}(B)) \\ &\subseteq f^{-1}(P_S^\beta Cl(B)) \backslash P_S^\gamma Int(f^{-1}(B)) \\ &\subseteq f^{-1}(P_S^\beta Cl(B)) \backslash P_S^\gamma Int(f^{-1}(P_S^\beta Int(B))) \\ &= f^{-1}(P_S^\beta Cl(B)) \backslash f^{-1}(P_S^\beta Int(B)) \\ &= f^{-1}(P_S^\beta Cl(B) \backslash P_S^\beta Int(B)) = f^{-1}(P_S^\beta Bd(B)). \end{aligned}$$

Therefore, $P_S^\gamma Bd(f^{-1}(B)) \subseteq f^{-1}(P_S^\beta Bd(B))$.

(2) $\Rightarrow$ (3): Let A be any subset of (X,τ). Then $f(A)$ is a subset of (Y,σ). Then by (2), we have $P_S^\gamma Bd(f^{-1}(f(A))) \subseteq f^{-1}(P_S^\beta Bd(f(A)))$ implies that $P_S^\gamma Bd(A) \subseteq f^{-1}(P_S^\beta Bd(f(A)))$ and hence $f(P_S^\gamma Bd(A)) \subseteq P_S^\beta Bd(f(A))$. This completes the proof.

(3) $\Rightarrow$ (1): Let E be any P_S^β-closed set in (Y,σ). Then $f^{-1}(E)$ is a subset of (X,τ). So by using part (3), we have

$$\begin{aligned} f(P_S^\gamma Bd(f^{-1}(E))) &\subseteq P_S^\beta Bd(f(f^{-1}(E))) \\ &\subseteq P_S^\beta Bd(E) \subseteq P_S^\beta Cl(E) = E. \end{aligned}$$

Hence $f(P_S^\gamma Bd(f^{-1}(E))) \subseteq E$. This implies that $P_S^\gamma Bd(f^{-1}(E)) \subseteq f^{-1}(E)$. Thus, by Lemma 3.5, $f^{-1}(E)$ is P_S^γ-closed set in X. Consequently f is $P_S^{(\gamma,\beta)}$-continuous.

Theorem 3.7. *Let* $f\colon (X,\tau) \to (Y,\sigma)$ *be any function, then:* $X \backslash P_S^\gamma C(f) = \bigcup\{P_S^\gamma Bd(f^{-1}(V)) : V \in P_S^\beta O(Y, f(x))$ *for each* $x \in X\}$, *where* $P_S^\gamma C(f)$ *denotes the set of points at which* f *is* $P_S^{(\gamma,\beta)}$*-continuous function.*

Proof: Let $x \in P_S^\gamma C(f)$. Then there exists $V \in P_S^\beta O(Y, f(x))$ such that $f(U) \nsubseteq V$ for every $U \in P_S^\gamma O(X,x)$. Hence $U \cap X \backslash f^{-1}(V) \neq \phi$ for every $U \in P_S^\gamma O(X,x)$. Therefore, $x \in P_S^\gamma Cl(X \backslash f^{-1}(V))$. Then

$$\begin{aligned} x \in f^{-1}(V) \cap P_S^\gamma Cl(X \backslash f^{-1}(V)) &\subseteq P_S^\gamma Cl(f^{-1}(V)) \cap P_S^\gamma Cl(X \backslash f^{-1}(V)) \\ &= P_S^\gamma Bd(f^{-1}(V)). \end{aligned}$$

Hence $X \backslash P_S^\gamma C(f) \subseteq \bigcup\{P_S^\gamma Bd(f^{-1}(V)) : V \in P_S^\beta O(Y, f(x))$ for each $x \in X\}$.

Conversely, let $x \notin X \backslash P_S^\gamma C(f)$. Then for each $V \in P_S^\beta O(Y, f(x))$, $f^{-1}(V) \in P_S^\gamma O(X,x)$. So $x \in P_S^\gamma Int(f^{-1}(V))$ and hence $x \notin P_S^\gamma Bd(f^{-1}(V))$ for every $V \in P_S^\beta O(Y, f(x))$. Therefore, $X \backslash P_S^\gamma C(f) \supseteq \bigcup\{P_S^\gamma Bd(f^{-1}(V)) : V \in P_S^\beta O(Y, f(x))$ for each $x \in X\}$.

Theorem 3.8. *If an injective function $f\colon (X,\tau)\to(Y,\sigma)$ is $P_S^{(\gamma,\beta)}$-continuous and the space (Y,σ) is P_S^{β}-T_2, then the space (X,τ) is P_S^{γ}-T_2.*

Proof: Let x_1 and x_2 be any distinct points of a space (X,τ). Since f is an injective function and (Y,σ) is P_S^{β}-T_2. Then there exist two P_S^{β}-open sets G_1 and G_2 in Y such that $f(x_1)\in G_1$, $f(x_2)\in G_2$ and $G_1\cap G_2=\phi$. Since f is $P_S^{(\gamma,\beta)}$-continuous, then $f^{-1}(G_1)$ and $f^{-1}(G_2)$ are P_S^{γ}-open sets in (X,τ) containing x_1 and x_2 respectively. Hence $f^{-1}(G_1)\cap f^{-1}(G_2)=\phi$. Therefore, (X,τ) is P_S^{γ}-T_2.

Theorem 3.9. *If an injective function $f\colon (X,\tau)\to(Y,\sigma)$ is $P_S^{(\gamma,\beta)}$-continuous and the space (Y,σ) is P_S^{β}-T_n, then the space (X,τ) is P_S^{γ}-T_n for $n\in\{0,1\}$.*

Proof: The proof is similar to Theorem 3.8.

Proposition 3.10. *A function $f\colon (X,\tau)\to(Y,\sigma)$ is $P_S^{(\gamma,\beta)}$-continuous if the graph function $g\colon (X,\tau)\to(X\times Y,\tau\times\sigma)$ is $P_S^{(\gamma,\lambda)}$-continuous and the mapping $\lambda\colon P_SO(X\times Y)\to P(X\times Y)$ is an operation on $P_SO(X\times Y)$.*

Proof: Let $x\in X$ and V be a P_S^{β}-open set of Y containing $f(x)$. Then $X\times V$ is a P_S^{λ}-open set of $X\times Y$ containing $g(x)$. Since g is $P_S^{(\gamma,\lambda)}$-continuous. So $g^{-1}(X\times V)=f^{-1}(V)$ is a P_S^{γ}-open set containing x. This shows that f is $P_S^{(\gamma,\beta)}$-continuous.

Definition 3.11. *A function $f\colon (X,\tau)\to(Y,\sigma)$ is called $P_S^{(\gamma,\beta)}$-open if $f(V)\in P_S^{\beta}O(Y,\sigma)$, for each $V\in P_S^{\gamma}O(X,\tau)$.*

Theorem 3.12. *Assume that a function $f\colon (X,\tau)\to(Y,\sigma)$ is surjective $P_S^{(\gamma,\beta)}$-open. If (X,τ) is P_S^{γ}-T_n, then (Y,σ) is P_S^{β}-T_n for $n\in\{0,1,2\}$.*

Proof: It is enough to proof for one case of n (say $n=2$) since the proofs of the other cases are similar.

Let f be a surjective $P_S^{(\gamma,\beta)}$-open function and $y_1,y_2\in Y$ such that $y_1\neq y_2$. Then there exist distinct points x_1 and x_2 of X such that $f(x_1)=y_1$ and $f(x_2)=y_2$. Since (X,τ) is P_S^{γ}-T_2 space, there exist P_S^{γ}-open sets V_1 and V_2 in X such that $x_1\in V_1$, $x_2\in V_2$ and $V_1\cap V_2=\phi$. Since f is $P_S^{(\gamma,\beta)}$-open, then $f(V_1)$ and $f(V_2)$ are P_S^{β}-open sets in (Y,σ) such that $y_1=f(x_1)\in f(V_1)$ and $y_2=f(x_2)\in f(V_2)$. This implies that $f(V_1)\cap f(V_2)=\phi$. Hence (Y,σ) is P_S^{β}-T_2.

Definition 3.13. *A function $f\colon (X,\tau)\to(Y,\sigma)$ is called $P_S^{(\gamma,\beta)}$-closed if the image of each P_S^{γ}-closed set of X is P_S^{β}-closed in Y.*

Theorem 3.14. *A surjective function $f\colon (X,\tau)\to(Y,\sigma)$ is $P_S^{(\gamma,\beta)}$-closed if and only if for each subset S of Y and each P_S^{γ}-open set O in X containing $f^{-1}(S)$, there exists a P_S^{β}-open set R in Y containing S such that $f^{-1}(R)\subseteq O$.*

Proof: Suppose that f is $P_S^{(\gamma,\beta)}$-closed function and let O be an P_S^{γ}-open set in X containing $f^{-1}(S)$, where S is any subset in Y. Then $f(X\backslash O)$ is P_S^{β}-open set in Y. If we put $R = Y\backslash f(X\backslash O)$. Then R is P_S^{β}-closed set in Y such that $S \subseteq R$ and $f^{-1}(R) \subseteq O$.

Conversely, let F be P_S^{γ}-closed set in X. Let $S = Y\backslash f(F) \subseteq Y$. Then $f^{-1}(S) \subseteq X\backslash F$ and $X\backslash F$ is P_S^{γ}-open set in X. By hypothesis, there exists a P_S^{β}-open set R in Y such that $S = Y\backslash f(F) \subseteq R$ and $f^{-1}(R) \subseteq X\backslash F$. For $f^{-1}(R) \subseteq X\backslash F$ implies $R \subseteq f(X\backslash F) \subseteq Y\backslash f(F)$. Hence $R = Y\backslash f(F)$. Since R is P_S^{β}-open set in Y. Then $f(F)$ is P_S^{β}-closed set in Y. Therefore, f is $P_S^{(\gamma,\beta)}$-closed function.

4 Functions with $P_S^{(\gamma,\beta)}$-Closed Graph

For a function $f\colon (X,\tau) \to (Y,\sigma)$, the subset $\{(x, f(x)) : x \in X\}$ of the product space $(X \times Y, \tau \times \sigma)$ is called the graph of f and is denoted by $G(f)$ [3]. In this section, we further investigate general operator approaches of closed graphs of functions.

Definition 4.1. *The graph $G(f)$ of $f\colon (X,\tau) \to (Y,\sigma)$ is called $P_S^{(\gamma,\beta)}$-closed if for each $(x,y) \notin G(f)$, there exist P_S^{γ}-open set $U \subseteq X$ and P_S^{β}-open set $V \subseteq Y$ containing x and y, respectively, such that $(U \times V) \cap G(f) = \phi$.*

The proof of the following lemma follows directly from the above definition.

Lemma 4.2. *The graph $G(f)$ of $f\colon (X,\tau) \to (Y,\sigma)$ is $P_S^{(\gamma,\beta)}$-closed if for each $(x,y) \notin G(f)$, there exist $U \in P_S^{\gamma}O(X,x)$ and $V \in P_S^{\beta}O(Y,y)$ such that $f(U) \cap V = \phi$.*

Theorem 4.3. *If a function $f\colon (X,\tau) \to (Y,\sigma)$ is $P_S^{(\gamma,\beta)}$-continuous and the space Y is P_S^{β}-T_2, then $G(f)$ is $P_S^{(\gamma,\beta)}$-closed.*

Proof: Let $(x,y) \notin G(f)$. This implies that $f(x) \neq y$. Since Y is P_S^{β}-T_2 space, then there exist two P_S^{β}-open sets V and W containing $f(x)$ and y respectively, such that $V \cap W = \phi$. Since f is $P_S^{(\gamma,\beta)}$-continuous function, so there exists a P_S^{γ}-open set U in X containing x such that $f(U) \subseteq V$. Therefore, $f(U) \cap W = \phi$ and hence by Lemma 4.2, $G(f)$ is $P_S^{(\gamma,\beta)}$-closed graph.

Theorem 4.4. *If $f\colon (X,\tau) \to (Y,\sigma)$ is sujective and has a $P_S^{(\gamma,\beta)}$-closed graph $G(f)$, then Y is P_S^{β}-T_1.*

Proof: Let $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. The surjectivity of f gives a $x_1 \in X$ such that $f(x_1) = y_1$. It follows that $(x_1, y_2) \notin G(f)$. The $P_S^{(\gamma,\beta)}$-closedness of $G(f)$, by Lemma 4.2 provides $U \in P_S^{\gamma}O(X,x_1)$ and $V \in P_S^{\beta}O(Y,y_2)$ such that $f(U) \cap V = \phi$, so we have a P_S^β-open set $V \subseteq Y$ such that $y_2 \in V$ but $y_1 \notin V$. Similarly, we get a P_S^β-open set in Y containing y_1, but does not contain y_2. Therefore, Y is P_S^β-T_1.

Theorem 4.5. *If $f\colon (X,\tau) \to (Y,\sigma)$ is $P_S^{(\gamma,\beta)}$-open surjective with a $P_S^{(\gamma,\beta)}$-closed graph $G(f)$, then Y is P_S^β-T_2.*

Proof: Let y_1 and y_2 be two distinct points in Y and f is surjective, then there is a $x \in X$ such that $f(x) = y_1$. Then we have $(x, y_2) \notin G(f)$. Since the graph $G(f)$ is $P_S^{(\gamma,\beta)}$-closed and by Lemma 4.2, there exist P_S^γ-open set $U \subseteq X$ and P_S^β-open set $V \subseteq Y$ such that $x \in U$, $y_2 \in V$ and $f(U) \cap V = \phi$. Since f is $P_S^{(\gamma,\beta)}$-open, then $f(U)$ is P_S^β-open in Y containing $f(x)$. This shows that Y is P_S^β-T_2 space.

Theorem 4.6. *A space X is P_S^γ-T_2 if and only if the identity function $id\colon (X,\tau) \to (X,\tau)$ has a $P_S^{(\gamma,\gamma)}$-closed graph $G(id)$.*

Proof: Let X be a P_S^γ-T_2 space. Since $id\colon (X,\tau) \to (X,\tau)$ is $P_S^{(\gamma,\gamma)}$-continuous. Then by Theorem 4.3, $G(id)$ is $P_S^{(\gamma,\gamma)}$-closed graph.

Conversely, let $G(id)$ be a $P_S^{(\gamma,\gamma)}$-closed graph. Then the surjectivity $P_S^{(\gamma,\gamma)}$-openness of id. This implies from Theorem 4.5 that X is P_S^γ-T_2.

Theorem 4.7. *If $f\colon (X,\tau) \to (Y,\sigma)$ is an injective function and has a $P_S^{(\gamma,\beta)}$-closed graph $G(f)$, then X is P_S^γ-T_1.*

Proof: Since f is injective for any pair of distinct points x_1, $x_2 \in X$, $f(x_1) \neq f(x_2)$. So $(x_1, f(x_2)) \notin G(f)$ and $G(f)$ is $P_S^{(\gamma,\beta)}$-closed graph, then by Lemma 4.2, gives $U \in P_S^\gamma O(X)$ and $V \in P_S^\beta O(Y)$ containing x_1 and $f(x_2)$ respectively, such that $f(U) \cap V = \phi$. This means that $f(x_2) \notin f(U)$ and hence $x_2 \notin U$. Similarly, we will get a P_S^γ-open set (say P) such that $x_2 \in P$ but $x_1 \notin P$. Thus, X is P_S^γ-T_1.

Theorem 4.8. *If an injective function $f\colon (X,\tau) \to (Y,\sigma)$ is $P_S^{(\gamma,\beta)}$-continuous and has a $P_S^{(\gamma,\beta)}$-closed graph $G(f)$, then X is P_S^γ-T_2.*

Proof: Let $x_1, x_2 \in X$ such that $x_1 \neq x_2$. By the injectivity of f gives $f(x_1) \neq f(x_2)$. Now $(x_1, f(x_2)) \notin G(f)$. Since the graph $G(f)$ is $P_S^{(\gamma,\beta)}$-closed. So by Lemma 4.2, there exist P_S^γ-open set U and P_S^β-open set V such that $x_1 \in U$, $f(x_2) \in V$ and $f(U) \cap V = \phi$. This implies that $U \cap f^{-1}(V) = \phi$ and $f^{-1}(V)$ is P_S^γ-open set in X containing x_2 since f is $P_S^{(\gamma,\beta)}$-continuous. Therefore, X is P_S^γ-T_2.

Remark 4.9. *The subsets $A, B \subseteq P_S^\gamma O(X)$ if and only if $A \times B \in P_S^\gamma O(X \times X)$.*

Theorem 4.10. *Let $f, g\colon (X,\tau) \to (Y,\sigma)$ be functions. If $G(f)$ is $P_S^{(\gamma,\beta)}$-closed and g is $P_S^{(\gamma,\beta)}$-continuous, then the subset*

$$\{(x_1, x_2) \in X \times X : f(x_1) = g(x_2)\} \text{ is } P_S^{\gamma}\text{-closed in } X \times X.$$

Proof: Let $A = \{(x_1, x_2) \in X \times X : f(x_1) = g(x_2)\}$. Suppose $(x_1, x_2) \notin A$. Then $f(x_1) \neq g(x_2)$ and hence $(x_1, g(x_2)) \notin G(f)$. Since $G(f)$ is $P_S^{(\gamma,\beta)}$-closed graph, by Lemma 4.2, there exist P_S^{γ}-open set $U \subseteq X$ and P_S^{β}-open set $V \subseteq Y$ such that $x_1 \in U$, $g(x_2) \in V$ and $f(U) \cap V = \phi$. Since g is $P_S^{(\gamma,\beta)}$-continuous, then by Theorem 3.3 (2), there exists a P_S^{γ}-open set $W \subseteq X$ containing x_2 such that $g(W) \subseteq V$ and hence $f(U) \cap g(W) = \phi$. Thus, we obtain $(U \times W)$ $\cap\ A = \phi$ and $(x_1, x_2) \in U \times W \in P_S^{\gamma}O(X \times X)$. Therefore, by Theorem 2.2, $(x_1, x_2) \notin P_S^{\gamma}Cl(A)$ and hence A is P_S^{γ}-closed set in $X \times X$.

Theorem 4.11. *If a function $f\colon (X,\tau) \to (Y,\sigma)$ is $P_S^{(\gamma,\beta)}$-continuous and the space (Y,σ) is P_S^{β}-T_2, then the set*
$A = \{(x_1, x_2) \in X \times X : f(x_1) = f(x_2)\}$ is P_S^{γ}-closed in the product space $X \times X$.

Proof: Since f is $P_S^{(\gamma,\beta)}$-continuous and (Y,σ) is P_S^{β}-T_2 space. Then by Theorem 4.3, $G(f)$ is $P_S^{(\gamma,\beta)}$-closed graph and hence by Theorem 4.10, the set A is P_S^{γ}-closed in the product space $X \times X$.

Theorem 4.12. *If the functions $f, g\colon (X,\tau) \to (Y,\sigma)$ are $P_S^{(\gamma,\beta)}$-continuous and the space (Y,σ) is P_S^{β}-T_2, then the set*

$$A = \{x \in X : f(x) = g(x)\} \text{ is } P_S^{\gamma}\text{-closed in } X.$$

Proof: The proof is follows from Theorem 4.11.

Theorem 4.13. *If $f\colon (X,\tau) \to (Y,\sigma)$ is a function with the $P_S^{(\gamma,\beta)}$-closed graph, then for each $x \in X$, $f(x) = \bigcap\{P_S^{\beta}Cl(f(U)) : U \in P_S^{\gamma}O(X,x)\}$.*

Proof: Suppose that the theorem is false. Then there exists a poiny $y \in Y$ such that $y \neq f(x)$ and $y \in \bigcap\{P_S^{\beta}Cl(f(U)) : U \in P_S^{\gamma}O(X,x)\}$. This implies that $y \in P_S^{\beta}Cl(f(U))$ for every $U \in P_S^{\gamma}O(X,x)$. So by Theorem 2.2, $f(U) \cap G \neq \phi$ for every $G \in P_S^{\beta}O(Y,y)$. Which is contradiction to the hypothesis that f is a function with $P_S^{(\gamma,\beta)}$-closed graph. Hence the theorem holds.

Theorem 4.14. *If $f\colon (X,\tau) \to (Y,\sigma)$ is $P_S^{(\gamma,\beta)}$-continuous and $G(g)$ is $P_S^{(\beta,\alpha)}$-closed graph of $g\colon (Y,\sigma) \to (Z,\rho)$, then $G(g \circ f)$ is $P_S^{(\gamma,\alpha)}$-closed graph of $g \circ f\colon (X,\tau) \to (Z,\rho)$ for any operation $\alpha\colon P_SO(Z) \to P(Z)$ on $P_SO(Z)$.*

Proof: Let $(x, z) \in X \times Z$ such that $z \neq g(y)$ where $y = f(x)$. Then $(y, z) \notin G(g)$. Since $G(g)$ is $P_S^{(\beta,\alpha)}$-closed graph, by Lemma 4.2, there exist $V \in P_S^{\beta}O(Y)$ and $W \in P_S^{\alpha}O(Z)$ containing y and z respectively, such that $g(V) \cap W = \phi$. Since f is $P_S^{(\gamma,\beta)}$-continuous, so there exists a P_S^{γ}-open set U in X containing x such that $f(U) \subseteq V$ which implies that $(g \circ f)(U) = g(f(U)) \subseteq g(V)$ and hence $g(f(U)) \cap W = \phi$. Thus, by Lemma 4.2, $G(g \circ f)$ is $P_S^{(\gamma,\alpha)}$-closed graph of $g \circ f$.

Definition 4.15. *A topological space (X, τ) is said to be P_S^{γ}-connected if it cannot be expressed as the union of two disjoint nonempty P_S^{γ}-open sets of X.*

Theorem 4.16. *If (X, τ) is P_S^{γ}-connected space and $f\colon (X, \tau) \to (Y, \sigma)$ is $P_S^{(\gamma,\beta)}$-continuous and has a $P_S^{(\gamma,\beta)}$-closed graph $G(f)$, then f is constant.*

Proof: Suppose that f is not constant. There exist disjoint points $x, y \in X$ such that $f(x) \neq f(y)$. Then $(y, f(x)) \notin G(f)$, but $G(f)$ is $P_S^{(\gamma,\beta)}$-closed graph, so by Lemma 4.2, we have $U \in P_S^{\gamma}O(X, y)$ and $V \in P_S^{\beta}O(Y, f(x))$ such that $f(U) \cap V = \phi$. Since f is $P_S^{(\gamma,\beta)}$-continuous, by Theorem 3.3 (2), there exists $W \in P_S^{\gamma}O(X, x)$ such that $f(W) \subseteq V$. Since U and W are disjoint P_S^{γ}-open sets of (X, τ). It follows that (X, τ) is not P_S^{γ}-connected. Therefore, f is constant.

Proposition 4.17. *If $f\colon (X, \tau) \to (Y, \sigma)$ is $P_S^{(\gamma,\beta)}$-continuous surjective function and (X, τ) is P_S^{γ}-connected space, then (Y, σ) is P_S^{β}-connected space.*

Proof: Suppose that (Y, σ) is not P_S^{β}-connected space. Then there exist disjoint P_S^{β}-open sets A and B of (Y, σ) such that $A \cup B = Y$. Since f is $P_S^{(\gamma,\beta)}$-continuous surjective, $f^{-1}(A)$ and $f^{-1}(B)$ are P_S^{γ}-open sets in (X, τ). Moreover, $f^{-1}(A) \cup f^{-1}(B) = X$, $f^{-1}(A) \neq \phi$ and $f^{-1}(B) \neq \phi$. This shows that (X, τ) is not P_S^{γ}-connected, which is a contradiction to the assumption that (X, τ) is P_S^{γ}-connected. By contradiction, (Y, σ) is P_S^{β}-connected.

5 Conclusion

In this paper, we introduced some classes of functions via the operation γ on $P_SO(X)$ and the set P_S^{γ}-open. Moreover, some basic properties of functions with a $P_S^{(\gamma,\beta)}$-closed graph had been studied.

References

[1] B.A. Asaad, Operation approaches on P_S-open sets and its separation axioms, *Journal University of Zakho*, 4(A)(2) (2016), 272-279.

[2] S.G. Crossley and S.K. Hildebrand, Semi-closure, *Texas Journal of Science*, 22(1971), 99-112.

[3] T. Husain, *Topology and Maps*, Plenum Press, New York, (1977).

[4] A.B. Khalaf and B.A. Asaad, P_S-open sets and P_S-continuity in topological spaces, *J. Duhok Univ.*, 12(2) (2009), 183-192.

[5] N. Levine, Semi-open sets and semi-continuity in topological spaces, *Amer. Math. Monthly*, 70(1) (1963), 36-41.

[6] A.S. Mashhour, M.E.A. El-Monsef and S.N. El-Deeb, On precontinuous and week precontinuous mappings, *Proc. Math. Phys. Soc. Egypt*, 53(1982), 47-53.