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# Characterizations of Quaternionic Mannheim Curves in Semi-Euclidean Space $\mathbf{E}_2^4$

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**Abstract**

*In this paper, we define quaternionic Mannheim curves in the semi-Euclidean space $E_2^4$ and we give some characterizations of them.*

**Keywords:** *Semi-Euclidean spaces, Semi-quaternionic characterizations, Quaternion algebra, Mannheim partner curves, Frenet frame.*

## 1 Introduction

The quaternion, on the 16th October 1843, was defined by Hamilton. His initial attempt is generalize the complex numbers by introducing a three-dimensional object failed in the sense that the algebra he constructed for these three-dimensional object did not have the desired properties. Hamilton discovered that the appropriate generalization is one in which the scalar (real) axis is left unchanged whereas the vector (imaginary) axis is supplemented by adding two further vectors axes.

In differential geometry, for the theory of space curves, the associated curve, the curves for which at the corresponding points of them one of the Frenet vectors of a curve coincides with the one of the Frenet vectors of the other curve have an important role for the characterizations of space curves,see [3]. The well-known examples of such curves are Bertrand curves and Mannheim curves. Space curves of which principal normals are the binormals of another curve are called Mannheim curves in the three dimensional Euclidean space. The notion of Mannheim curves was discovered by A.

Mannheim in 1878. The articles concerning the Mannheim curves are rather few. O. Tigano obtained the locus of Mannheim curves in the three dimensional Euclidean space [7].

Mannheim partner curves in the three dimensional Euclidean space and the three dimensional Minkowski space are studied and the necessary and sufficient conditions for the Mannheim partner curves are obtained, [4,6 ]. Recently, Mannheim curves are generalized and some characterizations and examples of generalized Mannheim curves in the four dimensional Euclidean space are introduced, [5].

For the study of a quaternionic curve, In 1987, the Serret-Frenet formulas for a quaternionic curve in $E^3$ and $E^4$ defined by Bharathi and Nagaraj [11] and then , Serret-Frenet formulas for quaternionic curves and quaternionic inclined curves have been defined in semi-Euclidean space by Çöken and Tuna, [1].

In 2007, the definition of Mannheim curves in Euclidean 3-space is given by H. Liu and F. Wang, [4]. A new kind of slant helix are defined in Euclidean space $E^4$ and semi-Euclidean space $E_2^4$, [8, 9]. They are called quaternionic $B_2$-slant helix in Euclidean space $E^4$ and semi- real quaternionic $B_2$-slant helix in semi-Euclidean space $E_2^4$. Güngör and Tosun gave some characterizations of quaternionic rectifying curve, [10]. Characterizations of the quaternionic Mannheim curves are defined in Euclidean space $E^4$, [14].

We can characterize some curves via relations between the Frenet vectors of them. For instance, Mannheim curve is a special curve and it is characterized by using the Frenet vectors of it and its Mannheim curve couple in the semi-Euclidean space $E_2^4$.

## 2 Preliminaries

Let $Q_v$ be the four-dimensional vector space over a field $v$ whose characteristic greater than 2. Let $e_i$ $(1 \leq i \leq 4)$ be a basis for the vector space. Let the rule of multiplication on $Q_v$ be defined on $e_i$ and extended to the whole of the vector space distributivity as in, [2]:

A semi-real quaternion is defined with $q = ae_1 + be_2 + ce_3 + d$, such that

1. $e_i \times e_i = -\varepsilon(e_i),\ 1 \leq i \leq 3$
2. $e_i \times e_j = \varepsilon(e_i)\varepsilon(e_j)e_k$ in $E_1^3$
3. $e_i \times e_j = -\varepsilon(e_i)\varepsilon(e_j)e_k$ in $E_2^4$.

And $(ijk)$ is an even permutation of (123) in the semi-Euclidean space $E_2^4$. Notice here that we define the set of all semi-real quaternions by $Q_v$, where $v = 1, 2$.

$$Q_v = \left\{ q \mid q = a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3 + d\ ; a, b, c, d \in \mathbb{R} \text{ and } e_1, e_2, e_3 \in \mathbb{R}^3 \right\}.$$

If $e_i$ is a spacelike or timelike vector, then $\varepsilon(e_i) = +1$ or $-1$, respectively.

We put $S_q = d$ and $V_q = ae_1 + be_2 + ce_3$. Then a quaternion $q$ can be written as $q = S_q + V_q$,where $S_q$ and $V_q$ are the scalar part and vectorial part of $q$, respectively. Now, we can give the product of two quaternions with following equation

$$p \times q = S_p S_q + \langle\, V_p, V_q \,\rangle + S_p V_q + S_q V_p + V_p \wedge V_q.$$

For every $p, q \in Q_v$, where we use the dot and cross products in semi-Euclidean space $E_1^3$. We see that the quaternionic product contains all the products of semi-Euclidean space $E_2^4$, [12]. There is a unique involutory anti-automorphism of the quaternion algebra, denoted by the symbol $\gamma$ and defined as follows:

$$\gamma q = -ae_1 - be_2 - ce_3 \text{for every } q = ae_1 + be_2 + ce_3 + d \in Q_v$$

which is called the Hamiltonian conjugation. This defines the symmetric nondegenerate valued bilinear form $h$ as follows:

$$h(p,q) = \frac{1}{2}\left[\varepsilon(p)\varepsilon(\gamma q)(p \times \gamma q) + \varepsilon(q)\varepsilon(\gamma p)(q \times \gamma p)\right] \text{ for } \ E_1^3,$$

$$h(p,q) = \frac{1}{2}\left[-\varepsilon(p)\varepsilon(\gamma q)(p \times \gamma q) - \varepsilon(q)\varepsilon(\gamma p)(q \times \gamma p)\right] \text{ for } E_2^4.$$

The norm of semi-real quaternion $q$ is given by

$$\|q\|^2 = |h_v(q,q)| = |\varepsilon(q)(q \times \gamma q)| = \left|-a^2 - b^2 + c^2 + d^2\right|$$

where if $h_v(p,q) = 0$ then $p$ and $q$ are called $h$-orthogonal.

The concept of a spatial quaternion will be made use throughout our work. $q$ is called a spatial quaternion whenever $q + \gamma q = 0$, [1, 13].

# 3 The Serret-Frenet Formulae for Quaternionic Curves in Semi-Euclidean Spaces

**Theorem 3.1** *The three-dimensional semi- Euclidean space $E_1^3$ is identified with the space of spatial quaternions $\{p \in Q_v \mid p + \gamma p = 0\}$ in an obvious manner. Let I=$[0,1]$ denote the unit interval of the real line $\mathbb{R}$ and*

$$\begin{aligned} \alpha \quad &: \quad I \subset \mathbb{R} \longrightarrow Q_v \\ s \quad &\longrightarrow \quad \alpha(s) = \sum_{i=1}^{3} \alpha_i(s) e_i \ , 1 \le i \le 3 \end{aligned}$$

*be the parameter along the smooth curve. Let* $\{t, n, b\}$ *be the Frenet trihedron of the differentiable semi-Euclidean space curve in the semi-Euclidean space* $E_1^3$. *Then Frenet equations are*

$$\left\{ \begin{array}{c} t' = \varepsilon_n k n \\ n' = \varepsilon_t \left[ \tau \varepsilon_n b - k t \right] \\ b' = -\varepsilon_b \tau n, \end{array} \right\}$$

*where* $h(t,t) = \varepsilon_t$, $h(n,n) = \varepsilon_n$ *and* $h\ (b,b) = \varepsilon_b$. *We may express Frenet formulae of the Frenet trihedron in the matrix form [13]:*

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_n k & 0 \\ -\varepsilon_t k & 0 & \varepsilon_n \tau \\ 0 & -\varepsilon_b \tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}.$$

**Theorem 3.2** *The four-dimensional semi-Euclidean spaces in* $E_2^4$ *is identified with the spaces of unit quaternions. Let*

$$\begin{array}{ccl} \alpha & : & I \subset \mathbb{R} \longrightarrow Q_v \\ s & \longrightarrow & \alpha(s) = \sum_{i=1}^{4} \alpha_i(s) e_i \ , 1 \leq i \leq 4, \ e_4 = 1 \end{array}$$

*be a smooth curve in* $E_2^4$ *with non-zero curvatures* $\{K,\ k,\ \tau - K\}$. *Let the parameter* $s$ *be chosen such that the tangent* $T(s) = \alpha'(s)$ *has unit magnitude. Let* $\{T; N; B; B_1\}$ *be the Frenet apparatus of the differentiable semi-Euclidean space curve in the semi-Euclidean spaces* $R_2^4$. *Then Frenet formulas are given by*

$$\left\{ \begin{array}{c} T'(s) = \varepsilon_N K(s) N(s) \\ N'(s) = k(s) \varepsilon_n B(s) - K(s) \varepsilon_N \varepsilon_t T(s) \\ B'(s) = -k(s) \varepsilon_t N(s) + \varepsilon_n \left[ \tau - K \varepsilon_T \varepsilon_t \varepsilon_N \right] B_1(s) \\ B_1'(s) = -\varepsilon_b \left[ \tau - K \varepsilon_T \varepsilon_t \varepsilon_N \right] B(s) \end{array} \right\}$$

*where* $K = \varepsilon_N \ \|T'(s)\|$ *and* $\|N'(s)\|^2 = |\varepsilon_N|$. *We may express Frenet formulae of the Frenet trihedron in the matrix form [14]:*

$$\begin{bmatrix} T' \\ N' \\ B' \\ B_1' \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_N K & 0 & 0 \\ -K \varepsilon_N \varepsilon_t & 0 & k \varepsilon_n & 0 \\ 0 & -k \varepsilon_t & 0 & \varepsilon_n \left[ \tau - K \varepsilon_T \varepsilon_t \varepsilon_N \right] \\ 0 & 0 & -\varepsilon_b \left[ \tau - K \varepsilon_T \varepsilon_t \varepsilon_N \right] & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \\ B_1 \end{bmatrix}.$$

# 4 Characterizations of Quaternionic Mannheim Curves in $\mathbf{E}_2^4$

In this section, we define quaternionic Mannheim curves in the semi-Euclidean space $E_2^4$ and investigate the properties of them.

**Definition 4.1** *A quaternionic curve $\alpha : I \subset R \longrightarrow E_2^4$ is a quaternionic Mannheim curve if there is a quaternionic curve $\alpha^* : I^* \subset R \longrightarrow E_2^4$ such that the first normal line at each point of $\alpha$ is included in the plane generated by the second normal line and the third normal line of $\alpha^*$ at the corresponding point under $\varphi$, where $\varphi$ is a bijective smooth function $\varphi{:}\alpha \longrightarrow \alpha^*$. In this case, the curve $\alpha^*$ is called the quaternionic Mannheim partner of $\alpha$.*

**Theorem 4.2** *The distance between corresponding points of a quaternionic Mannheim curve and of its quaternionic Mannheim partner curve with arc-length $s$ and $s^*$, respectively, in $Q_v$ is a constant.*

**Proof:** From the definition (1), quaternionic Mannheim partner curve $\alpha^*$ of $\alpha$ is given by the following equation $\alpha^*(s) = \alpha(s) + \lambda(s)N(s)$ for some smooth function $\lambda(s)$. A smooth function : $\phi : s \in I \longrightarrow \phi(s) = s^* \in I^*$ is defined by

$$\phi(s) = \int_0^s \left\| \frac{d\alpha(s)}{ds} \right\| ds = s^*,$$

where bijection $\varphi : \alpha \longrightarrow \alpha^*$ is defined by $\varphi(\alpha(s)) = \alpha^*(\phi(s))$. Because the second Frenet vector at each point of $\alpha$ is included the plane generated by the third Frenet vector and the fourth Frenet vector of $\alpha^*$ at corresponding point under $\varphi$, for each $s \in I$, the Frenet vector $N(s)$ is given by the linear combination of Frenet vectors $B^*(\phi(s))$ and $B_1^*(\phi(s))$ of $\alpha^*$, that is, we can write

$$N(s) = f_1(s)B^*(\phi(s)) + f_2 B_1^*(\phi(s)),$$

where $f_1(s)$ and $f_2(s)$ are smooth functions on $I$. So we can write,

$$\alpha^*(\phi(s)) = \alpha(s) + \lambda(s)N(s). \tag{1}$$

By taking the derivative of equation (1) with respect to $s$ and by applying the Frenet formulas, we get

$$(\alpha^*(\phi(s))' = \alpha'(s) + \lambda'(s)N(s) + \lambda(s)N'(s)$$

$$\left\{ \begin{array}{c} T^*(\phi(s))\phi'(s) = \\ (1 - K(s)\varepsilon_N\varepsilon_t\lambda(s))T(s) + \lambda'(s)N(s) + \lambda(s)k(s)\varepsilon_n B(s) \end{array} \right\} \tag{2}$$

where $\acute{\phi}(s) = \frac{ds^*}{ds}$. f we consider

$$h(T^*(\phi(s)),\ f_1(s)B^*(\phi(s)) + f_2B_1^*(\phi(s))) = 0$$

and equation (2), then we have $\acute{\lambda}(s) = 0$. This means that $\lambda$ is a nonzero constant. On the other hand, from the distance function between two points,we have $d(\alpha^*(s^*), \alpha(s)) = |\lambda|$.

Namely, $d(\alpha^*(s^*), \alpha(s))$ is a constant. This completes the proof.

**Theorem 4.3** *If a quaternionic curve $\alpha : I \subset R \longrightarrow E_2^4$ in $Q_v$ is a quaternionic Mannheim curve, then the principal curvature $K(s)$ and the torsion $k(s)$ of $\alpha$ satisfy the equality:*

$$K(s) = \lambda\varepsilon_t \left\{ \frac{\varepsilon_n}{\varepsilon_N} k^2(s) + K^2(s) \right\} \tag{3}$$

*where $\lambda$ is nonezero constant.*

**Proof:** Since $\lambda$ is a non-zero constant, equation (2) is reduced to

$$\left\{ \begin{array}{c} \acute{\phi}(s)T^*(\phi(s)) \\ = (1 - \lambda K(s)\varepsilon_N\varepsilon_t)T(s) + (\lambda k(s)\varepsilon_n)B(s). \end{array} \right\} \tag{4}$$

By taking the derivative of the both sides of equation (4) with respect to s , we have

$$T^*(\varphi(s)) = \left\{ \begin{array}{c} \left(\frac{1-\lambda K(s)\varepsilon_N\varepsilon_t}{\acute{\varphi}(s)}\right)' T(s) + \\ \left(\frac{K(s)\varepsilon_N - \lambda k^2(s)\varepsilon_n\varepsilon_t - \lambda\varepsilon_t(K(s)\varepsilon_N)^2}{\acute{\varphi}(s)}\right) N(s) + \\ \left(\frac{\lambda(s)k(s)\varepsilon_n}{\varphi(s)}\right)' B(s) + \left(\frac{\varepsilon_n^2(\tau - K\varepsilon_t\varepsilon_T\varepsilon_N}{\acute{\varphi}(s)}\right) B_1(s). \end{array} \right\} \tag{5}$$

By the fact:

$$h(N^*(\phi(s)),\ f_1(s)B^*(\phi(s)) + f_2B_1^*(\phi(s))) = 0$$

we have that coefficient of $N$ in the above equation is zero, that is,

$$\lambda\varepsilon_t\varepsilon_n k^2(s) + \varepsilon_N K(s) + \lambda\varepsilon_t K^2(s) = 0$$

Thus, we have

$$K(s) = \lambda\varepsilon_t \left\{ \frac{\varepsilon_n}{\varepsilon_N} k^2(s) + K^2(s) \right\}$$

for $s \in I$. This completes the proof.

**Theorem 4.4** *Let $\alpha : I \subset \mathbb{R} \longrightarrow E_2^4$ be quaternionic curve with arc-length $s$. For non-constant principal curvature function $K(s)$ and the torsion $k(s)$ of a quaternionic curve $\alpha$ in $Q_v$, if $\alpha$ satisfies the equation (3), then a curve*

$$\alpha^*(s) = \alpha(s) + \lambda N(s)$$

*is a quaternionic Mannheim partner curve $\alpha$.*

**Proof:** Let $s^*$ be the arc-length of a curve $\alpha^*$. That is, $s^*$ is given the following form:

$$s^* = \int_0^s \left\| \frac{d\alpha(s)}{ds} \right\| ds$$

for $s \in I$. We can write a smooth function $\phi : s \in I \longrightarrow \phi(s) = s^* \in I^*$. By the assumption of the curvature function $K(s)$ and $k(s)$, we have

$$\acute{\phi}(s) = \sqrt{(1 - \lambda K(s)\varepsilon_N \varepsilon_t)^2 + (\lambda k \varepsilon_n)^2}$$

$$\acute{\phi}(s) = \sqrt{|1 - \lambda K(s)\varepsilon_N \varepsilon_t|}$$

for $s \in I$.Then we can write

$$\begin{aligned} \alpha^*(s^*) &= \alpha^*(\phi(s)) \\ &= \alpha(s) + \lambda N(s). \end{aligned}$$

for the quaternionic curve $\alpha^*$. Then, by taking the derivative of both sides of the above equation respect to with $s$, we have

$$T^*(\phi(s)) = \frac{1 - K(s)\varepsilon_N \varepsilon_t \lambda(s)}{\acute{\phi}(s)} T(s) + \frac{\lambda(s)k(s)\varepsilon_n}{\acute{\phi}(s)} B(s).$$

It follows

$$T^*(\varphi(s)) = \sqrt{1 - K(s)\varepsilon_N \varepsilon_t \lambda(s)}\, T(s) + \frac{\lambda(s)k(s)\varepsilon_n}{\sqrt{1 - K(s)\varepsilon_N \varepsilon_t \lambda(s)}} B(s). \tag{6}$$

After the differentiation of the above equation with respect to $s$ and by applying the Frenet formulas, we have

$$T^*(\varphi(s)) = \left\{ \begin{array}{c} (\sqrt{1 - K(s)\varepsilon_N \varepsilon_t \lambda(s)})' \, T(s) \\ + \left( \frac{K(s)\varepsilon_N - \lambda k^2(s)\varepsilon_n \varepsilon_t - \lambda \varepsilon_t + (K(s)\varepsilon_N)^2}{\sqrt{1 - K(s)\varepsilon_N \varepsilon_t \lambda(s)}} \right) N(s) \\ + \left( \frac{\lambda(s)k(s)\varepsilon_n}{\sqrt{1 - K(s)\varepsilon_N \varepsilon_t \lambda(s)}} \right)' B(s) \\ + \left( \frac{\varepsilon_n^2(\tau - K\varepsilon_t \varepsilon_T + \varepsilon_N)}{\sqrt{1 - K(s)\varepsilon_N \varepsilon_t \lambda(s)}} \right) B_1(s). \end{array} \right\} \tag{7}$$

From assumption,

$$\frac{\varepsilon_N K(s)(1-\lambda K(s)) - \lambda\varepsilon_t\varepsilon_n k^2(s)}{\sqrt{1-K(s)\varepsilon_N\varepsilon_t\lambda(s)}} = 0.$$

We find that the coefficient of $N(s)$ in the above equality vanishes. On the other hand, the vector $N^*(\phi(s))$ is given by linear combination of $T(s)$, $B(s)$ and $B_1(s)$ for each $s \in I$. And the vector $T^*(\phi(s))$ is given by linear combination of $T(s)$ and $B(s)$ for each $s \in I$ in the Eq. (6). Since the curve $\alpha^*$ is quaternionic curve in $E_2^4$, the vector $N(s)$ is given by linear combination of $B^*(s^*)$ and $B_1^*(s^*)$. Thus, the second Frenet curve at each point of $\alpha$ is included in the plane generated the third Frenet vector and the fourth Frenet vector of $\alpha^*$ at corresponding point under $\varphi$, where a bijection $\varphi$ is defined by $\varphi(\alpha(s)) = \alpha^*(\phi(s))$. This completes the proof.

**Theorem 4.5** *Let $\alpha : I \subset \mathbb{R} \longrightarrow E_2^4$ be quaternionic curve with arc-length $s$ with nonezero bitorsion $(\tau - K\varepsilon_T\varepsilon_t\varepsilon_N)$. For a curve $\alpha^*$ in $Q_v$ , such that the first normal line of $\alpha$ is linearly dependent on the third normal line of $\alpha^*$ at the corresponding points $\alpha(s)$ and $\alpha^*(s^*)$ if and only if the principal curvature $K$ and the torsion $k$ of $\alpha$ are constants.*

**Proof:** Let the first normal line $N$ of $\alpha$ be linearly dependent on the third normal line $B_1^*$ of $\alpha^*$ at the corresponding points $\alpha(s)$ and $\alpha^*(s^*)$. Then the curve $\alpha^*$ is given by

$$\alpha^*(s^*) = \alpha(s) + \lambda N(s).$$

By taking the derivative of the above equation with respect to $s$ and applying Frenet formulas, we obtain

$$T^*(\phi(s))\phi\'(s) = \left\{ \begin{array}{c} (1-K(s)\varepsilon_N\varepsilon_t\lambda(s))T(s)+ \\ \lambda\'(s)N(s) + \lambda(s)k(s)\varepsilon_n B(s). \end{array} \right\}$$

Since $N = \pm\, B_1^*$ , $< T^*(s^*), N(s) >= 0$ which implies from (2) we obtain $\lambda\'(s) = 0$. Thus, equation (2) becomes equation (4), by taking differentiation both sides of equation (4) with respect to $s$, we have

$$T^*(\phi(s)) = \left\{ \begin{array}{c} \left(\frac{1-\lambda K(s)\varepsilon_N\varepsilon_t}{\phi\'(s)}\right)\' T(s)+ \\ \left(\frac{K(s)\varepsilon_N - \lambda k^2(s)\varepsilon_n\varepsilon_t - \lambda\varepsilon_t(K(s)\varepsilon_N)^2}{\phi\'(s)}\right) N(s)+ \\ \left(\frac{\lambda(s)k(s)\varepsilon_n}{\phi\'(s)}\right)\' B(s) + \left(\frac{\varepsilon_n^2(\tau - K\varepsilon_t\varepsilon_T\varepsilon_N}{\phi\'(s)}\right) B_1(s). \end{array} \right\}$$

Since $N = \pm\, B_1^*$ and $< N^*(s^*), B_1^*(s) >= 0$ . We have from above equation,

$$\lambda = \frac{K(s)\varepsilon_N}{\varepsilon_t(\varepsilon_n k(s)^2 + \varepsilon_N K(s)^2)}.$$

On the other hand, the differentiation of equation (5) with respect to $s$ gives

$$\{\phi'(s)T^*(\phi(s)) =\}$$

$$\left\{\begin{array}{c}\left(\frac{1-\lambda K(s)\varepsilon_N\varepsilon_t}{\phi'(s)}\right)'T(s)+\left(\frac{1-\lambda K(s)\varepsilon_N\varepsilon_t}{\phi'(s)}\right)'T'(s)+\\ \left(\frac{K(s)\varepsilon_N-\lambda k^2(s)\varepsilon_n\varepsilon_t-\lambda\varepsilon_t(K(s)\varepsilon_N)^2}{\phi'(s)}\right)'N(s)+\\ \left(\frac{K(s)\varepsilon_N-\lambda k^2(s)\varepsilon_n\varepsilon_t-\lambda\varepsilon_t(K(s)\varepsilon_N)^2}{\phi'(s)}\right)N'(s)+\left(\frac{\lambda(s)k(s)\varepsilon_n}{\phi'(s)}\right)'B(s)+\\ \left(\frac{\lambda(s)k(s)\varepsilon_n}{\phi'(s)}\right)'B'(s)+\left(\frac{\varepsilon_n^2(\tau-K\varepsilon_t\varepsilon_T\varepsilon_N}{\phi'(s)}\right)'B_1(s)+\left(\frac{\varepsilon_n^2(\tau-K\varepsilon_t\varepsilon_T\varepsilon_N}{\phi'(s)}\right)B_1'(s).\end{array}\right\}$$

As $< N(s), T^*\ (s^*) >= 0$ and $< N(s), B_1^*\ (s) >= 0$ , we have from above equation,

$$\left\{\begin{array}{c}(K'(s)\varepsilon_N-2\lambda k(s)k'(s)\varepsilon_n\varepsilon_t-2\lambda K(s)K'(s)\varepsilon_t\varepsilon_N^2\\ -\lambda k^2(s)\varepsilon_n\varepsilon_t+\varepsilon_N K(s)-\lambda K^2(s)\varepsilon_N^2)\varphi'\\ +(K(s)\varepsilon_N-\lambda k^2(s)\varepsilon_n\varepsilon_t-\lambda K^2(s)\varepsilon_t\varepsilon_N^2)\varphi'=0\end{array}\right\} \tag{8}$$

By differentiating equation (8) with respect to $s$, we get

$$K'(s)\varepsilon_N-2\lambda\varepsilon_t(k(s)k'(s)\varepsilon_n-K'(s)K(s)\varepsilon_N^2=0 \tag{9}$$

which implies from (5) and (9) we easily show that $K'(s)=0$, that is, $K(s)$ is a constant. Also, from (9) $k(s)$ is a constant. Conversely, let us suppose that the principal curvature $K$ and the torsion $k$ of a quaterninoic curve $\alpha$ in $Q_v$ are constants. Then

$$\frac{K\varepsilon_N}{\varepsilon_t(\varepsilon_n k^2+\varepsilon_N K^2)}$$

is a constant. This completes the proof.

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