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Some Applications of Fractional Calculus Operators to a Certain Subclass of Analytic Functions Defined by Integral Operator Involving Generalized Hypergeometric Function

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Abstract

The purpose of the present paper is to prove various distortion theorems for the fractional calculus of functions in the class $M_\lambda^{\zeta,\eta}[(\alpha_j, A_j)_{1,q}; (\mu_j, \rho_j)_{1,m}; (\beta_j, B_j)_{m+1,p}]$ *consisting of analytic and univalent functions with negative coefficients defined by integral operator involving generalized hypergeometric function. Furthermore, a distortion theorem for a fractional integral operator of functions in the class* $M_\lambda^{\zeta,\eta}[(\alpha_j, A_j)_{1,q}; (\mu_j, \rho_j)_{1,m}; (\beta_j, B_j)_{m+1,p}]$ *is shown.*

Keywords: *Analytic, Univalent, Fractional calculus, Hypergeometric function, Mittag-Leffler function, Noor integral operator.*

1 Introduction

Let H be the class of functions analytic in $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ and S be the subclass of H consisting of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1}$$

For complex parameters

$$\alpha_1, ..., \alpha_q(\frac{\alpha_j}{A_j} \neq 0, -1, -2, ...; j = 1, ..., q),$$

and

$$\beta_1, ..., \beta_p(\frac{\beta_j}{B_j} \neq 0, -1, -2, ...; j = 1, ..., p),$$

the Fox-Wright generalization ${}_q\Psi_p[z]$ of the hypergeometric ${}_qF_p$ function is defined by [4, 16, 13]

$$ {}_q\Psi_p \left[\begin{array}{l} (\alpha_1, A_1), ..., (\alpha_q, A_q); \\ (\beta_1, B_1), ..., (\beta_p, B_p); \end{array} z \right] =_q \Psi_p[\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,p}; z]$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma(\alpha_1 + kA_1)...\Gamma(\alpha_q + kA_q)z^k}{\Gamma(\beta_1 + kB_1)...\Gamma(\beta_p + kB_p)k!}$$

$$= \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{q} \Gamma(\alpha_j + kA_j)z^k}{\prod_{j=1}^{p} \Gamma(\beta_j + kB_j)k!},$$

where $A_j > 0$ for all $j = 1, ..., q, B_j > 0$ for all $j = 1, ..., p$ and $1 + \sum_{j=1}^{p} B_j - \sum_{j=1}^{q} A_j \geq 0$ for suitable values $|z|$. For special case, when $A_j = 1$ for all $j = 1, ..., q$, and $B_j = 1$ for all $j = 1, ..., p$ we have the following relationship

$${}_qF_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_p; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k...(\alpha_q)_k z^k}{(\beta_1)_k...(\beta_p)_k k!}$$

$$= \Omega_p\Psi_p[\alpha_j, 1)_{1,q}; (\beta_j, 1)_{1,p}; z],$$

$$q \leq p + 1; q, p \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, z \in \mathcal{U},$$

where

$$\Omega = \frac{\Gamma(\beta_1)...\Gamma(\beta_p)}{\Gamma(\alpha_1)\Gamma(\alpha_q)},$$

and $(a)_k$ is the Pochhammer symbol defined by

$$(a)_k = \frac{\Gamma(a + k)}{\Gamma(a)} = \begin{cases} 1, & k = 0 \\ a(a + 1)(a + 2)...(a + k - 1), & k \in \mathbb{N}. \end{cases}$$

For special cases of ${}_q\Psi_p$ are the entire functions

$${}_1\Psi_1(z) = \sum_{k=0}^{\infty} \frac{\Gamma(1 + Ak)z^k}{\Gamma(1 + Bk)k!}, B \geq A \geq 0,$$

that when $A = 1$ and $B = \lambda$, reduces to the classical Mittag- Leffler function (see [1])

$$E_\lambda(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\lambda k + 1)}, \lambda > 0.$$

Another special case of ${}_q\Psi_p$ is the function

$${}_2\Psi_1(z) = \sum_{k=0}^{\infty} \frac{\Gamma(1 + Ak)\Gamma(1 + \alpha k)z^k}{\Gamma(1 + \beta + Bk)k!}, A, B, \alpha, \beta > 0, B \geq A + \alpha,$$

that when $A = 0, \alpha = 1, B = \lambda$ and $\beta = \delta - 1$ becomes the generalized Mittag-Leffler function

$$E_{\lambda,\delta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\lambda k + \delta)}, \lambda, \delta > 0.$$

The Mittage-Leffler function plays an important role in analysis and fractional calculus (see [3, 6, 12). Some other work related to Fox-Wright and hypergeometric functions can be seen in [18] and [19] for different purposes.
In [7], Kiryakova, introduced the so-called multi-index Mittag-Leffler function as follows.
For $m > 1$ (integers) $\rho_1, ..., \rho_m > 0$ and $\mu_1, ..., \mu_m$ (real numbers)

$$E_{(1/\rho_i),(\mu_i)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu_1 + \frac{k}{\rho_1}\Gamma(\mu_m + \frac{k}{\rho_m}}$$

$$=_1 \Psi_m(z) \left[\begin{matrix} (1,1); \\ ((\mu_i + \frac{n}{\rho_i})_1^m; \end{matrix} \; z \right]$$

$$= H^{1,1}_{1,m+1} \left[\begin{matrix} (0,1); \\ -z| \\ (0,1), (1 - \mu_i, \frac{1}{\rho_i})_1^m; \end{matrix} \right].$$

Putting $(\beta_m, B_m) = (\mu_m, \frac{1}{\rho_m}), m < p$, we have the generalized Wright function

$${}_q\Phi_p = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{q} \Gamma(\alpha_j + kA_j)z^k}{\prod_{j=1}^{m} \Gamma(\mu_j + \frac{1}{\rho_i}) \prod_{j=m+1}^{p} \Gamma(\beta_j + kB_j)k!}. \quad (2)$$

Note that when $m = 1$ in equation (2), ${}_q\Phi_p(z)$ reduces to the generalized Wright function due to Mainardi and Pagnini [8]. We introduce a function $(z_q\Phi_p(z))^{-1}$ given by

$$[z_q\Phi_p(z)] * [z_q\Phi_p(z)]^{-1} = \frac{z}{(1 - z)^{\lambda+1}}$$

$$= z + \sum_{k=2}^{\infty} \frac{(\lambda+1)_{k-1}}{(k-1)!} z^k, (\lambda > -1)$$

and obtain the following operator:

$$I_\lambda[(\alpha_j, A_j)_{1,q}; (\mu_j, \rho_j)_{1,m}; (\beta_j, B_j)_{m+1,p}]f(z) = [z_q\Phi_p(z)]^{-1} * f(z),$$

where $f \in A, z \in \mathcal{U}$ and

$$[z_q\Phi_p(z)]^{-1} = z + \sum_{k=2}^{\infty} \frac{\prod_{j=1}^{m} \Gamma(\mu_j + (k-1)\frac{1}{\rho_j}) \prod_{j=m+1}^{p} \Gamma(\beta_j + (k-1)B_j)}{\prod_{j=1}^{q} \Gamma(\alpha_j + (k-1)A_j)} (\lambda+1)_{k-1} z^k.$$

A comprtation gives

$$I_\lambda[(\alpha_j, A_j)_{1,q}; (\mu_j, \rho_j)_{1,m}; (\beta_j, B_j)_{m+1,p}]f(z) = \qquad (3)$$

$$= z + \sum_{k=2}^{\infty} \frac{\prod_{j=1}^{m} \Gamma(\mu_j + (k-1)\frac{1}{\rho_j}) \prod_{j=m+1}^{p} \Gamma(\beta_j + (k-1)B_j)}{\prod_{j=1}^{q} \Gamma(\alpha_j + (k-1)A_j)} (\lambda+1)_{k-1} a_k z^k,$$

where

$$\frac{\prod_{j=1}^{m} \Gamma(\mu_j) \prod_{j=m+1}^{p} \Gamma(\beta_j)}{\prod_{j=1}^{q} \Gamma(\alpha_j)} = 1.$$

From (3) we have the following result.

Lemma 1.1 *Let $f(z) \in S$. Then*

1. $I_0[(1,1)_{1,1}; (0, k-1)_{1,m}; (1, \frac{1}{k-1})_{m+1,p}]f(z) = f(z).$
2. $I_1[(1,1)_{1,1}; (0, k-1)_{1,m}; (1, \frac{1}{k-1})_{m+1,p}]f(z) = zf'(z).$
3. $z[I_\lambda[(\alpha_j, A_j)_{1,q}; (\mu_j, \rho_j)_{1,m}; (\beta_j, B_j)_{m+1,p}]f(z)]' =$
   $(\lambda+1)I_{\lambda+1}[(\alpha_j, A_j)_{1,q}; (\mu_j, \rho_j)_{1,m}; (\beta_j, B_j)_{m+1,p}]f(z)$
   $-\lambda I_\lambda[(\alpha_j, A_j)_{1,q}; (\mu_j, \rho_j)_{1,m}; (\beta_j, B_j)_{m+1,p}]f(z).$

Operator (3) reduces to the one defined and studied by the authors [5] when $(\mu_j, \frac{1}{\rho_j})_{1,m} = (\beta_j, B_j)_{1,m}$ which is also a generalization of Noor integral operator [9].

In the following definition, we introduce new classes of analytic functions containing generalized integral operator (3):

Definition 1.2 *Let $f(z) \in S$, then $f(z) \in S_\lambda^{\zeta,\eta}[(\alpha_j, A_j)_{1,q}; (\mu_j, \rho_j)_{1,m}; (\beta_j, B_j)_{m+1,p}]$ if and only if*

$$\Re\left\{\frac{\frac{[I_\lambda[(\alpha_j,A_j)_{1,q};(\mu_j,\rho_j)_{1,m};(\beta_j,B_j)_{m+1,p}]f(z)]'}{I_\lambda[(\alpha_j,A_j)_{1,q};(\mu_j,\rho_j)_{1,m};(\beta_j,B_j)_{m+1,p}]f(z)}}{\zeta\frac{[I_\lambda[(\alpha_j,A_j)_{1,q};(\mu_j,\rho_j)_{1,m};(\beta_j,B_j)_{m+1,p}]f(z)]'}{I_\lambda[(\alpha_j,A_j)_{1,q};(\mu_j,\rho_j)_{1,m};(\beta_j,B_j)_{m+1,p}]f(z)} + (1-\zeta)}\right\} > \eta, \qquad (5)$$

for some $\eta (0 \le \eta < 1), \xi (0 \le \xi < 1)$ and for all $z \in \mathcal{U}$.

Let T denote the subclass of S consisting of all functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k (a_k \geq 0). \tag{6}$$

Furthermore, we define the class

$$M_\lambda^{\zeta,\eta}[(\alpha_j, A_j)_{1,q}; (\mu_j, \rho_j)_{1,m}; (\beta_j, B_j)_{m+1,p}] = S_\lambda^{\zeta,\eta}[(\alpha_j, A_j)_{1,q}; (\mu_j, \rho_j)_{1,m}; (\beta_j, B_j)_{m+1,p}] \cap T.$$

We note that, by specializing the parameters λ, ξ and η we obtain the following subclasses studied by various authors:

1. $M_0^{\zeta,\eta}[(1,1)_{1,1}; (0, k-1)_{1,m}; (1, \frac{1}{k-1})_{m+1,p}] = T(\xi, \eta)$
   $M_1^{\zeta,\eta}[(1,1)_{1,1}; (0, k-1)_{1,m}; (1, \frac{1}{k-1})_{m+1,p}] = C(\xi, \eta)$ (Altintas and Owa [2]);

2. $M_0^{0,\eta}[(1,1)_{1,1}; (0, k-1)_{1,m}; (1, \frac{1}{k-1})_{m+1,p}] = T^*(\eta)$ and
   $M_1^{0,\eta}[(1,1)_{1,1}; (0, k-1)_{1,m}; (1, \frac{1}{k-1})_{m+1,p}] = C(\eta)$(Silverman [13])

2 Coefficient Estimates

Theorem 2.1 *Let the function $f(z)$ defined by (6). Then $f(z) \in M_\lambda^{\zeta,\eta}[(\alpha_j, A_j)_{1,q}; (\mu_j, \rho_j)_{1,m}; (\beta_j, B_j)_{m+1,p}]$ if and only if*

$$\sum_{k=2}^{\infty} H_{k-1}[((\lambda+1)_k - (\lambda)_k) - \eta(\lambda+1)_{k-1}[1 + \zeta(k-1)]]a_k| \leq 1 - \eta, \tag{7}$$

where

$$H_{k-1} = \frac{\prod_{j=1}^{m} \Gamma(\mu_j + (k-1)\frac{1}{\rho_j}) \prod_{j=m+1}^{p} \Gamma(\beta_j + (k-1)B_j)}{\prod_{j=1}^{q} \Gamma(\alpha_j + (k-1)A_j)}. \tag{8}$$

Proof: Assume that the inequality (7) holds and let $|z| = 1$. Then we have

$$|\frac{\frac{[I_\lambda[(\alpha_j,A_j)_{1,q};(\mu_j,\rho_j)_{1,m};(\beta_j,B_j)_{m+1,p}]f(z)]'}{I_\lambda[(\alpha_j,A_j)_{1,q};(\mu_j,\rho_j)_{1,m};(\beta_j,B_j)_{m+1,p}]f(z)}}{\zeta\frac{[I_\lambda[(\alpha_j,A_j)_{1,q};(\mu_j,\rho_j)_{1,m};(\beta_j,B_j)_{m+1,p}]f(z)]'}{I_\lambda[(\alpha_j,A_j)_{1,q};(\mu_j,\rho_j)_{1,m};(\beta_j,B_j)_{m+1,p}]f(z)} + (1-\zeta)} - 1| =$$

$$|\frac{\sum_{k=2}^{\infty}(1-\zeta)H_{k-1}(1-k)((\lambda+1)_k - (\lambda)_k)a_k z^{k-1}}{1 - \sum_{k=2}^{\infty} H_{k-1}[1+\zeta(k-1)](\lambda+1)_{k-1})a_k z^{k-1}}|$$

$$\leq \frac{\sum_{k=2}^{\infty}(1-\zeta)H_{k-1}(1-k)((\lambda+1)_k - (\lambda)_k)a_k}{1 - \sum_{k=2}^{\infty} H_{k-1}[1+\zeta(k-1)](\lambda+1)_{k-1})a_k}|$$

$$\leq 1 - \eta.$$

This shows that the value of $\frac{\frac{[I_\lambda[(\alpha_j,A_j)_{1,q};(\mu_j,\rho_j)_{1,m};(\beta_j,B_j)_{m+1,p}]f(z)]'}{I_\lambda[(\alpha_j,A_j)_{1,q};(\mu_j,\rho_j)_{1,m};(\beta_j,B_j)_{m+1,p}]f(z)}}{\zeta\frac{[I_\lambda[(\alpha_j,A_j)_{1,q};(\mu_j,\rho_j)_{1,m};(\beta_j,B_j)_{m+1,p}]f(z)]'}{I_\lambda[(\alpha_j,A_j)_{1,q};(\mu_j,\rho_j)_{1,m};(\beta_j,B_j)_{m+1,p}]f(z)}+(1-\zeta)}$ lies in a circle centered at $w = 1$ whose radius is $1-\eta$. Hence $f(z)$ satisfies the condition (5).

Conversely, assume that the function $f(z)$ defined by (6) is in the class $M_\lambda^{\zeta,\eta}[(\alpha_j, A_j)_{1,q}; (\mu_j, \rho_j)_{1,m}; (\beta_j, B_j)_{m+1,p}]$. Then

$$\Re\{\frac{\frac{[I_\lambda[(\alpha_j,A_j)_{1,q};(\mu_j,\rho_j)_{1,m};(\beta_j,B_j)_{m+1,p}]f(z)]'}{I_\lambda[(\alpha_j,A_j)_{1,q};(\mu_j,\rho_j)_{1,m};(\beta_j,B_j)_{m+1,p}]f(z)}}{\zeta\frac{[I_\lambda[(\alpha_j,A_j)_{1,q};(\mu_j,\rho_j)_{1,m};(\beta_j,B_j)_{m+1,p}]f(z)]'}{I_\lambda[(\alpha_j,A_j)_{1,q};(\mu_j,\rho_j)_{1,m};(\beta_j,B_j)_{m+1,p}]f(z)}+(1-\zeta)}\}$$

$$\Re\{\frac{1-\sum_{k=2}^{\infty} H_{k-1}((\lambda+1)_k-(\lambda)_k)a_k z^{k-1}}{1-\sum_{k=2}^{\infty} H_{k-1}[1+\zeta(k-1)](\lambda+1)_{k-1}a_k z^{k-1}}\}, \tag{9}$$

for $z \in \mathcal{U}$.

Choose values of z on the real axis so that $\frac{\frac{[I_\lambda[(\alpha_j,A_j)_{1,q};(\mu_j,\rho_j)_{1,m};(\beta_j,B_j)_{m+1,p}]f(z)]'}{I_\lambda[(\alpha_j,A_j)_{1,q};(\mu_j,\rho_j)_{1,m};(\beta_j,B_j)_{m+1,p}]f(z)}}{\zeta\frac{[I_\lambda[(\alpha_j,A_j)_{1,q};(\mu_j,\rho_j)_{1,m};(\beta_j,B_j)_{m+1,p}]f(z)]'}{I_\lambda[(\alpha_j,A_j)_{1,q};(\mu_j,\rho_j)_{1,m};(\beta_j,B_j)_{m+1,p}]f(z)}+(1-\zeta)}$ is real.

Upon clearing the denominator in (9) and letting $z \to 1^-$ through real values, we obtain

$$1-\sum_{k=2}^{\infty} H_{k-1}((\lambda+1)_k-(\lambda)_k)a_k \geq \eta\{1-\sum_{k=2}^{\infty} H_{k-1}[1+\zeta(k-1)](\lambda+1)_{k-1})a_k\},$$

which gives (7). Hence the proof is complete.

Corollary 2.2 *Let the function $f(z)$ defined by (6) be in the class $M_\lambda^{\zeta,\eta}[(\alpha_j, A_j)_{1,q}; (\mu_j, \rho_j)_{1,m}; (\beta_j, B_j)_{m+1,p}]$. Then*

$$a_k \leq \frac{1-\eta}{H_{k-1}[((\lambda+1)_k-(\lambda)_k)-\eta(\lambda+1)_{k-1}[1+\zeta(k-1)]]}, k \geq 2$$

where H_{k-1} is defined in (8).

The result is sharp with the extremal function $f(z)$ given by

$$f(z) = z - \frac{1-\eta}{H_{k-1}[((\lambda+1)_k-(\lambda)_k)-\eta(\lambda+1)_{k-1}[1+\zeta(k-1)]]} z^k (k \geq 2).$$

3 Some Properties of the Class $M_\lambda^{\zeta,\eta}[(\alpha_j, A_j)_{1,q}; (\mu_j, \rho_j)_{1,m}; (\beta_j, B_j)_{m+1,p}]$

Theorem 3.1 *Let $0 \leq \eta < 1, 0 \leq \zeta_1 \leq \zeta_2$. Then*

$$M_\lambda^{\zeta_1,\eta}[(\alpha_j, A_j)_{1,q}; (\mu_j, \rho_j)_{1,m}; (\beta_j, B_j)_{m+1,p}] \subseteq M_\lambda^{\zeta_2,\eta}[(\alpha_j, A_j)_{1,q}; (\mu_j, \rho_j)_{1,m}; (\beta_j, B_j)_{m+1,p}].$$

Proof: It follows from Theorem 2.1 that

$$\sum_{k=2}^{\infty} H_{k-1}[((\lambda+1)_k - (\lambda)_k) - \eta(\lambda+1)_{k-1}[1+\zeta_2(k-1)]]a_k| \leq$$

$$\sum_{k=2}^{\infty} H_{k-1}[((\lambda+1)_k - (\lambda)_k) - \eta(\lambda+1)_{k-1}[1+\zeta_1(k-1)]]a_k| \leq 1-\eta,$$

for $f(z) \in M_{\lambda}^{\zeta_1,\eta}[(\alpha_j, A_j)_{1,q}; (\mu_j, \rho_j)_{1,m}; (\beta_j, B_j)_{m+1,p}]$.
Hence $f(z) \in M_{\lambda}^{\zeta_2,\eta}[(\alpha_j, A_j)_{1,q}; (\mu_j, \rho_j)_{1,m}; (\beta_j, B_j)_{m+1,p}]$.

Theorem 3.2 *Let* $0 \leq \eta < 1, 0 \leq \zeta_1 < 1$*. Then*

$M_{\lambda+1}^{\zeta,\eta}[(\alpha_j, A_j)_{1,q}; (\mu_j, \rho_j)_{1,m}; (\beta_j, B_j)_{m+1,p}] \subset M_{\lambda}^{\zeta,\eta}[(\alpha_j, A_j)_{1,q}; (\mu_j, \rho_j)_{1,m}; (\beta_j, B_j)_{m+1,p}]$.

The proof follows immediately from Theorem 2.1.

4 Distortion Theorem

Theorem 4.1 *Let the function* $f(z)$ *defined by (6) be in the class* $M_{\lambda}^{\zeta,\eta}[(\alpha_j, A_j)_{1,q}; (\mu_j, \rho_j)_{1,m}; (\beta_j, B_j)_{m+1,p}]$*. Then*

$$|f(z)| \geq |z| - \frac{1-\eta}{H_1[((\lambda+1)_2 - (\lambda)_2) - \eta(\lambda+1)_1(1+\zeta)]}|z|^2 \tag{10}$$

$$|f(z)| \leq |z| + \frac{1-\eta}{H_1[((\lambda+1)_2 - (\lambda)_2) - \eta(\lambda+1)_1(1+\zeta)]}|z|^2. \tag{11}$$

for $z \in \mathcal{U}$ *where* H_{k-1} *is defined in (8). Then equalities in (10) and (11) are attained for the function* $f(z)$ *given by*

$$f(z) = z - \frac{1-\eta}{H_1[((\lambda+1)_2 - (\lambda)_2) - \eta(\lambda+1)_1(1+\zeta)]}z^2. \tag{12}$$

Proof: Let $f(z) \in M_{\lambda}^{\zeta,\eta}[(\alpha_j, A_j)_{1,q}; (\mu_j, \rho_j)_{1,m}; (\beta_j, B_j)_{m+1,p}]$.
Then in view of Theorem 2.1, we have

$$H_1[((\lambda+1)_2 - (\lambda)_2) - \eta(\lambda+1)_1[1+\zeta]] \sum_{k=2}^{\infty} a_k \leq$$

$$\sum_{k=2}^{\infty} H_{k-1}[((\lambda+1)_k - (\lambda)_k) - \eta(\lambda+1)_{k-1}[1+\zeta(k-1)]]a_k \leq 1-\eta.$$

This yields

$$\sum_{k=2}^{\infty} a_k \leq \frac{1-\eta}{H_1[((\lambda+1)_2-(\lambda)_2)-\eta(\lambda+1)_1[1+\zeta]]}.$$

Now

$$|f(z)| = |z - \sum_{k=2}^{\infty} a_k z^k|$$

$$\geq |z| - |z|^2 \sum_{k=2}^{\infty} |a_k|$$

$$\geq |z| - \frac{1-\eta}{H_1[((\lambda+1)_2-(\lambda)_2)-\eta(\lambda+1)_1[1+\zeta]]}|z|^2.$$

Also,

$$|f(z) \leq |z| + \frac{1-\eta}{H_1[((\lambda+1)_2-(\lambda)_2)-\eta(\lambda+1)_1[1+\zeta]]}|z|^2.$$

Hence the proof is complete.

Corollary 4.2 *Under the hypothesis of Theorem 4.1, $f(z)$ is included in a disk with center at the origin and radius r given by*

$$r = 1 + \frac{1-\eta}{H_1[((\lambda+1)_2-(\lambda)_2)-\eta(\lambda+1)_1[1+\zeta]]}.$$

The result is sharp with the extremal function $f(z)$ given by (12).

Theorem 4.3 *Let the function $f(z)$ defined by (6) be in the class $M_\lambda^{\zeta,\eta}[(\alpha_j, A_j)_{1,q}; (\mu_j, \rho_j)_{1,m}; (\beta_j, B_j)_{m+1,p}]$. Then we have*

$$|I_\lambda[(\alpha_j, A_j)_{1,q}; (\mu_j, \rho_j)_{1,m}; (\beta_j, B_j)_{m+1,p}]f(z)| \leq |z| + (1-\eta)|z|^2 \tag{13}$$

$$|I_\lambda[(\alpha_j, A_j)_{1,q}; (\mu_j, \rho_j)_{1,m}; (\beta_j, B_j)_{m+1,p}]f(z)| \geq |z| + (1-\eta)|z|^2, \tag{14}$$

for $z \in \mathcal{U}$ where H_{k-1} is defined in (8). Then equalities in (13) and (14) are attained for the function $f(z)$ given by

$$f(z) = z - (1-\eta)z^2.$$

Proof: By using Theorem 2.1, one can verify that

$$\sum_{k=2}^{\infty} H_{k-1}(\lambda+1)_{k-1}a_k \leq \sum_{k=2}^{\infty} H_{k-1}[((\lambda+1)_k-(\lambda)_k)-\eta(\lambda+1)_{k-1}[1+\zeta(k-1)]]a_k \leq 1-\eta,$$

that is, that

$$\sum_{k=2}^{\infty} H_{k-1}(\lambda+1)_{k-1}a_k \leq 1-\eta.$$

It follows that

$$|I_{\lambda}[(\alpha_j, A_j)_{1,q}; (\mu_j, \rho_j)_{1,m}; (\beta_j, B_j)_{m+1,p}]f(z)| = |z - \sum_{k=2}^{\infty} a_k z^k|$$

$$\leq |z| + |z|^2 \sum_{k=2}^{\infty} H_{k-1}(\lambda+1)_{k-1}|a_k|$$

$$\leq |z| + |z|^2 \sum_{k=2}^{\infty} H_{k-1}(\lambda+1)_{k-1}|a_k|$$

$$\leq |z| + (1-\eta)|z|^2.$$

The other assertion can be proved as follows:

$$|I_{\lambda}[(\alpha_j, A_j)_{1,q}; (\mu_j, \rho_j)_{1,m}; (\beta_j, B_j)_{m+1,p}]f(z)| = |z - \sum_{k=2}^{\infty} a_k z^k|$$

$$\geq |z| - |z|^2 \sum_{k=2}^{\infty} H_{k-1}(\lambda+1)_{k-1}|a_k|$$

$$\geq |z| - |z|^2 \sum_{k=2}^{\infty} H_{k-1}(\lambda+1)_{k-1}|a_k|$$

$$\geq |z| - (1-\eta)|z|^2.$$

This competes the proof of Theorem 4.2.

5 Fractional Calculus

We begin with the statements of the following definitions of fractional calculus (that is, fractional derivatives and fractional integrals) which were defined by Owa ([10], [11]) and were used recently by Srivastava and Owa [14] and many researchers.

Definition 5.1 *The fractional integral of order δ is defined, for a function $f(z)$, by*

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(\delta)} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\delta}} d\xi (\delta > 0),$$

where $f(z)$ is analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z-\xi)^{1-\delta}$ is removed by requiring $\log(z-\xi)$ to be real when $z-\xi > 0$.

Definition 5.2 *The fractional derivative of order δ is defined, for a function $f(z)$, by*

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\delta}} d\xi (0 \leq \delta < 1),$$

where $f(z)$ is constrained, and the multiplicity of $(z-\xi)^{-\delta}$ is removed as in Definition 5.1.

Definition 5.3 *Under the hypothesis of Definition 5.2, the fractional derivative of order $n+1$ is defined by*

$$D_z^{n+\delta} f(z) = \frac{d^n}{dz^n} D_z^{\delta} f(z) (0 \leq \delta < 1; n \in \mathbb{N}_0).$$

Theorem 5.4 *Let the function $f(z)$ defined by (6) be in the class $M_{\lambda}^{\zeta,\eta}[(\alpha_j, A_j)_{1,q}; (\mu_j, \rho_j)_{1,m}; (\beta_j, B_j)_{m+1,p}]$. Then we have*

$$|D_z^{-\delta} f(z)| \geq \frac{|z|^{1+\delta}}{\Gamma(2+\delta)} \{1 - \frac{2(1-\eta)}{(2+\delta) H_1[((\lambda+1)_2 - (\lambda)_2) - \eta(\lambda+1)_1[1+\zeta]]} |z|\},$$

and

$$|D_z^{-\delta} f(z)| \leq \frac{|z|^{1+\delta}}{\Gamma(2+\delta)} \{1 + \frac{2(1-\eta)}{(2+\delta) H_1[((\lambda+1)_2 - (\lambda)_2) - \eta(\lambda+1)_1[1+\zeta]]} |z|\}$$

for $\delta > 0$ and $z \in \mathcal{U}$. The result is sharp .

Proof: Let

$$F(z) = \Gamma(2+\delta) z^{-\delta} D_z^{-\delta} f(z)$$

$$f(z) = z - \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\delta)}{\Gamma(k+1+\delta)} a_k z^k$$

$$f(z) = z - \sum_{k=2}^{\infty} \Psi_1(k,\delta) a_k z^k,$$

where

$$\Psi_1(k,\delta) = \frac{\Gamma(k+1)\Gamma(2+\delta)}{\Gamma(k+1+\delta)} (k \geq 2).$$

Since

$$0 < \Psi_1(k,\delta) \leq \Psi_2(2,\delta) = \frac{2}{2+\delta}. \qquad (15).$$

In view of Theorem 2.1, we have

$$H_1[((\lambda+1)_2 - (\lambda)_2) - \eta(\lambda+1)_1[1+\zeta]] \sum_{k=2}^{\infty} a_k \leq$$

$$\sum_{k=2}^{\infty} H_{k-1}[((\lambda+1)_k - (\lambda)_k) - \eta(\lambda+1)_{k-1}(1+\zeta(k-1))]a_k \leq 1-\eta,$$

which evidently yields

$$\sum_{k=2}^{\infty} a_k \leq \frac{1-\eta}{H_1[((\lambda+1)_2 - (\lambda)_2) - \eta(\lambda+1)_1[1+\zeta]]}. \tag{16}$$

Therefore, by using (15) and (16), we can see that

$$|F(z)| \geq |z| - |z|^2 \Psi_1(2,\delta) \sum_{k=2}^{\infty} |a_k|$$

$$\geq |z| - \frac{2(1-\eta)}{(2+\delta)H_1[((\lambda+1)_2 - (\lambda)_2) - \eta(\lambda+1)_1[1+\zeta]]}|z|^2,$$

and

$$|F(z)| \leq |z| + |z|^2 \Psi_1(2,\delta) \sum_{k=2}^{\infty} |a_k|$$

$$|F(z) \leq |z| + \frac{2(1-\eta)}{(2+\delta)H_1[((\lambda+1)_2 - (\lambda)_2) - \eta(\lambda+1)_1[1+\zeta]]}|z|^2,$$

which prove the inequalities of Theorem 5.1. Further, equalities are attained for the function $f(z)$ defined by

$$D_z^{-\delta} f(z) = \frac{z^{1+\delta}}{\Gamma(2+\delta)}\{1 - \frac{2(1-\eta)}{(2+\delta)H_1[((\lambda+1)_2 - (\lambda)_2) - \eta(\lambda+1)_1(1+\zeta)]}|z|\},$$

or

$$f(z) = z - \frac{1-\eta}{H_1[((\lambda+1)_2 - (\lambda)_2) - \eta(\lambda+1)_1(1+\zeta)]}z^2 \qquad (17).$$

Hence the proof is complete.

Corollary 5.5 *Under the hypothesis of Theorem 5.1, $D_z^{-\delta} f(z)$ is included in a disk with center at the origin and radius r_1 given by*

$$r_1 = \frac{1}{\Gamma(2+\delta)}\{1 + \frac{2(1-\eta)}{(2+\delta)H_1[((\lambda+1)_2 - (\lambda)_2) - \eta(\lambda+1)_1(1+\zeta)]}\},$$

Theorem 5.6 *Let the function $f(z)$ defined by (6) be in the class $M_\lambda^{\zeta,\eta}[(\alpha_j, A_j)_{1,q}; (\mu_j, \rho_j)_{1,m}; (\beta_j, B_j)_{m+1,p}]$. Then we have*

$$|D_z^{\delta} f(z)| \geq \frac{|z|^{1-\delta}}{\Gamma(2-\delta)}\{1 - \frac{2(1-\eta)}{(2-\delta)H_1[((\lambda+1)_2 - (\lambda)_2) - \eta(\lambda+1)_1(1+\zeta)]}|z|\}$$

$$|D_z^{\delta} f(z)| \leq \frac{|z|^{1-\delta}}{\Gamma(2-\delta)}\{1 + \frac{2(1-\eta)}{(2-\delta)H_1[((\lambda+1)_2 - (\lambda)_2) - \eta(\lambda+1)_1(1+\zeta)]}|z|\}$$

for $0 \leq \delta < 0$ and $z \in \mathcal{U}$. The result is sharp .

Proof: Let

$$G(z) = \Gamma(2-\delta) z^{\delta} D_z^{\delta} f(z)$$

$$f(z) = z - \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\delta)}{\Gamma(k+1-\delta)} a_k z^k$$

$$f(z) = z - \sum_{k=2}^{\infty} \Phi_1(k,\delta) a_k z^k,$$

where

$$\Phi_1(k,\delta) = \frac{\Gamma(k+1)\Gamma(2-\delta)}{\Gamma(k+1-\delta)} (k \geq 2).$$

Since

$$0 < \Phi_1(k,\delta) \leq \Phi_2(k,\delta) = \frac{2}{2-\delta}. \qquad (18).$$

In view of Theorem 2.1, we have

$$H_1[((\lambda+1)_2 - (\lambda)_2) - \eta(\lambda+1)_1[1+\zeta]] \sum_{k=2}^{\infty} a_k \leq$$

$$\sum_{k=2}^{\infty} H_{k-1}[((\lambda+1)_k - (\lambda)_k) - \eta(\lambda+1)_{k-1}[1+\zeta(k-1)]] a_k \leq 1-\eta,$$

which evidently yields

$$\sum_{k=2}^{\infty} a_k \leq \frac{1-\eta}{H_1[((\lambda+1)_2 - (\lambda)_2) - \eta(\lambda+1)_1[1+\zeta]]}. \qquad (19)$$

Therefore, by using (18) and (19), we can see that

$$|G(z)| \geq |z| - |z|^2 \Phi_1(2,\delta) \sum_{k=2}^{\infty} |a_k|$$

$$\geq |z| - \frac{2(1-\eta)}{(2-\delta)H_1[((\lambda+1)_2 - (\lambda)_2) - \eta(\lambda+1)_1[1+\zeta]]} |z|^2,$$

and

$$|G(z)| \leq |z| + |z|^2 \Phi_1(2,\delta) \sum_{k=2}^{\infty} |a_k|$$

$$|G(z) \leq |z| + \frac{2(1-\eta)}{(2-\delta)H_1[((\lambda+1)_2 - (\lambda)_2) - \eta(\lambda+1)_1[1+\zeta]]} |z|^2,$$

which prove the inequalities of Theorem 5.2. Further, equalities are attained for the function $f(z)$ defined by

$$D_z^{\delta} f(z) = \frac{|z|^{1-\delta}}{\Gamma(2-\delta)} \{1 - \frac{2(1-\eta)}{(2-\delta)H_1[((\lambda+1)_2 - (\lambda)_2) - \eta(\lambda+1)_1(1+\zeta)]} |z|\},$$

that is, by (17).
We complete the assertion of Theorem 5.2.

Corollary 5.7 *Under the hypothesis of Theorem 5.2, $D_z^\delta f(z)$ is included in a disk with center at the origin and radius r_2 given by*

$$r_2 = \frac{1}{\Gamma(2-\delta)}\{1 + \frac{2(1-\eta)}{(2-\delta)H_1[((\lambda+1)_2 - (\lambda)_2) - \eta(\lambda+1)_1(1+\zeta)]}\},$$

Corollary 5.8 *For every $f(z) \in M_\lambda^{\zeta,\eta}[(\alpha_j, A_j)_{1,q}; (\mu_j, \rho_j)_{1,m}; (\beta_j, B_j)_{m+1,p}]$, we have*

$$\frac{|z|^2}{2}[1 - \frac{2(1-\eta)}{3H_1[((\lambda+1)_2 - (\lambda)_2) - \eta(\lambda+1)_1(1+\zeta)]}|z|] \leq$$

$$|\int_0^z f(t)dt| \leq \frac{|z|^2}{2}[1 + \frac{2(1-\eta)}{3H_1[((\lambda+1)_2 - (\lambda)_2) - \eta(\lambda+1)_1(1+\zeta)]}|z|],$$

and

$$|z|[1 - \frac{(1-\eta)}{3H_1[((\lambda+1)_2 - (\lambda)_2) - \eta(\lambda+1)_1(1+\zeta)]}|z|] \leq$$

$$|f(t)| \leq |z|[1 + \frac{(1-\eta)}{3H_1[((\lambda+1)_2 - (\lambda)_2) - \eta(\lambda+1)_1(1+\zeta)]}|z|],$$

Proof: By definition 5.1 and Theorem 5.1 for $\delta = 1$, we have $D_z^{-1} f(z) = \int_0^z f(t)dt$, the result is true. Also by Definition 5.2 and Theorem 5.2 for $\delta = 0$, we have

$$D_z^0 f(z) = \frac{d}{dz}\int_0^z f(t)dt = f(z).$$

Hence the proof is complete.

6 Fractional Integral Operator

We need the following definition of fractional integral operator given by Srivastava, Saigo and Owa [15].

Definition 6.1 *For real numbers $\beta_1 > 0, \gamma_1$ and η_1, the fractional integral operator $I_{0,z}^{\beta_1,\gamma_1,\eta_1}$ is defined by*

$$I_{0,z}^{\beta_1,\gamma_1,\eta_1} f(z) = \frac{z^{-\beta_1-\gamma_1}}{\Gamma(\beta_1)}\int_0^z (z-t)^{\beta_1-1} F(\beta_1+\gamma_1, -\eta_1; \beta_1; 1-\frac{t}{z})f(t)dt,$$

where $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin with the order

$$f(z) = O(|z|^\epsilon), z \to 0,$$

where $\epsilon > \max(0, \gamma_1 - \eta_1) - 1$, *and*

$$F(a, b, ; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k (1)_k} z^k,$$

where $(v)_k$ *is the Pochhammer symbol defined by*

$$(v)_k = \frac{\Gamma(v+k)}{\Gamma(v)} = \begin{cases} 1, & k = 0 \\ v(v+1)(v+2)...(v+k-1), & k \in \mathbb{N}, \end{cases}$$

the multiplicity of $(z-t)^{\beta_1 - 1}$ *is removed by requiring* $\log(z-t)$ *to be real when* $z - t > 0$.

Remark 6.2 *For* $\gamma_1 = -\beta_1$, *we note that*

$$I_{0,z}^{\beta_1, \gamma_1, \eta_1} f(z) = D_z^{-\beta_1} f(z).$$

In order to prove our result for the fractional operator, we have to recall here the following Lemma due to Srivastava, Saigo and Owa [15].

Lemma 6.3 *If* $\beta_1 > 0$ *and* $k > \gamma_1 - \eta_1 - 1$, *then*

$$I_{0,z}^{\beta_1, \gamma_1, \eta_1} z^k = \frac{\Gamma(k+2)\Gamma(k - \gamma_1 + \eta_1 + 1)}{\Gamma(k - \gamma_1 + 1)\Gamma(k + \beta_1 + \eta_1 + 1)} z^{k - \gamma_1}.$$

With the aid of Lemma 6.1, we prove

Theorem 6.4 *Let* $\beta_1 < 0, \gamma_1 < 2, \beta_1 + \eta_1 > -2, \gamma_1 - \eta_1 < 2$ *and* $\gamma_1(\beta_1 + \eta_1) \leq 3\beta_1$. *If the function* $f(z)$ *defined by (6) is in the class* $M_\lambda^{\zeta,\eta}[(\alpha_j, A_j)_{1,q}; (\mu_j, \rho_j)_{1,m}; (\beta_j, B_j)_{m+1,p}]$, *then*

$$|I_{0,z}^{\beta_1, \gamma_1, \eta_1} f(z)| \geq \frac{\Gamma(2 - \gamma_1 + \eta_1)|z|^{1-\gamma_1}}{\Gamma(2-\gamma_1)\Gamma(2 + \beta_1 + \eta_1)}$$

$$\{1 - \frac{(1-\eta)(2 - \gamma_1 + \eta_1)}{H_1[((\lambda+1)_2 - (\lambda)_2) - \eta(\lambda+1)_1(1+\zeta)](2-\gamma_1)(2+\beta_1+\eta_1)} |z|\}, \quad (20)$$

$$|I_{0,z}^{\beta_1, \gamma_1, \eta_1} f(z)| \leq \frac{\Gamma(2 - \gamma_1 + \eta_1)|z|^{1-\gamma_1}}{\Gamma(2-\gamma_1)\Gamma(2 + \beta_1 + \eta_1)}$$

$$\{1 + \frac{(1-\eta)(2 - \gamma_1 + \eta_1)}{H_1[((\lambda+1)_2 - (\lambda)_2) - \eta(\lambda+1)_1(1+\zeta)](2-\gamma_1)(2+\beta_1+\eta_1)} |z|\}, \quad (21)$$

for $z \in \mathcal{U}_0$, *where*

$$\mathcal{U}_0 = \begin{cases} \mathcal{U} & (\gamma_1 \leq 1) \\ \mathcal{U} - \{0\} & (\gamma_1 > 1). \end{cases}$$

The equalities in (20) and (21) are attained by the function $f(z)$ *given by (17).*

Proof: By using Lemma 6.1, we have

$$I_{0,z}^{\beta_1,\gamma_1,\eta_1} f(z) = \frac{\Gamma(2-\gamma_1+\eta_1)}{\Gamma(2-\gamma_1)\Gamma(2+\beta_1+\eta_1)} z^{1-\gamma_1} - \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(k-\gamma_1+\eta_1+1)}{\Gamma(k-\gamma_1+1)\Gamma(k+\beta_1+\eta_1+1)} a_k z^{k-\gamma_1}.$$

Letting

$$H(z) = \frac{\Gamma(2-\gamma_1)\Gamma(2+\beta_1+\eta_1)}{\Gamma(2-\gamma_1+\eta_1)} z^{\gamma_1}$$

$$= z - \sum_{k=2}^{\infty} h(z) a_k z^k,$$

where

$$h(k) = \frac{(2-\gamma_1+\eta_1)_{k-1}(1)_k}{(2-\gamma_1)_{k-1}(2+\beta_1+\eta_1)_{k-1}} (k \geq 2),$$

we can see that $h(k)$ is non- increasing for integers $k \geq 2$, and we have

$$0 < h(k) \leq h(2) = \frac{2(2-\gamma_1+\eta_1)}{(2-\gamma_1)(2+\beta_1+\eta_1)}. \tag{22}$$

Therefore, by using (16) and (22), we have

$$|H(z)| \geq |z| - |z|^2 h(2) \sum_{k=2}^{\infty} a_k$$

$$\geq |z| - \frac{2(1-\eta)(2-\gamma_1+\eta_1)}{H_1[((\lambda+1)_2-(\lambda)_2)-\eta(\lambda+1)_1[1+\zeta]](2-\gamma_1)(2+\beta_1+\eta_1)} |z|^2,$$

and

$$|H(z)| \leq |z| + |z|^2 h(2) \sum_{k=2}^{\infty} a_k$$

$$\leq |z| + \frac{2(1-\eta)(2-\gamma_1+\eta_1)}{(2-\delta)H_1[((\lambda+1)_2-(\lambda)_2)-\eta(\lambda+1)_1[1+\zeta]](2-\gamma_1)(2+\beta_1+\eta_1)} |z|^2.$$

This completes the proof of Theorem 6.1.

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