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On Coincidence and Common Fixed Point for Nonlinear Generalized Hybrid Contractions

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Abstract

The purpose of this paper is to prove some coincidence point theorems for non-linear hybrid contraction involving two pairs of single-valued and multi-valued mappings on complete metric space.

Keywords: *Coincidentally commuting mapping, Hybrid Contraction, Multi-Valued Mappings, Metrical fixed point.*

1 Introduction

Nadler [8] was the first mathematician who obtained a set-valued version of Banach contraction principle. Since then there is multitude of metrical fixed point theorem for set valued mappings which are indeed extension of various singled-valued metrical fixed point theorems.The work of Asina-Massa-Rous [1], Circ [3], Bos and Mukherjee [2], Reich [11] [12], Kaulkud and Pai [7] are special mention in this context. Hausdorff metric is ordinary distance functions between points and set.

2 Preliminaries and Notations

A nonempty subset S of a metric space (X, d) is said to be proximinal if for each $x \in X$ there exists a point $y \in S$ such that $d(x, y) = d(x, S)$. It is well known that every compact set is proximinal.We denote

$$CB(X) = \{S : S \textit{ is closed bounded subset of } X\},$$
$$PB(X) = \{S : S \textit{ is proximinal bounded subet of } X\},$$
$$C(X) = \{S : S \textit{ is compact subset of } X\}$$

Since every proximinal set is closed, we have $C(X) \subseteq PB(X) \subseteq CB(X)$. Kaneko and Sessa [6]extended the notion of weak commutativity for single-valued mappings to the settings of single-valued and multi-valued mappings whereas for compatible mappings the same is done by Singh et al [13]. Now we need to recall relevant definitions.

Definition 2.1 *[6] The mappings T and F are said to be weakly commuting it for all $x \in X, fTx \in CB(X)$ and $H(Tfx, fTx) \leq d(fx, Tx)$, where H is the Hausdorff metric defined on $CB(X)$.*

The Hausdorff H on $CB(X)$ induced by the metric d is defined as

$$H(A,B) = max\ \{sup_{x\in A}\ d(x,\ B),\ sup_{y\in B}\ d(y, A)\}$$

for all $A,\ B \in CB(X)$,where $d(x, A) = inf_{y\in A}\ d(x,\ y)$.

It is well known that $(CB(X), H)$ is a metric space, and if a metric space (X, d) is complete, then so is $(CB(X), H)$.

Definition 2.2 *[6] The mappings T and F are said to compatible if and only if $fTx \in CB(X)$ for $x \in X$ and $H(Tfx_n, fTx_n) \to 0$ as $n \to \infty$, whenever $\{x_n\} \subset X$ such that $Tx_n \to M \in CB(X)$ and $fx_n \to t \in M$ as $n \to \infty$.*

Kaneko and Sessa [6] has furnish an example which shows that compatibility does not implies weak commutativity. Pathak [9] introduced the concept of weak compatible mappings for a hybrid pair of single-valued and multi-valued mappings as follows:

Definition 2.3 *[9] The mappings f and T are said to be f-weak compatible if $fT(X) \in CB(X)$ for all $x \in X$ and the following limits exists and satisfy the relevant inequality.*

$$lim_{n\to\infty}H(fTx_n, Tfx_n) \leq lim_{n\to\infty}H(Tfx_n, Tx_n),$$

$$lim_{n\to\infty}d(fTx_n, fx_n) \leq lim_{n\to\infty}H(Tfx_n, Tx_n),$$

where$\{x_n\}$ is a sequence in X such that $f(x_n) \to t$ and $Tx_n \to M \in CB(X)$ as $n \to \infty$.

Compatible pairs are weakly compatible but not conversely. Examples supporting this fact can be found in Pathak [9]

Definition 2.4 *[4] Let K be a non empty subsets of a metric space (X, d) where $F : K \to CB(X)$ and $T : K \to X$. Then the pair (F, T) is said to weakly commuting if for every x, y in K such that $x \in Fy$ and $Ty \in K$, imply that $d(Tx, FTy) \in d(Ty, Fy)$.*

Definition 2.5 *[4] Let (X, d) be a metric space. A mappings $T : X \to CB(X)$ is said to be continuous at $x_0 \in X$ if for any $\in > 0$ there exists a $\delta > 0$ such that $H(Tx, Tx_0) < \in$ whenever $d(x, x_0) < \delta$. If T is continuous at every point of X, then we say that T is continuous on X.*

Definition 2.6 *[5] A pair of mappings (S, T) is said to be coincidently commuting (resp.weakly compatible) if they commute at coincidence points.*

Lemma 2.7 *[8] Let $A, B \in CB(X)$ and $k > 1$.Then for each $a \in A$, there exists a point $b \in B$ such that $d(a, b) \leq kH(A, B)$.*

3 Main Result

In this section we give some coincidence and fixed points theorems for non-linear hybrid generalized contractions using the notion of weak compatible mappings introduce by Pathak et al [10].

Theorem 3.1 *Let S, T be two multi-valued continuous mappings of a complete metric space (X, d) in $CB(X)$, whereas I, J be two continuous self mappings of X. Suppose that (S, I) and (T, J) are compatible mappings with $S(X) \subset J(X)$ and $T(X) \subset I(X)$ satisfying*

$$H(Sx, Ty) \leq h[aL(Ix, Jy) + (1 - a)N(Ix, Jy)] \qquad (3.1.1),$$

for all x, y *in* X, $(0 \leq h < 1, 0 \leq a \leq 1)$, where

$L(Ix, Jy) = max\{d(Ix, Jy), d(Ix, Sx), d(Jy, Ty), \frac{1}{2}[d(Ix, Ty)+d(Jy, Sx)]\}$ and

$$N(Ix, Jy) = [max\{d^2(Ix, Jy), d(Ix, Sx)d(Jy, Ty), d(Ix, Ty)d(Jy, Sx), \frac{1}{2}[d(Ix, Sx)d(Jy, Sx)], \frac{1}{2}[d(Ix, Ty)d(Jy, Ty)\}]^{\frac{1}{2}}$$

Then there exists a point $t \in X$ such that $It = Jt \in St \cap Tt$, i.c the point t is a coincidence point of I, J, S *and* T.

Proof: Assume $k = \frac{1}{\sqrt{h}}$. Let $x_0 \in$ X and y_1 be an arbitrary point in Sx_0.Then there is $x_1 \in X$ such that $Jx_1 = y_1$ which is possible as S(X)$\subset J(X)$. By

Lemma 2.7 we can find a $y_2 \in Tx_1$ such that $d(y_1, y_2) \leq kH(Sx_0, Tx_1)$. Let us set $y_2 = Ix_2$ as $T(X) \subset I(X)$. Thus in general one can choose $y_{2n+2} = Ix_{2n+2} \in Tx_{2n+1}$ and $y_{2n+1} = Jx_{2n+1} \in Sx_{2n}$ such that $d(y_{2n+2}, dy_{2n+1}) \leq kH(Sx_{2n}, Tx_{2n+1})$ for $n = 1, 2, 3$..........If $h = 0$, the result is obvious, hence we consider the case when $h \neq 0$. Now, for n ≥ 1 we have

$$d(y_{2n+2}, y_{2n+1}) = d(Jx_{2n+1}, Ix_{2n+2}) \leq kH(Sx_{2n}, Tx_{2n+1})$$

$$\leq \sqrt{h}[aL(Ix_{2n}, Jx_{2n+1}) + (1-a)N(Ix_{2n}, Jx_{2n+1})],$$

where

$$L(Ix_{2n}, Jx_{2n+1}) = max\{d(Ix_{2n}, Jx_{2n+1}), d(Ix_{2n}, Sx_{2n}),$$

$$d(Jx_{2n+1}, Tx_{2n+1}), \frac{1}{2}[d(Ix_{2n}, Tx_{2n+1}) + d(Jx_{2n+1}, Sx_{2n})]\}$$

$$\leq max\{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, d(y_{2n+2})\}$$

and

$$N(Ix_{2n}, Jx_{2n+1}) \leq [max\{d^2(Ix_{2n}, Jx_{2n+1}), d(Ix_{2n}, Sx_{2n})d(Jx_{2n+1}, Tx_{2n+1}),$$

$$d(Ix_{2n}, Tx_{2n+1})d(Jx_{2n+1}, Sx_{2n}), \frac{1}{2}(d(Ix_{2n}, Sx_{2n})d(Jx_{2n+1}, Sx_{2n})),$$

$$\frac{1}{2}d(Ix_{2n}, Tx_{2n+1})d(Jx_{2n+1}, Tx_{2n+1})\}]^{\frac{1}{2}}$$

$$\leq [max\{d^2(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1})d(y_{2n+1}, y_{2n+2}), 0, 0,$$

$$\frac{1}{2}((d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}))d(y_{2n+1}, y_{2n+2})\}]^{\frac{1}{2}}.$$

$$\leq [max\{d^2(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1})d(y_{2n+1}, y_{2n+2}), d^2(y_{2n+1}, y_{2n+2})\}]^{\frac{1}{2}}.$$

Suppose on contrary that $d(y_{2n+1}, y_{2n+2}) > \sqrt{h}d(y_{2n}, y_{2n+1})$ for some $n \in N$. Then we have $d(y_{2n+1}, y_{2n+2}) < d(y_{2n+1}, y_{2n+2})$ which is contradiction and so

$$d(y_{2n+1}, y_{2n+2}) \leq \sqrt{h}d(y_{2n+1}, y_{2n}) \qquad (3.1.2)$$

Similarly one can show that

$$d(y_{2n+1}, y_{2n}) \leq \sqrt{h}d(y_{2n}, y_{2n-1})$$

which in general yields that

$d(y_{n+1}, y_n) \leq \sqrt{h}d(y_n, y_{n-1})$ for all n establishing that the sequence y_n described by

$$\{Ix_0, Jx_1, Ix_2, Jx_n..........Jx_{2n-1}, Ix_{2n}, Jx_{2n+1}....................\} \tag{3.1.3}$$

is a Cauchy sequence and get limit t in X. Hence the sequences $\{Ix_{2n}\}$ and $\{Jx_{2n+1}\}$ which are subsequences of $\{y_n\}$ also converge to the point t. Also by the fact that $H(Sx_{2n}, Tx_{2n+1}) \leq hd(Ix_{2n}, Jx_{2n+1})$ together with (3.1.3)one can conclude that

$$\{Sx_0, Tx_1, Sx_2, Tx_2, ...Tx_{2n-1}, Sx_{2n}, Tx_{2n+1}............\} \tag{3.1.4}$$

is a Cauchy sequence in $(CB(X), H)$. Hence he sequences $\{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ converge to some M in $CB(X)$. Now, one can have

$$d(t, M) \leq d(t, Ix_{2n}) + d(Ix_{2n}, M) \leq d(t, Ix_{2n}) + H(Tx_{2n-1}, M) \to 0 \; as \; n \to \infty,$$

establishing that $t \in M$ as M is closed. Now, by the weak compatibility of (S, I), one can write

$$lim_{n\to\infty} H(ISx_{2n}, SIx_{2n}) \leq lim_{n\to\infty} H(SIx_{2n}, Sx_{2n}) \tag{3.1.5}$$

$$lim_{n\to\infty} d(ISx_{2n}, Ix_{2n}) \leq lim_{n\to\infty} H(SIx_{2n}, Sx_{2n}) \tag{3.1.6}$$

Using the above mentioned inequality, we obtained

$$lim_{n\to\infty} \mathrm{d}(IIx_{2n}, Ix_{2n}) \leq lim_{n\to\infty} d(IIx_{2n}, ISx_{2n}) + lim_{n\to\infty} \mathrm{d}(ISx_{2n}, Ix_{2n})$$

$$\leq lim_{n\to\infty} d(IIx_{2n}, ISx_{2n}) + lim_{n\to\infty} H(SIx_{2n}, Sx_{2n}) \tag{3.1.7}$$

Since S and I are continuous, making $n \to \infty$,(3.1.5) (3.1.6)(3.1.7) we get

$$H(I(M), St) \leq H(St, M) \; and \; d(It, t) \leq H(St, M)$$

Similarly using the continuity and weak compatibility of the pair (T, J) one can show that

$$H(J(M), Tt) \leq H(Tt, M) \text{ and } d(Jt, t) \leq H(Tt, M)$$

Now

$$\begin{aligned} d(Jt, Tt) &\leq d(Jt, JIx_{2n}) + d(JIx_{2n}, Tt) \\ &\leq d(Jt, JIx_{2n}) + H(JTx_{2n-1}, Tt) \\ &\leq d(Jt, JIx_{2n}) + H(JTx_{2n-1}, TJx_{2n-1}) + d(TJx_{2n-1}, Tt) \end{aligned}$$

Which on letting n$\to \infty$, reduces to

$$d(Jt, Tt) \leq H(Tt, M)$$

Now using (3.1.1) we have

$$H(Sx_{2n}, Tt) \leq h[aL(Ix_{2n}, Jt) + (1-a)N(Ix_{2n}, Jt)],$$

Where

$$L(Ix_{2n}, Jt) \leq max\{d(Ix_{2n}, Jt), d(Ix_{2n}, Sx_{2n}), d(Jt, Tt), \frac{1}{2}[d(Ix_{2n}, Tt) + d(Jt, Ix_{2n}) + d(Ix_{2n}, Sx_{2n})]\}$$

which on letting $n \to \infty$, reduce to

$$lim_{n\to\infty} L(Ix_{2n}, Jt) \leq max\{H(Tt, M), 0, H(Tt, M), \frac{1}{2}[H(Tt, M) + H(Tt, M) + 0]\}$$
$$= H(M, Tt)$$

and

$$N(Ix_{2n}, Jt) \leq max\{d^2(Ix_{2n}, Jt), d(Ix_{2n}, Sx_{2n})d(Jt, Tt),$$

$$d(Ix_{2n}, Tt)[d(Jt, Ix_{2n}) + d(Ix_{2n}, Sx_{2n})],$$

$$\frac{1}{2}d(Ix_{2n}, Sx_{2n})[d(Jt, Ix_{2n}) + d(Ix_{2n}, Sx_{2n})], \frac{1}{2}[d(Ix_{2n}, Tt)d(Jt, Tt)\}]^{\frac{1}{2}}.$$

which on letting $n \to \infty$, reduces to

$$lim_{n\to\infty} N(Ix_{2n}, Jt) \leq [max\{d^2(t, Jt), d(t, M)d(Jt, Tt), d(t, Tt)[d(Jt, t) + d(t, M)],$$

$$\frac{1}{2}d(t, M)[d(Jt, t) + d(t, M)], \frac{1}{2}[d(t, Tt)d(Jt, Tt)\}]^{\frac{1}{2}}$$

$$\leq [max\{H^2(Tt, M), 0, H(Tt, M)[H(Tt, M) + 0], 0, \frac{1}{2}H^2(Tt, M)\}]^{\frac{1}{2}},$$

$$\leq H(M, Tt) \tag{3.1.8}$$

Thus

$$H(M, Tt) = lim_{n\to\infty} H(Sx_{2n}, Tt)$$

$$\leq h[a\ lim_{n\to\infty} L(Ix_{2n}, Jt) + (1-a) lim_{n\to\infty} N(Ix_{2n}, Jt)]$$

$$\leq h[aH(M, Tt) + (1-a)H(M, Tt)] = hH(M, Tt)$$

which implies that $H(M, Tt) = 0$. Therefore $d(Jt, Tt) = 0$ which in turn yields $Jt \in Tt$ as Tt is closed. Similarly, one can also show that $It \in St$.

Now it remains to show that $It = Jt$. For this we consider

$$d(It, Jt) \leq d(It, SIx_{2n}) + H(SIx_{2n}, TJx_{2n-1}) + d(TJx_{2n-1}, Jt)$$

$$\leq d(It, SIx_{2n}) + d(TJx_{2n-1}, Jt) + h[a\ max d(I^2x_{2n}, J^2x_{2n-1}), d(I^2x_{2n}, SIx_{2n}),$$

$$d(J^2x_{2n-1}, TJx_{2n-1}), \tfrac{1}{2}[d(I^2x_{2n}, Jt) + d(Jt, TJx_{2n-1}) + d(J^2x_{2n-1}, It) + d(It, SIx_{2n})]$$

$$+(1-a)[max\{d^2(I^2x_{2n}, J^2x_{2n-1}), d(I^2x_{2n}, SIx_{2n})d(J^2x_{2n-1}, TJx_{2n-1}),$$

$$(d(I^2x_{2n}, J^2x_{2n-1}) + d(J^2x_{2n-1}, TJx_{2n-1}))(d(J^2x_{2n-1}, I^2x_{2n}) + d(I^2x_{2n}, SIx_{2n})),$$

$$\frac{1}{2} d(I^2x_{2n}, SIx_{2n}) d(J^2x_{2n-1}, SIx_{2n}),$$

$$\frac{1}{2}[d(I^2x_{2n}, J^2x_{2n-1}) + d(J^2x_{2n-1}, TJx_{2n-1})] d(J^2x_{2n-1}, TJx_{2n-1})\})]^{\frac{1}{2}}$$

which on letting $n \to \infty$, reduces

$$d(It, Jt) \leq hd(It, Jt)$$

yielding thereby $It = Jt$

Thus we have shown that $It = Jt \in St \cap Tt$ establishing that t is a coincidence point of $I, J, S\ and\ T$.

This completes the proof.

In order to obtain a fixed point result corresponding to Theorem 3.1 one requires additional hypotheses. In this regard the following lemma from Pathak et al[10] is useful.

Lemma 3.2 *[10] Let* $S, T : X \to CB(X)$ *and* $I, J : X \to X$ *be continuous mappings if* $Iw = Jw \in Tw \cap Sw$ *for some* $w \in X$ *and Theorem 3.1 holds for all* x, y *in* X*, then* $JTw = TJw$*, and* $ISw = SIw$.

Proof: Let $x_n = w \; for \; all \; n \in N$. Hence if Iw=Jw $\in Tw \cap Sw$, then by weak compatibility of (S, I) and (T, J) one can have

$$H(ISw, SIw) \leq H(SIw, Sw) \qquad (3.2.1),$$

$$H(JTw, TJw) \leq H(TJw, Tw),$$

$$d(I^2w, Jw) \leq d(I^2w, ISw) + d(ISw, Iw) + d(Iw, Jw) \leq H(SIw, Sw),$$

and similarly

$$d(Iw, J^2w) \leq H(SIw, Sw).$$

Now

$$H(SIw, Sw) = H(SIw, Tw)$$

$$\leq h[aL(I^2w, Jw) + (1-a)N(I^2w, Jw)] \qquad (3.2.2)$$

where

$$L(I^2w, Jw) = max\{d(I^2w, Jw), d(I^2w, SIw), d(Jw, Tw), \frac{1}{2}[d(I^2w, Tw)+d(Jw, SIw)]\}$$

$$\leq max\{H(SIw, Sw), H(SIw, Sw), 0, H(SIw, Sw)\},$$

and

$$N(I^2w, Jw) = [max\{d^2(I^2w, Jw), d(I^2w, SIw)d(Jw, Tw), d(I^2w, Tw)d(Jw, SIw),$$

$$\frac{1}{2}[d(I^2w, SIw)d(Jw, SIw), \frac{1}{2}[d(I^2w, Tw)d(Jw, Tw)\}]^{\frac{1}{2}}$$

$$\leq [max\{H^2(SIw, Sw), 0, H^2(SIw, Sw), \frac{1}{2}H^2(SIw, Sw), 0, \}]^{\frac{1}{2}},$$

$$= H(SIw, Sw)$$

which in turn yields that

$$H(SIw, Sw) = H(SIw, Tw) \leq h[a.H(SIw, Sw) + (1-a)H(SIw, Sw)]$$

$$= hH(SIw, Sw)$$

which is a contradiction.Therefore, we have $SIw = Sw$.Hence from (3.2.1)

$$SIw = ISw$$

Similarly we can show that $TJw = JTw$.

Now we formulate a fixed point theorem as follows:

Theorem 3.3 *Let S, T, I and J satisfy all the conditions of Theorem 3.1. Assume that for each $x \in X$ either*

$(i) Ix \neq I^2x \Rightarrow Ix \notin Sx (resp, Jx \neq J^2x \Rightarrow Jx \notin Tx$

$(ii) Ix \in Sx \Rightarrow I^nx \to w for some\ w \in X (resp Jx \in Tx \Rightarrow J^nx \to w'$

for some $w' \in X$, then S, T, I and J have a common fixed point in X.

Proof: By Theorem 3.1 there exists a point z in X such that $Iz = Jz \in Sz \cap Tz$. Since $Iz \in Sz$, Lemma 3.2 yields $ISz = SIz$. If (i) holds, $Iz = I^2z \in ISz = SIz$. Thus $w = Iz$ is the fixed point of I and S.

If (ii) holds, then it is clear that $Iw = w$ as I is continuous. Now we assert that $I^nz \in SI^{n-1}z$ for each n. To verify this, we consider $I^2z = IIz \in ISz = SIz$. Using Lemma3.2 $(w = Iz)$ we can have $I^3z = II^2z \in I(ISz) = SI^2z$. Thus inductively we get $I^nz = SI^{n-1}z$ and hence the continuity implies that

$$d(w, Sw) \leq d(w, I^nz) + d(I^nz, Sw)$$

$$\leq d(w, I^nz) + d(SI^{n-1}z, Sw)$$

which tends to zero as n $\to \infty$. Hence $w = Iw \in Sw\ as\ Sw$ is closed. Similarly one can show that $w' = Jw' \in Tw'$.

Now using contraction condition, one can obtains

$$d(w, w') = d(Iw, Jw')$$

$$= H(Sw, Tw')$$

$$\leq h[ad(Iw, Jw') + (1 - a)d(Iw, Jw')]$$

$$\leq hd(w, w')$$

implying thereby $w = w$'

Thus we prove have that $w = Iw = Jw \in Sw \cap Tw$. Hence w is a common fixed point of S,T,I and J.

If we replace weak compatibility[6],[10] by weak commutativity due to Hadzic-Gajic [4], then the continuity of S and T can be relaxed and no additional hypotheses are needed to ensure the existence of coincidence point which appears to be a noted improvement over Theorem 3.1.

Theorem 3.4 *Let S, T, I, J, X and $CB(X)$ be the same as in Theorem 3.1. If we replace the weak compatibility with weak commutativity in Theorem 3.1 with I and J continuous then there is a point t in X such that $It = Jt \in St \cap Tt$.*

Proof: Proceeding as in Theorem 3.1, we can show that the subsequences Ix_{2n}, Jx_{2n+1} converge to some t in X whereas the sequences Sx_{2n}, Tx_{2n+1} converge to some M in $CB(X)$.

Since J is continuous, sequence JIx_{2n} converges Jt. Now, using the weak commutativity of (T, J), we have $Ix_{2n} \in Tx_{2n-1}$ and so

$$d(JIx_{2n}, TJx_{2n-1}) = d(JTx_{2n-1}, TJx_{2n-1}) \leq d(Jx_{2n-1}, Tx_{2n-1}) \leq d(Ix_{2n}, Jx_{2n-1})$$

which on letting $n \to \infty$, reduce to

$$d(Jt, TJx_{2n-1}) \to 0$$

Similarly, using the continuity of I and weak commutativity of (S, I), we can show that

$$d(It, SIx_{2n}) \to 0 \; as \; n \to \infty.$$

Now consider

$$d(It, Jt) \leq d(It, SIx_{2n}) + H(SIx_{2n}, TJx_{2n-1}) + d(TJx_{2n-1}, Jt)$$

$$\leq d(It, SIx_{2n}) + d(Jt, TJx_{2n-1}) + h\{[a \, maxd(I^2x_{2n}, J^2x_{2n-1}), d(I^2x_{2n}, SIx_{2n}),$$

$$d(J^2x_{2n-1}, TJx_{2n-1}), \tfrac{1}{2}[d(I^2x_{2n}, Jt) + d(Jt, TJx_{2n-1}) + d(J^2x_{2n-1}, It) + d(It, SIx_{2n})]\}$$

$$+(1-a)[max\{d^2(I^2x_{2n}, J^2x_{2n-1}), d(I^2x_{2n}, SIx_{2n})d(J^2x_{2n-1}, TJx_{2n-1}),$$

$$[d(I^2x_{2n}, TJx_{2n-1})d(J^2x_{2n-1}, SIx_{2n})], \frac{1}{2}d(I^2x_{2n}, SIx_{2n})d(J^2x_{2n-1}, SIx_{2n}),$$

$$\frac{1}{2}[I^2x_{2n}, TJx_{2n-1})d(J^2x_{2n-1}, TJx_{2n-1})\})]^{\frac{1}{2}}$$

which on letting $n \to \infty$, reduces

$d(It, Jt) \leq hd(It, Jt)$, yielding thereby $It = Jt$.

Now

$$d(Jt, St) \leq d(Jt, TJx_{2n-1}) + H(TJx_{2n-1}, St)$$

$$\leq d(Jt, JTx_{2n-1}) + d(JTx_{2n-1}, TJx_{2n-1}) + H(ST, TJx_{2n-1},)$$

$$\leq d(Jt, JTx_{2n-1}) + d(JTx_{2n-1}, TJx_{2n-1}) + h[a\ max\{d(It, J^2x_{2n-1}),$$

$$d(It, St)d(J^2x_{2n-1}, TJx_{2n-1}), \tfrac{1}{2}[d(It, TJx_{2n-1}) + d(J^2x_{2n-1}, St)]\}$$

$$+(1-a)[maxd^2(I^2t, Jx^2_{2n-1}), d(It, St)d(J^2x_{2n-1}, TJx_{2n-1})$$

,

$$d(It, TJx_{2n-1})d(J^2x_{2n-1}, St)], \frac{1}{2}d(It, St)d(J^2x_{2n-1}, St)$$

,

$$\frac{1}{2}d(It, TJx_{2n-1})d(J^2x_{2n-1}, TJx_{2n-1})]\}]^{\frac{1}{2}}$$

which on letting $n \to \infty$, reduce to

$$d(Jt, St) \leq hd(Jt, St),$$

yielding thereby $Jt \in St$, as St is closed.

Similarly one can show that $It \in Tt$. Thus we have shown that $It = Jt \in St \cap Tt$.

Remark:

(a) If we replace $CB(X)$ *by* $PB(X)$(with $ISx, JTx \in PB(X)$)and choose $L(x, y) = d(Ix, Jy), a = 1$ in Theorem 3.1 then we get an improve version of Corollary 2.2 of Pathak et al.[10] as it involves four mappings instead of two.

(b) If we choose $a = 1$ in Theorem,then we get sharpen version of Theorem 2 of [9] which in tern generalizes the main result of Kaneko and Sessa[6]

Related Example: We present example to discuss the validity of the hypotheses of main results.

Example: Let $X = [0, \infty)$ be endowed with the Euclidean metric $d(x, y) = \mid x - y \mid$.Let $I(X)=\frac{3}{2}(x^4 + x^2)$, $J(X)= \frac{3}{2}(x^2 + x)$, $T(X) = [0, x^2 + 2], S(X) = [0, x^4 + 2]$ for each $x \geq 0$. Then I,J,S and T are continuous and $I(X) = J(X) = T(X) = S(X)$. Since $Sx_n, Tx_n \to [0, 3]$ and $Ix_n, Jx_n \to 3$ if $x_n \to 1$. We observe by the verification that

$$d(ISx_n, Ix_n) \to 0, \qquad H(ISx_n, SIx_n) \to 52, \qquad H(SIx_n, Sx_n) \to 80,$$
$$d(JTx_n, Jx_n) \to 0, \qquad H(JTx_n, TJx_n) \to 7, \qquad H(TJx_n, Tx_n) \to 8,$$

Therefore (S, I) and (T, J) are weak compatible but they are not compatible. Also since

$$H(Sx, Ty) = \mid x^4 - y^2 \mid$$

$$\tfrac{(x^2+y)}{(x^2+y+1)} \mid x^2 - y \mid\mid x^2 + y + 1 \mid$$

$$\frac{2(x^2 + y)3}{3(x^2 + y + 1)2} \mid x^4 - y^2 + x^2 - y \mid$$

$$\leq \frac{2}{3} d(Ix, Jy) = h[aL(Ix, Jy) + (1 - a)N(Ix, Jy)]$$

for all $x, y \ \in X$, where $h \in [\frac{2}{3}, 1]$ and $0 \leq a \leq 1$. Thus all the conditions of Theorem 3.1 are satisfied and 0 is the unique common fixed point of S ,T,I and J.

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