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Coefficient Estimates for λ–Bazilevič Functions of Bi-univalent Functions

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Abstract

In this paper, we introduce two new subclasses of the function class Σ of λ–Bazilevič functions of bi-univalent functions defined in the open unit disc. We find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses. The results presented in this paper would generalize some recent works of Xu et al. and Ali et al.

Keywords: *Analytic functions, Univalent functions, Bazilevič functions, Bi-univalent functions, Coefficient estimates.*

1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disc $U = \{z \in C : |z| < 1\}$. We also denote by $\mathcal{S}$ the subclass of the normalized analytic function class $\mathcal{A}$ consisting of all functions in $\mathcal{A}$ which are also univalent in U (see [1-4]). Familiar subclasses of starlike functions of order $\xi (0 \leq \xi < 1)$ and convex functions of order ξ for which either of the quantity

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \xi \quad \text{and} \quad \Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \xi.$$

The class consisting these two functions are given by $\mathcal{S}^*(\xi)$ and $\mathcal{K}(\xi)$, respectively. For a constant $\beta \in (-\pi/2, \pi/2)$, a function f is univalent on U and satisfies the condition that $\Re\{e^{i\theta}zf'(z)/f(z)\} > 0$ in U. We denote this class by $\mathcal{TS}^*$(see [2]).

It is well known that every function $f(z) \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \quad (z \in U)$$

and

$$f(f^{-1}(\omega)) = \omega, \quad (|\omega| < r_0(f), r_0(f) \geq \frac{1}{4}).$$

In fact, the inverse function is given by

$$f^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \cdots \qquad (2)$$

A function $f \in \mathcal{S}$ is bi-univalent in U if both f and f^{-1} are univalent in U. We denote by Σ the class of all bi-univalent functions in U given by the Taylor-Maclaurin series expansion (1). Lewin [5] investigated the class Σ of bi-univalent functions and obtained the bound for the second coefficient. Several authors have subsequently studied similar problems in this direction (see [6]). Srivastava et al.[7], and Frasin and Aouf [8] introduced subclasses of bi-univalent functions and obtained bounds for the initial coefficients. Recently, Xu et al. [9], Goyal and Goswami [10] and Ali et al.[11] introduced and investigated subclasses of bi-univalent functions and obtained bounds for the initial coefficients.

Let f and g be analytic functions in U, we say that f is subordinate to g, written as $f(z) \prec g(z)$ if there exists a Schwarz function $\omega(z)$ in U, with $\omega(0) = 0$ and $|\omega(z)| < 1 (z \in U)$, such that $f(z) = g(\omega(z))$. In particular, when g is univalent, then the above subordination is equivalent to $f(0) = 0$ and $f(U) \subseteq g(U)$.

Let

$$H(U) = \{h : U \to C, \Re\{h(z)\} > 0 \;\text{ and }\; h(0) = 1, h(\overline{z}) = \overline{h(z)} (z \in U)\}.$$

Assume that φ is an analytic univalent function with positive part in U, $\varphi(U)$ is symmetric with respect to the real axis and starlike with respect to $\varphi(0) = 1$, and $\varphi'(0) > 0$. Such a function has series expansion of the form

$$\varphi(z) = 1 + B_1z + B_2z^2 + \cdots, \quad (B_1 > 0). \qquad (3)$$

Obviously, $\varphi(U) \subseteq H(U)$.

Wang et al.[13] (also see Li [14]) introduced and investigated the class of $\lambda-$Bazilevič functions consists of functions $f \in \mathcal{A}$ satisfying the subordination:

$$(1-\lambda)\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{g(z)}\right)^{\alpha+i\mu} + \lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\left(\frac{f'(z)}{g'(z)}\right)^{\alpha+i\mu} \prec \frac{1+Az}{1+Bz}$$

$$(\alpha \geq 0, \lambda \geq 0, \mu, A, B \in R \text{ and } A \neq B, -1 \leq B \leq 1; g \in S^*(\xi)).$$

In this paper, using the subordination, we introduce the following two classes of $\lambda-$Bazilevič functions of bi-univalent functions.

Definition 1.1 *Let the function $f(z)$, defined by (1), be in the analytic function class $\mathcal{A}$. We say that $f(z) \in U_\alpha^{p,q}(\beta, b, \lambda)$ if the following conditions are satisfied:*

$$f(z) \in \Sigma$$

and

$$\left\{\frac{e^{i\beta}}{\cos\beta}\left[1+\frac{1}{b}\left((1-\lambda)\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha} + \lambda\left(1+\frac{zf''(z)}{f'(z)}\right)(f'(z))^{\alpha} - 1\right)\right] - i\tan\beta\right\} \in p(U) \quad (4)$$

and

$$\left\{\frac{e^{i\beta}}{\cos\beta}\left[1+\frac{1}{b}\left((1-\lambda)\frac{\omega g'(\omega)}{g(\omega)}\left(\frac{g(\omega)}{\omega}\right)^{\alpha} + \lambda\left(1+\frac{\omega g''(\omega)}{g'(\omega)}\right)(g'(\omega))^{\alpha} - 1\right)\right] - i\tan\beta\right\} \in q(U) \quad (5)$$

$$(p(z), q(\omega) \in H(U); z, \omega \in U)$$

where $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2}); b \in C\backslash\{0\}; \alpha \geq 0, \lambda \geq 0; n \in N_0$, the function $g(\omega) = f^{-1}(\omega)$ is given by (2).

Definition 1.2 *Let the function $f(z)$ of the form (1), be in the analytic function class $\mathcal{A}$. We say that $f(z) \in L_\alpha^{\varphi}(\beta, b, \lambda)$ if the following conditions are satisfied:*

$$f(z) \in \Sigma$$

and

$$\frac{e^{i\beta}}{\cos\beta}\left[1+\frac{1}{b}\left((1-\lambda)\frac{zf'(z)}{f(z)}(\frac{f(z)}{z})^{\alpha} + \lambda(1+\frac{zf''(z)}{f'(z)})(f'(z))^{\alpha} - 1\right)\right] - i\tan\beta \prec \varphi(z)$$

and

$$\frac{e^{i\beta}}{\cos\beta}\left[1+\frac{1}{b}\left((1-\lambda)\frac{\omega g'(\omega)}{g(\omega)}(\frac{g(\omega)}{\omega})^{\alpha}+\lambda(1+\frac{\omega g''(\omega)}{g'(\omega)})(g'(\omega))^{\alpha}-1\right)\right]$$

$$-i\tan\beta \prec \varphi(\omega)$$

$$(g(\omega)=f^{-1}(\omega); z,\omega\in U).$$

For $f(z)\in L_{\alpha}^{\varphi}(\beta,b,\lambda)$ and $\varphi=(\frac{1+z}{1-z})^{\eta}(0<\eta\leq 1)$, Definition 1.1 readily yields the following class $B_{\alpha}^{\eta}(\beta,b,\lambda)$ satisfying:

$$f(z)\in\Sigma$$

and

$$\left|\arg\frac{e^{i\beta}}{\cos\beta}\left[1+\frac{1}{b}\left((1-\lambda)\frac{zf'(z)}{f(z)}(\frac{f(z)}{z})^{\alpha}+\lambda(1+\frac{zf''(z)}{f'(z)})(f'(z))^{\alpha}-1\right)\right]\right|<\frac{\eta\pi}{2}$$

and

$$\left|\arg\frac{e^{i\beta}}{\cos\beta}\left[1+\frac{1}{b}\left((1-\lambda)\frac{\omega g'(\omega)}{g(\omega)}(\frac{g(\omega)}{\omega})^{\alpha}+\lambda(1+\frac{\omega g''(\omega)}{g'(\omega)})(g'(\omega))^{\alpha}-1\right)\right]\right|<\frac{\eta\pi}{2}$$

$$(g(\omega)=f^{-1}(\omega); 0<\eta\leq 1; z,\omega\in U)$$

For $f(z)\in L_{\alpha}^{\varphi}(\beta,b,\lambda)$ and $\varphi=(\frac{1+Az}{1+Bz})(-1\leq B<A\leq 1)$, Definition 1.2 readily yields the following class $L_{\alpha}^{A,B}(\beta,b,\lambda)$ satisfying:

$$f(z)\in\Sigma$$

and

$$\frac{e^{i\beta}}{\cos\beta}\left[1+\frac{1}{b}\left((1-\lambda)\frac{zf'(z)}{f(z)}(\frac{f(z)}{z})^{\alpha}+\lambda(1+\frac{zf''(z)}{f'(z)})(f'(z))^{\alpha}-1\right)\right]$$

$$-i\tan\beta \prec \frac{1+Az}{1+Bz}$$

and

$$\frac{e^{i\beta}}{\cos\beta}\left[1+\frac{1}{b}\left((1-\lambda)\frac{\omega g'(\omega)}{g(\omega)}(\frac{g(\omega)}{\omega})^{\alpha}+\lambda(1+\frac{\omega g''(\omega)}{g'(\omega)})(g'(\omega))^{\alpha}-1\right)\right]$$

$$-i\tan\beta \prec \frac{1+A\omega}{1+B\omega}$$

$$(g(\omega)=f^{-1}(\omega); z,\omega\in U).$$

For suitable choices of p, q and by specializing the parameters $b, \lambda, \alpha, \eta, \beta$ involved in the class $U_{\alpha}^{p,q}(\beta, b, \lambda)$, $L_{\alpha}^{\varphi}(\beta, b, \lambda)$ and $B_{\alpha}^{\eta}(\beta, b, \lambda)$, we also obtain the following subclasses which were studied in many earlier works:

(1)$S_{\Sigma}(\xi) = L_0^{1-2\xi,-1}(0,1,0)$(Bi-Starlike function)(Brannan and Taha [6]);
(2)$K_{\Sigma}(\xi) = L_0^{1-2\xi,-1}(0,1,1)$(Bi-Starlike function)(Brannan and Taha [6]);
(3)$U(p,q) = U_1^{p,q}(0,1,0)$ (Xu et al. [9]);
(4)$M_{\Sigma}(p,\lambda) = U_0^{p,p}(0,1,\lambda)$(General Bi-Mocanu-convex function of Ma-Minda) (Ali et al.[11]);
(5)$B_{\Sigma}(\alpha,\varphi) = L_{\alpha}^{\varphi}(0,1,0)$(Bi-Bazilevič functions of Ma-Minda type [16]).

In this paper, estimates on the initial coefficients for class $U_{\alpha}^{p,q}(\beta, b, \lambda)$, $L_{\alpha}^{\varphi}(\beta, b, \lambda)$and $B_{\alpha}^{\eta}(\beta, b, \lambda)$ are obtained. Several related classes are also considered, and a connection to earlier known results is made.

2 Coefficient Bounds for the Function Class $U_{\alpha}^{p,q}(\beta, b, \lambda)$

Theorem 2.1 *Suppose that $f(z) \in \mathcal{A}$ of the form (1), be in the class $U_{\alpha}^{p,q}(\beta, b, \lambda)$. Then*

$$|a_2| \leq \min\left\{\frac{|b|\cos\beta\sqrt{|p'(0)|^2 + |q'(0)|^2}}{(\alpha+1)(\lambda+1)}, \sqrt{\frac{(|p''(0)| + |q''(0)|)|b|\cos\beta}{2(\alpha+2)[\lambda+\alpha(1+3\lambda)+1]}}\right\} \quad (6)$$

and

$$|a_3| \leq \min\left\{\frac{(|p''(0)| + |q''(0)|)|b|\cos\beta}{4(\alpha+2)(2\lambda+1)} + \frac{|b|^2\cos^2\beta(|p'(0)|^2 + |q'(0)|^2)}{2(\alpha+1)^2(\lambda+1)^2}, \right.$$
$$\left. \frac{|b|\cos\beta}{4(\alpha+2)} \cdot \frac{|p''(0)|[5\lambda+\alpha(3\lambda+1)+3] + |q''(0)|(3\lambda+1)|1-\alpha|}{(2\lambda+1)[\lambda+\alpha(3\lambda+1)+1]}\right\}. \quad (7)$$

Proof. It follows from the conditions (4) and (5) that

$$e^{i\beta}\left\{1 + \frac{1}{b}\left[(1-\lambda)\frac{zf'(z)}{f(z)}(\frac{f(z)}{z})^{\alpha} + \lambda(1 + \frac{zf''(z)}{f'(z)})(f'(z))^{\alpha} - 1\right]\right\}$$
$$= p(z)\cos\beta + i\sin\beta \quad (8)$$

and

$$e^{i\beta}\left\{1 + \frac{1}{b}\left[(1-\lambda)\frac{\omega g'(\omega)}{g(\omega)}(\frac{g(\omega)}{\omega})^{\alpha} + \lambda(1 + \frac{\omega g''(\omega)}{g'(\omega)})(g'(\omega))^{\alpha} - 1\right]\right\}$$

$$= q(\omega)\cos\beta + i\sin\beta, \tag{9}$$

where $p(z) \in H(U), q(\omega) \in H(U)$. Furthermore, the functions $p(z)$ and $q(\omega)$ have the following series expansions

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots, p_m = \frac{p^{(m)}(0)}{m!} \quad (m \in N) \tag{10}$$

and

$$q(\omega) = 1 + q_1 \omega + q_2 \omega^2 + \cdots, q_m = \frac{q^{(m)}(0)}{m!} \quad (m \in N) \tag{11}$$

respectively. Now, in view of the series expansions (10) and (11), by equating the coefficients in (8) and (9), we get

$$\frac{e^{i\beta}}{b}\left[(\alpha+1)(\lambda+1)\right]a_2 = p_1\cos\beta, \tag{12}$$

$$\frac{e^{i\beta}}{b}\left[(\alpha+2)(2\lambda+1)a_3 + \frac{(\alpha-1)(\alpha+2)(3\lambda+1)}{2}a_2^2\right] = p_2\cos\beta, \tag{13}$$

$$-\frac{e^{i\beta}}{b}\left[(\alpha+1)(\lambda+1)\right]a_2 = q_1\cos\beta, \tag{14}$$

and

$$\frac{e^{i\beta}}{b}\left[(\alpha+2)(2\lambda+1)(2a_2^2 - a_3) + \frac{(\alpha-1)(\alpha+2)(3\lambda+1)}{2}a_2^2\right] = q_2\cos\beta. \tag{15}$$

We find from (12) and (14) that

$$p_1 = -q_1 \tag{16}$$

and

$$2e^{i2\beta}\left[\frac{(\alpha+1)(\lambda+1)}{b}\right]^2 a_2^2 = (p_1^2 + q_1^2)\cos^2\beta. \tag{17}$$

Also, from (13) and (15), we obtain

$$\frac{e^{i\beta}}{b}(\alpha+2)[\lambda + \alpha(3\lambda+1)+1]\,a_2^2 = (p_2+q_2)\cos\beta. \tag{18}$$

Therefore, we find from (17) and (18) that

$$a_2^2 = \frac{(p_1^2+q_1^2)b^2\cos^2\beta}{2e^{2i\beta}(\alpha+1)^2(\lambda+1)^2} \tag{19}$$

and

$$a_2^2 = \frac{(p_2+q_2)b\cos\beta}{e^{i\beta}(\alpha+2)[\lambda+\alpha(3\lambda+1)+1]}. \tag{20}$$

Since $p_1 = p'(0), p_2 = \frac{p''(0)}{2}, q_1 = q'(0), q_2 = \frac{q''(0)}{2}$, it follows from (19) and (20) that

$$|a_2| \leq \frac{|b| \cos \beta \sqrt{|p'(0)|^2 + |q'(0)|^2}}{(\alpha+1)(\lambda+1)}$$

and

$$|a_2| \leq \sqrt{\frac{(|p''(0)| + |q''(0)|)|b| \cos \beta}{2(\alpha+2)[\lambda + \alpha(1+3\lambda)+1]}},$$

which gives us the desired estimate on $|a_2|$ as asserted in (6).

Next, in order to find the bound on $|a_3|$, by subtracting (13) from (15), we get

$$\frac{e^{i\beta}}{b} \left[2(\alpha+2)(2\lambda+1)\right] (a_3 - a_2^2) = (p_2 - q_2) \cos \beta. \tag{21}$$

Thus, upon substituting the value of a_2^2 from (16) and (19) into (21), it follows that

$$a_3 = \frac{(p_2 - q_2) b \cos \beta}{2e^{i\beta}(\alpha+2)(2\lambda+1)} + \frac{b^2 \cos^2 \beta (p_1^2 + q_1^2)}{2e^{2i\beta}(\alpha+1)^2(\lambda+1)^2},$$

which yields

$$|a_3| \leq \frac{(|p''(0)| + |q''(0)|)|b| \cos \beta}{4(\alpha+2)(2\lambda+1)} + \frac{|b|^2 \cos^2 \beta (|p'(0)|^2 + |q'(0)|^2)}{(\alpha+1)^2(\lambda+1)^2}.$$

On the other hand, by using (16) and (10) in (21), we obtain

$$a_3 = \frac{b \cos \beta}{2e^{i\beta}(\alpha+2)} \cdot \frac{[5\lambda + \alpha(1+3\lambda) + 3]p_2 + (3\lambda+1)(1-\alpha)q_2}{(2\lambda+1)[\lambda + \alpha(1+3\lambda)+1]},$$

it follows that

$$|a_3| \leq \frac{|b| \cos \beta}{4(\alpha+2)} \cdot \frac{[5\lambda + \alpha(1+3\lambda) + 3]|p''(0)| + (3\lambda+1)|1-\alpha||q''(0)|}{(2\lambda+1)[\lambda + \alpha(1+3\lambda)+1]}.$$

This completes the proof of Theorem 2.1.

For $b = 1, \beta = 0$, Theorem 2.1 readily yields the following coefficient estimates for $U_{\alpha}^{p,q}(0, 1, \lambda)$.

Corollary 2.2 *Suppose that $f(z) \in \mathcal{A}$ of the form (1), be in the class $U_{\alpha}^{p,q}(0, 1, \lambda)$. Then*

$$|a_2| \leq \min \left\{ \frac{\sqrt{|p'(0)|^2 + |q'(0)|^2}}{(\alpha+1)(\lambda+1)}, \sqrt{\frac{|p''(0)| + |q''(0)|}{2(\alpha+2)[\lambda + \alpha(3\lambda+1)+1]}} \right\}$$

and

$$|a_3| \leq \min \left\{ \frac{|p''(0)| + |q''(0)|}{4(\alpha+2)(2\lambda+1)} + \frac{(|p'(0)|^2 + |q'(0)|^2)}{2(\alpha+1)^2(\lambda+1)^2}, \right.$$
$$\left. \frac{|p''(0)|[5\lambda + \alpha(3\lambda+1)+3] + |q''(0)|(3\lambda+1)|1-\alpha|}{4(\alpha+2)(2\lambda+1)[\lambda + \alpha(3\lambda+1)+1]} \right\}.$$

For $b=1,\beta=0,\alpha=0,\lambda=0$, we obtain the results in [15] by S. Bulut.

For $b=1,\beta=0,\alpha=1,\lambda=0$, Theorem 2.1 readily improve coefficient estimates for $U(p,q)$ in [9] as follows.

Corollary 2.3 *Suppose that $f(z)\in\mathcal{A}$ of the form (1), be in the class $U(p,q)$. Then*

$$|a_2|\leq\min\left\{\frac{|p'(0)|}{2},\sqrt{\frac{|p''(0)|+|q''(0)|}{12}}\right\}$$

and

$$|a_3|\leq\min\left\{\frac{|p''(0)|+|q''(0)|}{12}+\frac{(p'(0))^2}{4},\frac{|p''(0)|}{6}\right\}.$$

3 Coefficient Bounds for the Function Class $L^{\varphi}_{\alpha}(\beta,b,\lambda)$ and $B^{\eta}_{\alpha}(\beta,b,\lambda)$

In order to prove our main results, we first recall the following lemmas.

Lemma 3.1 *(see [12]) If $p(z)\in\mathcal{P}$, then $|p_k|\leq 2$ for each k, where $\mathcal{P}$ is the family of all functions $p(z)$ analytic in U for which $\Re\{p(z)\}>0, p(z)=1+p_1z+p_2z^2+\cdots$ for $z\in U$.*

Theorem 3.2 *Suppose that $f(z)\in\mathcal{A}$ of the form (1), be in the class $L^{\varphi}_{\alpha}(\beta,b,\lambda)$. Then*

$$|a_2|\leq\min\left\{\frac{|B_1||b|\cos\beta}{(\alpha+1)(\lambda+1)},\sqrt{\frac{2|b|\cos\beta(|B_1|+|B_2-B_1|)}{(\alpha+2)[\lambda+\alpha(3\lambda+1)+1]}},\right.$$
$$\left.\frac{|B_1|\sqrt{2|B_1||b|\cos\beta}}{\sqrt{|B_1^2b\cos\beta(\alpha+2)[\lambda+\alpha(3\lambda+1)+1]-2(B_2-B_1)e^{i\beta}(\alpha+1)^2(\lambda+1)^2|}}\right\} \tag{22}$$

and

$$|a_3|\leq\min\left\{\frac{|B_1||b|\cos\beta}{(\alpha+2)(2\lambda+1)}+\frac{|B_1|^2|b|^2\cos^2\beta}{(\alpha+1)^2(\lambda+1)^2},Q_1,Q_2\right\}, \tag{23}$$

where

$$Q_1=\frac{|b|\cos\beta\{[5\lambda+(\alpha+|1-\alpha|)(1+3\lambda)+3]|B_1|+4(2\lambda+1)|B_2-B_1|\}}{2(\alpha+2)(2\lambda+1)[\lambda+\alpha(3\lambda+1)+1]},$$

$$Q_2=\frac{|B_1b|\{B_1^2|b|\cos\beta(\alpha+2)[5\lambda+(\alpha+|1-\alpha|)(3\lambda+1)+3]+Q_3\}}{2(\alpha+2)(2\lambda+1)|B_1^2b(\alpha+2)[\lambda+\alpha(3\lambda+1)+1]-Q_4|},$$

$$Q_3=4|B_2-B_1|(\alpha+1)^2(\lambda+1)^2$$

and

$$Q_4=2(B_2-B_1)(1+i\tan\beta)(1+\alpha)^2(1+\lambda)^2.$$

Proof. Let $f \in L_{\alpha}^{\varphi}(\beta, b, \lambda)$, consider the analytic functions $u, v : U \longrightarrow U$, with $u(0) = v(0) = 0$, such that

$$e^{i\beta}\left\{1+\frac{1}{b}\left[(1-\lambda)\frac{zf'(z)}{f(z)}(\frac{f(z)}{z})^{\alpha}+\lambda(1+\frac{zf''(z)}{f'(z)})(f'(z))^{\alpha}-1\right]\right\}$$
$$=\varphi(u(z))\cos\beta+i\sin\beta \tag{24}$$

and

$$e^{i\beta}\left\{1+\frac{1}{b}\left[(1-\lambda)\frac{\omega g'(\omega)}{g(\omega)}(\frac{g(\omega)}{\omega})^{\alpha}+\lambda(1+\frac{\omega g''(\omega)}{g'(\omega)})(g'(\omega))^{\alpha}-1\right]\right\}$$
$$=\varphi(v(\omega))\cos\beta+i\sin\beta, \tag{25}$$

where $g := f^{-1}$.
Define the function m and n by

$$m(z)=\frac{1+u(z)}{1-u(z)}=1+m_1z+m_2z^2+\cdots,$$

and

$$n(z)=\frac{1+v(z)}{1-v(z)}=1+n_1z+n_2z^2+\cdots,$$

it follows that

$$u(z)=\frac{m(z)-1}{m(z)+1}=\frac{m_1}{2}z+\frac{1}{2}(m_2-\frac{m_1^2}{2})z^2+\cdots \tag{26}$$

and

$$v(z)=\frac{n(z)-1}{n(z)+1}=\frac{n_1}{2}z+\frac{1}{2}(n_2-\frac{n_1^2}{2})z^2+\cdots \tag{27}$$

It is clear that m and n are analytic in U and $m(0) = n(0) = 1$. Since $u, v : U \longrightarrow U$, the function m and n have positive real part in U, by virtue of Lemma 3.1, we have $|m_i| \leq 2$ and $|n_i| \leq 2 \quad (i = 1, 2, \cdots)$.

From (24),(25),(26) and (27), it follows that

$$e^{i\beta}\left\{1+\frac{1}{b}\left[(1-\lambda)\frac{zf'(z)}{f(z)}(\frac{f(z)}{z})^{\alpha}+\lambda(1+\frac{zf''(z)}{f'(z)})(f'(z))^{\alpha}-1\right]\right\}$$
$$=\varphi(\frac{m(z)-1}{m(z)+1})\cos\beta+i\sin\beta \tag{28}$$

and

$$e^{i\beta}\left\{1+\frac{1}{b}\left[(1-\lambda)\frac{\omega g'(\omega)}{g(\omega)}(\frac{g(\omega)}{\omega})^{\alpha}+\lambda(1+\frac{\omega g''(\omega)}{g'(\omega)})(g'(\omega))^{\alpha}-1\right]\right\}$$

$$= \varphi(\frac{n(\omega)-1}{n(\omega)+1})\cos\beta + i\sin\beta. \tag{29}$$

According to (3), it is evident that

$$\varphi(u(z)) = 1 + \frac{m_1 B_1}{2} z + (\frac{m_2 B_1}{2} + \frac{m_1^2(B_2 - B_1)}{4}) z^2 + \cdots$$

and

$$\varphi(v(\omega)) = 1 + \frac{n_1 B_1}{2} \omega + (\frac{n_2 B_1}{2} + \frac{n_1^2(B_2 - B_1)}{4}) \omega^2 + \cdots$$

Since

$$e^{i\beta}\left\{1 + \frac{1}{b}\left[(1-\lambda)\frac{zf'(z)}{f(z)}(\frac{f(z)}{z})^\alpha + \lambda(1 + \frac{zf''(z)}{f'(z)})(f'(z))^\alpha - 1\right]\right\}$$

$$= e^{i\beta} + \frac{e^{i\beta}(\alpha+1)(\lambda+1)}{b} a_2 z + \frac{e^{i\beta}}{b}\Big[(\alpha+2)(2\lambda+1)a_3 + \frac{(\alpha-1)(\alpha+2)(3\lambda+1)}{2} a_2^2\Big] z^2 + \cdots$$

and

$$e^{i\beta}\left\{1 + \frac{1}{b}\left[(1-\lambda)\frac{\omega g'(z)}{g(z)}(\frac{g(\omega)}{\omega})^\alpha + \lambda(1 + \frac{\omega g''(\omega)}{g'(\omega)})(g'(\omega))^\alpha - 1\right]\right\}$$

$$= e^{i\beta} - \frac{e^{i\beta}(\alpha+1)(\lambda+1)}{b} a_2 \omega + \frac{e^{i\beta}}{b}\Big[(\alpha+2)(2\lambda+1)(2a_2^2 - a_3) + \frac{(\alpha-1)(\alpha+2)(3\lambda+1)}{2} a_2^2\Big] \omega^2 + \cdots$$

By equating the coefficients in (28) and (29), we get

$$\frac{e^{i\beta}(\alpha+1)(\lambda+1)}{b\cos\beta} a_2 = \frac{m_1 B_1}{2}, \tag{30}$$

$$\frac{e^{i\beta}}{b\cos\beta}[(\alpha+2)(1+2\lambda)a_3 + \frac{(\alpha-1)(\alpha+2)(1+3\lambda)}{2} a_2^2] = \frac{m_2 B_1}{2} + \frac{m_1^2(B_2 - B_1)}{4} \tag{31}$$

$$-\frac{e^{i\beta}(\alpha+1)(\lambda+1)}{b\cos\beta} a_2 = \frac{n_1 B_1}{2}, \tag{32}$$

and

$$\frac{e^{i\beta}}{b\cos\beta}[(\alpha+2)(2\lambda+1)(a_2^2 - a_3) + \frac{(\alpha-1)(\alpha+2)(3\lambda+1)}{2} a_2^2]$$

$$= \frac{n_2B_1}{2} + \frac{n_1^2(B_2 - B_1)}{4} \tag{33}$$

We find from (30) and (32) that

$$m_1 = -n_1 \tag{34}$$

and

$$\frac{2e^{2i\beta}(\alpha+1)^2(\lambda+1)^2}{b^2\cos^2\beta}a_2^2 = \frac{B_1^2}{4}(m_1^2+n_1^2). \tag{35}$$

Also, from (31) and (33), we obtain

$$\frac{e^{i\beta}}{b\cos\beta}(\alpha+2)[\lambda+\alpha(1+3\lambda)+1]a_2^2 = \frac{(m_2+n_2)B_1}{2} + \frac{(B_2-B_1)(m_1^2+n_1^2)}{4} \tag{36}$$

From (35) and (36), we have

$$a_2^2 = \frac{(m_2+n_2)B_1^3b^2\cos^2\beta}{2be^{i\beta}B_1^2\cos\beta(\alpha+2)[\lambda+\alpha(3\lambda+1)+1] - 4(B_2-B_1)e^{2i\beta}(\alpha+1)^2(\lambda+1)^2} \tag{37}$$

Since $|m_i| \leq 2$ and $|n_i| \leq 2 \quad (i = 1, 2)$, it follows from (35), (36) and (37) that

$$|a_2| \leq \frac{|B_1||b|\cos\beta}{(\alpha+1)(1+\lambda)},$$

$$|a_2| \leq \sqrt{\frac{2|b|\cos\beta(|B_1|+|B_2-B_1|)}{(\alpha+2)[1+\lambda+\alpha(1+3\lambda)]}},$$

and

$$|a_2| \leq \frac{|B_1|\sqrt{2|B_1|}|b|\cos\beta}{\sqrt{|B_1^2b\cos\beta(\alpha+2)[1+\lambda+\alpha(1+3\lambda)] - 2(B_2-B_1)e^{i\beta}(1+\alpha)^2(1+\lambda)^2|}},$$

which yields the desired estimate on $|a_2|$ as asserted in (22).

Next, in order to find the bound on $|a_3|$, by subtracting (31) from (33), we get

$$\frac{2e^{i\beta}}{b\cos\beta}(\alpha+2)(2\lambda+1)(a_3-a_2^2) = \frac{(m_2-n_2)B_1}{2} \tag{38}$$

Substituting value of a_2^2 from (35), (36) and (37) in (38), we get

$$a_3 = \frac{(m_2-n_2)B_1b\cos\beta}{4e^{i\beta}(\alpha+2)(2\lambda+1)} + \frac{B_1^2b^2\cos^2\beta(m_1^2+n_1^2)}{8e^{2i\beta}(\alpha+1)^2(\lambda+1)^2}, \tag{39}$$

$$a_3 = \frac{b\cos\beta\{([5\lambda+\alpha(3\lambda+1)+3]m_2+(3\lambda+1)(1-\alpha)n_2)B_1+W_3\}}{4e^{i\beta}(\alpha+2)(2\lambda+1)[\lambda+\alpha(1+3\lambda)+1]} \tag{40}$$

and

$$a_3 = \frac{B_1 b\cos\beta(W_1 m_2 + W_2 n_2)}{4e^{i\beta}(\alpha+2)(2\lambda+1)\{B_1^2 b\cos\beta(\alpha+2)[\lambda+\alpha(1+3\lambda)+1]-W_4\}}, \quad (41)$$

where

$$W_1 = (\alpha+2)[5\lambda+\alpha(1+3\lambda)+3]B_1^2 b\cos\beta - 2(B_2-B_1)e^{i\beta}(\alpha+1)^2(\lambda+1)^2,$$

$$W_2 = (\alpha+2)(3\lambda+1)(1-\alpha)B_1^2 b\cos\beta + 2(B_2-B_1)e^{i\beta}(\alpha+1)^2(\lambda+1)^2,$$

$$W_3 = 2(2\lambda+1)(B_2-B_1)m_1^2$$

and

$$W_4 = 2(B_2-B_1)e^{i\beta}(\alpha+1)^2(\lambda+1)^2.$$

Using (39), (40) and (41), we have

$$|a_3| \le \frac{|B_1||b|\cos\beta}{(\alpha+2)(2\lambda+1)} + \frac{|B_1|^2|b|^2\cos^2\beta}{(\alpha+1)^2(\lambda+1)^2}, \quad (42)$$

$$|a_3| \le \frac{|b|\cos\beta\{[5\lambda+(\alpha+|1-\alpha|)(3\lambda+1)+3]|B_1|+4(2\lambda+1)|B_2-B_1|\}}{2(\alpha+2)(2\lambda+1)[\lambda+\alpha(1+3\lambda)+1]} \quad (43)$$

and

$$|a_3| \le \frac{|B_1||b|\cos\beta\{B_1^2|b|(\alpha+2)[5\lambda+(\alpha+|1-\alpha|)(3\lambda+1)+3]+W_5\}}{2(\alpha+2)(2\lambda+1)|B_1^2 b(\alpha+2)[\lambda+\alpha(1+3\lambda)+1]-W_6|}, \quad (44)$$

where

$$W_5 = 4|B_2-B_1|(\alpha+1)^2(\lambda+1)^2$$

and

$$W_6 = 2(B_2-B_1)(1+i\tan\beta)(1+\alpha)^2(1+\lambda)^2.$$

From (42), (43) and (44), we obtain the desired estimate on $|a_3|$ given in (23). This is the end of Theorem 3.2.

Let $b=1, \beta=0, \lambda=0$, Theorem 3.2 improves Theorem 2.8 in [6] by E. Deniz as follows.

Corollary 3.3 *Suppose that $f \in \mathcal{A}$ of the form (1), be in the class $B_\Sigma(\alpha,\varphi)$. Then*

$$|a_2| \le \min\left\{\frac{|B_1|}{\alpha+1}, \sqrt{\frac{2(|B_1|+|B_2-B_1)|)}{(\alpha+2)(\alpha+1)}}, \frac{|B_1|\sqrt{2|B_1|}}{\sqrt{|B_1^2(\alpha+2)(\alpha+1)-2(B_2-B_1)(\alpha+1)^2|}}\right\}$$

and

$$|a_3| \le \min\left\{\frac{|B_1|}{\alpha+2}+\frac{|B_1|^2}{(1+\alpha)^2}, \frac{(\alpha+|1-\alpha|+3)|B_1|+4|B_2-B_1|}{2(\alpha+2)(\alpha+1)},\right.$$

$$\left.\frac{|B_1|\{(\alpha+2)(\alpha+|1-\alpha|+3)B_1^2+4|B_2-B_1|(\alpha+1)^2\}}{2(\alpha+2)|B_1^2(\alpha+2)(\alpha+1)-2(B_2-B_1)(\alpha+1)^2|}\right\}.$$

Also, let $b=1, \beta=0, \alpha=0$ in Theorem 3.2, we obtain the following Corollary, which improves Theorem 2.3 in [11].

Corollary 3.4 *Suppose that $f(z)\in\mathcal{A}$ of the form (1), be in the class $M_\Sigma(\lambda,\varphi)$. Then*

$$|a_2| \le \min\left\{\frac{|B_1|}{(\lambda+1)}, \sqrt{\frac{(|B_1|+|B_2-B_1|)}{(\lambda+1)}}, \frac{|B_1|\sqrt{|B_1|}}{\sqrt{|B_1^2(\lambda+1)-(B_2-B_1)(1+\lambda)^2|}}\right\}$$

and

$$|a_3| \le \min\left\{\frac{|B_1|}{2(2\lambda+1)}+\frac{|B_1|^2}{(\lambda+1)^2}, \frac{|B_1|+|B_2-B_1|}{\lambda+1},\right.$$

$$\left.\frac{|B_1|[2(2\lambda+1)B_1^2+(\lambda+1)^2|B_2-B_1|]}{2(2\lambda+1)(\lambda+1)|B_1^2-(B_2-B_1)(\lambda+1)|}\right\}.$$

By setting $\varphi(z)=(\frac{1+z}{1-z})^\eta$ $(0<\eta\le 1)$ in Theorem 3.2, we get the following Corollary:

Corollary 3.5 *Suppose that $f(z)\in\mathcal{A}$ of the form (1), be in the class $B_\alpha^\eta(\beta,b,\lambda)$. Then*

$$|a_2| \le \min\left\{\frac{2\eta|b|\cos\beta}{(\alpha+1)(\lambda+1)}, 2\sqrt{\frac{\eta|b|\cos\beta(2-\eta)}{(\alpha+2)[\lambda+\alpha(1+3\lambda)+1]}},\right.$$

$$\left.\frac{2\eta|b|\cos\beta}{\sqrt{|b\eta\cos\beta(\alpha+2)[\lambda+\alpha(3\lambda+1)+1]+e^{i\beta}(1-\eta)[(\alpha+1)^2(\lambda+1)]^2|}}\right\}$$

and

$$|a_3| \le \min\left\{\frac{2\eta|b|\cos\beta}{(\alpha+2)(2\lambda+1)}+\frac{4\eta^2|b|^2\cos^2\beta}{(\alpha+1)^2(\lambda+1)^2}, N_\lambda(\alpha,\beta,\eta,b)), Q_\lambda(\alpha,\beta,\eta,b)\right\},$$

where

$$N_\lambda(\alpha,\beta,\eta,b)=\frac{\eta|b|\cos\beta}{(\alpha+2)}\cdot\frac{[5\lambda+(\alpha+|1-\alpha|)(3\lambda+1)+3]+4(2\lambda+1)(1-\eta)}{(2\lambda+1)[\lambda+\alpha(3\lambda+1)+1]},$$

$$Q_\lambda(\alpha,\beta,\eta,b)=\frac{\eta|b|\{\eta|b|\cos\beta(\alpha+2)[5\lambda+(\alpha+|1-\alpha|)(3\lambda+1)+3]+W_7\}}{(\alpha+2)(2\lambda+1)|b\eta(\alpha+2)[\lambda+\alpha(3\lambda+1)+1]+W_8|},$$

$$W_7=2(1-\eta)(\alpha+1)^2(\lambda+1)^2$$

and

$$W_8=(1-\eta)(1+i\tan\beta)(\alpha+1)^2(\lambda+1)^2.$$

Especially, for $b=1,\beta=0,\alpha=0$, Corollary 3.5 readily yields the following coefficient estimates for $B_0^\eta(0,1,\lambda)$,

Corollary 3.6 *Suppose that $f(z)\in\mathcal{A}$ of the form (1), be in the class $B_0^\eta(0,1,\lambda)$. Then*

$$|a_2|\le\min\left\{\frac{2\eta}{(\lambda+1)},\sqrt{\frac{2\eta(2-\eta)}{(\lambda+1)}},\frac{2\eta}{\sqrt{(\lambda+1)|2\eta+(1-\eta)(\lambda+1)|}}\right\}$$

and

$$|a_3|\le\min\left\{\frac{\eta}{(2\lambda+1)}+\frac{4\eta^2}{(\lambda+1)^2},\frac{2\eta(2-\eta)}{(\lambda+1)},\right.$$
$$\left.\frac{\eta\{4\eta(2\lambda+1)+(1-\eta)(\lambda+1)^2\}}{(2\lambda+1)(\lambda+1)|2\eta+(1-\eta)(\lambda+1)|}\right\}.$$

By setting $\varphi(z)=\frac{1+(1-2\gamma)z}{1-z}(0<\gamma\le1)$ in Theorem 3.2, we get the following Corollary:

Corollary 3.7 *Suppose that $f(z)\in\mathcal{A}$ of the form (1), be in the class $L_\alpha^{1-2\gamma,-1}(\beta,b,\lambda)$. Then*

$$|a_2|\le\min\left\{\frac{2(1-\gamma)|b|\cos\beta}{(\alpha+1)(\lambda+1)},\sqrt{\frac{4|b|\cos\beta(1-\gamma)}{(\alpha+2)[\lambda+\alpha(3\lambda+1)+1]}},\right.$$
$$\left.\frac{2\sqrt{(1-\gamma)}}{\sqrt{|(\alpha+2)[\lambda+\alpha(1+3\lambda)+1]|}}\right\}$$

and

$$|a_3|\le\min\left\{\frac{2(1-\gamma)|b|\cos\beta}{(\alpha+2)(2\lambda+1)}+\frac{4(1-\gamma)^2|b|^2\cos^2\beta}{(\alpha+1)^2(\lambda+1)^2},M_1,M_2\right\},$$

where

$$M_1=\frac{(1-\gamma)|b|\cos\beta[5\lambda+(\alpha+|1-\alpha|)(3\lambda+1)+3]}{(\alpha+2)(2\lambda+1)[\lambda+\alpha(1+3\lambda)+1]}$$

and

$$M_2=\frac{(1-\gamma)|b|\cos\beta\{[5\lambda+(\alpha+|1-\alpha|)(3\lambda+1)+3]\}}{(\alpha+2)(2\lambda+1)|[\lambda+\alpha(3\lambda+1)+1]|}.$$

Especially, for $b=1,\beta=0,\alpha=1,\lambda=0$, Corollary 3.7 readily improves the result in [9].

Corollary 3.8 *Suppose that $f(z)\in\mathcal{A}$ of the form (1), be in the class $L_1^{1-2\gamma,-1}(0,1,0)$. Then*

$$|a_2|\leq\min\left\{(1-\gamma),\sqrt{\frac{2(1-\gamma)}{3}}\right\}$$

and

$$|a_3|\leq\min\left\{\frac{2(1-\gamma)}{3}+(1-\gamma)^2,\frac{2(1-\gamma)}{3}\right\}=\frac{2(1-\gamma)}{3}.$$

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