



*Gen. Math. Notes, Vol. 24, No. 2, October 2014, pp.10-17*  
*ISSN 2219-7184; Copyright ©ICSRS Publication, 2014*  
*www.i-csrs.org*  
*Available free online at <http://www.geman.in>*

## Ruled Weingarten Surfaces Related to Dual Spherical Curves

İlkay Arslan Güven<sup>1</sup>, Semra Kaya Nurkan<sup>2</sup> and Murat Kemal Karacan<sup>3</sup>

<sup>1</sup>University of Gaziantep, Department of Mathematics  
Şehitkamil, 27310, Gaziantep, Turkey  
E-mail: [iarslan@gantep.edu.tr](mailto:iarslan@gantep.edu.tr)

<sup>2,3</sup>University of Uşak, Department of Mathematics  
64200, Uşak, Turkey

<sup>2</sup>E-mail: [semrakaya\\_gs@yahoo.com](mailto:semrakaya_gs@yahoo.com)

<sup>3</sup>E-mail: [mkkaracan@yahoo.com](mailto:mkkaracan@yahoo.com)

(Received: 8-4-14 / Accepted: 17-6-14)

### Abstract

*We study ruled surfaces in  $\mathbb{R}^3$  which are obtained from dual spherical indicatrix curves of dual Frenet vector fields. We find the Gaussian and mean curvatures of the ruled surfaces and give some results of being Weingarten surface.*

**Keywords:** *Weingarten surface, ruled surface, dual curves.*

## 1 Introduction

E. Study established a relationship of directed lines to dual unit vectors and he defined a mapping which is called Study mapping. This mapping exists one-to-one correspondence between the dual points of a dual unit sphere in  $\mathbb{D}$ -Module and the directed lines in  $\mathbb{R}^3$ .

A differentiable curve on the dual unit sphere, depending on a real parameter  $s$ , represents a differentiable family of straight lines in  $\mathbb{R}^3$  which is called ruled surface. This correspondence allows us to study the properties of a ruled surface on the geometry of dual spherical curves on a dual unit sphere [5].

Ruled surfaces of Weingarten type which have a nontrivial relation holds between the Gaussian curvature and mean curvature, were studied by many

scientists in [2, 7, 8]. Also properties and inventions of ruled linear Weingarten surfaces in Minkowski 3-space were investigated [1, 2, 3, 4, 8].

In this paper we take a unit speed curve on the dual unit sphere and move each vector of its Frenet frame to the center of dual unit sphere. These Frenet vectors generates spherical representation curves on the dual unit sphere. So we have ruled surfaces in  $\mathbb{R}^3$  which have been corresponded to representation curves, by Study mapping. Each ruled surface is determined by a parametrization

$$\varphi(s, v) = \alpha(s) + v\vec{X}(s)$$

where  $\alpha$  is the base curve and  $\vec{X}$  is director vector field which will be Frenet vector field of dual curve in this paper.

We study on these ruled surfaces that which one is Weingarten and minimal surface. The conditions of being Weingarten and minimal ruled surfaces were given by theorems.

## 2 Preliminaries

Dual numbers were defined in the 19<sup>th</sup> century by W.K. Clifford. The set of all dual numbers  $\mathbb{D}$  consists of elements in the form of  $A = a + \varepsilon a^*$ , where  $a$  and  $a^*$  are real numbers,  $\varepsilon$  is the dual unit with the property of  $\varepsilon^2 = 0$ . The set

$$\mathbb{D} = \{A = a + \varepsilon a^* \mid a, a^* \in \mathbb{R}\}$$

forms a commutative ring with the following operations

$$\begin{aligned} i) (a + \varepsilon a^*) + (b + \varepsilon b^*) &= (a + b) + \varepsilon(a^* + b^*) \\ ii) (a + \varepsilon a^*). (b + \varepsilon b^*) &= a.b + \varepsilon(ab^* + ba^*). \end{aligned}$$

The division of two dual numbers  $A = a + \varepsilon a^*$  and  $B = b + \varepsilon b^*$  provided  $b \neq 0$  can be defined as

$$\frac{A}{B} = \frac{a + \varepsilon a^*}{b + \varepsilon b^*} = \frac{a}{b} + \varepsilon \frac{a^*b - ab^*}{b^2}.$$

The set  $\mathbb{D}^3$  is a module on the ring  $\mathbb{D}$  which is called  $\mathbb{D}$ -Module or dual space and is denoted by

$$\mathbb{D}^3 = \mathbb{D} \times \mathbb{D} \times \mathbb{D} = \left\{ \begin{array}{l} \vec{A} \mid \vec{A} = (a_1 + \varepsilon a_1^*, a_2 + \varepsilon a_2^*, a_3 + \varepsilon a_3^*) \\ \quad = (a_1 + a_2 + a_3) + \varepsilon (a_1^* + a_2^* + a_3^*) \\ \quad = a + \varepsilon a^*, \quad a \in \mathbb{R}^3, a^* \in \mathbb{R}^3 \end{array} \right\}$$

where the elements are dual vectors. For  $a \neq 0$ , the norm  $\|\vec{A}\|$  of  $\vec{A}$  is defined by

$$\|\vec{A}\| = \sqrt{\langle \vec{A}, \vec{A} \rangle} = \|\vec{a}\| + \varepsilon \frac{\langle \vec{a}, \vec{a}^* \rangle}{\|\vec{a}\|}.$$

Now let us give basic concepts of a dual curve and dual Frenet frame.

Let

$$\begin{aligned} \widehat{\alpha} : I &\longrightarrow \mathbb{D}^3 \\ s &\longrightarrow \vec{\alpha}(s) = \vec{\alpha}(s) + \varepsilon \vec{\alpha}^*(s) \end{aligned}$$

be a dual curve with arc-length parameter  $s$ . Then

$$\frac{d\vec{\alpha}}{d\widehat{s}} = \frac{d\vec{\alpha}}{ds} \cdot \frac{ds}{d\widehat{s}} = \vec{T}$$

is called the unit tangent vector of  $\widehat{\alpha}(s)$ . The derivative of  $\vec{T}$  is

$$\frac{d\vec{T}}{d\widehat{s}} = \frac{d\vec{T}}{ds} \cdot \frac{ds}{d\widehat{s}} = \frac{d^2\vec{\alpha}}{ds^2} = \widehat{\kappa} \vec{N}$$

and the norm of the vector  $\frac{d\vec{T}}{d\widehat{s}}$  is called curvature function of  $\vec{\alpha}(s)$ . Here  $\widehat{\kappa} : I \longrightarrow \mathbb{D}$  is never pure-dual. Then the unit principal normal vector of  $\vec{\alpha}(s)$  is defined as

$$\vec{N} = \frac{1}{\widehat{\kappa}} \cdot \frac{d\vec{T}}{d\widehat{s}}$$

The vector  $\vec{B} = \vec{T} \times \vec{N}$  is called the binormal vector of  $\vec{\alpha}(s)$ . Also we call the vectors  $\vec{T}, \vec{N}, \vec{B}$  dual Frenet trihedron of  $\vec{\alpha}(s)$  at the point  $\widehat{\alpha}(s)$ . The derivatives of dual Frenet vectors  $\vec{T}, \vec{N}, \vec{B}$  can be written in matrix form as

$$\begin{bmatrix} \vec{T}' \\ \vec{N}' \\ \vec{B}' \end{bmatrix} = \begin{bmatrix} 0 & \widehat{\kappa} & 0 \\ -\widehat{\kappa} & 0 & \widehat{\tau} \\ 0 & -\widehat{\tau} & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{bmatrix}$$

which are called Frenet formulas [6]. The function  $\widehat{\tau} : I \longrightarrow \mathbb{D}$  such that  $\frac{d\vec{B}}{d\widehat{s}} = -\widehat{\tau} \vec{N}$  is called the torsion of  $\vec{\alpha}(s)$ .

A ruled surface in  $\mathbb{R}^3$  is swept up by a straight line  $\ell$  which is moving along a curve  $\alpha$ . It is defined by the parametrization  $\varphi(s, v) = \alpha(s) + v\vec{X}(s)$ , where  $\alpha$  is differentiable base curve and  $\vec{X}$  is a nowhere vanishing director vector field of  $\ell$ . The lines  $\ell$  are called the rulings of the surface. A ruled surface is said to be developable if the Gaussian curvature of surface  $K$  is zero. The tangent plane of developable surface is constant along a fixed ruling. If the mean curvature of surface  $H$  is zero, then the ruled surface is called minimal surface.

A Weingarten surface is a surface for which the Gaussian curvature  $K$  and the mean curvature  $H$  satisfy a nontrivial relation  $\Phi(H, K) = 0$ . For

ruled surfaces in  $\mathbb{E}^3$  Dini and Beltrami expressed a theorem in 1865 which says that any non-developable ruled Weingarten surface in Euclidean 3-space  $\mathbb{E}^3$  is a piece of a helicoidal ruled surface, defined as the orbit of a straight line under the action of a 1-parameter group of screw motions. In particular, the Gaussian curvature is nowhere zero if it is nonzero at some point. The only minimal ruled surface is the classical right helicoid. In [7], this theorem is stated directly as; among the ruled surfaces, the class of Weingarten surfaces is the set of all developable surfaces and all helicoidal ruled surfaces.

### 3 Gaussian and Mean Curvatures of Ruled Surfaces

In this section we will compute the Gaussian and mean curvatures of ruled surfaces which are obtained from dual spherical indicatrix curves of dual Frenet vector fields. According to the theorem in [7], we will say that the developable surfaces are Weingarten surfaces. Also we know that if Gaussian and mean curvature of the ruled surface are zero, then the ruled surface is called developable and minimal surface, respectively. So we will investigate the conditions of Gaussian and mean curvatures to being zero.

A curve  $\hat{\alpha}$  is taken on dual unit sphere and dual spherical representation curves are formed on dual unit sphere by moving the Frenet vectors to the center of dual unit sphere. These representation curves which we will denote as  $(X)$  are corresponded to ruled surfaces in  $\mathbb{R}^3$ . The Frenet vectors of  $\hat{\alpha}$ ; unit tangent vector, unit principal normal vector and unit binormal vector are  $\hat{T} = T + \varepsilon T^*$ ,  $\hat{N} = N + \varepsilon N^*$ ,  $\hat{B} = B + \varepsilon B^*$ , respectively. Also curvature and torsion of  $\hat{\alpha}$  will be denoted as  $\hat{\kappa} = \kappa + \varepsilon \kappa^*$  and  $\hat{\tau} = \tau + \varepsilon \tau^*$ . We get the equations below from Frenet formulas and properties of Frenet vectors:

$$\begin{aligned} T \times N &= B & \text{and} & & T^* &= N^* \times B + N \times B^* \\ N \times B &= T & & & N^* &= B^* \times T + B \times T^* \\ B \times T &= N & & & B^* &= T^* \times N + T \times N^* \end{aligned}$$

and the derivatives are

$$\begin{aligned} T' &= \kappa N & \text{and} & & T^{*'} &= \kappa N^* + \kappa^* N \\ N' &= -\kappa T + \tau B & & & N^{*'} &= -\kappa T^* - \kappa^* T + \tau B^* + \tau^* B \\ B' &= -\tau N & & & B^{*'} &= -\tau N^* - \tau^* N. \end{aligned}$$

Now we will express the dual vectoral equations of each ruled surface. We denote ruled surfaces which are corresponded to dual spherical representation

curves as  $\phi_X(s, v)$ . The equations are

$$\begin{aligned}\phi_T(s, v) &= T(s) \times T^*(s) + vT(s) \quad , \quad T^*(s) = \beta_T(s) \times T(s) \\ \phi_N(s, v) &= N(s) \times N^*(s) + vN(s) \quad , \quad N^*(s) = \beta_N(s) \times N(s) \\ \phi_B(s, v) &= B(s) \times B^*(s) + vB(s) \quad , \quad B^*(s) = \beta_B(s) \times B(s)\end{aligned}$$

where  $\beta_X(s)$  is the base curve of ruled surface and  $X^*(s) = \beta_X(s) \times X(s)$  is vectoral moment of the vector  $X$  , also  $s$  is not arc-length parameter of base curve.

Let find the Gaussian curvatures of each ruled surface. We will calculate Gaussian curvatures from the matrix of shape operator of ruled surfaces.

We take up first ruled surface whose dual vectoral equation is

$$\phi_T(s, v) = T(s) \times T^*(s) + vT(s) \quad , \quad T^*(s) = \beta_T(s) \times T(s).$$

The director vector of this ruled surfaces is  $T$  and let the normal vector be  $N_T$ . Since the derivatives

$$\begin{aligned}(\phi_T)_v &= T \\ (\phi_T)_s &= -\kappa\langle\beta_T, N\rangle T - \kappa\langle\beta_N, T\rangle N + \kappa^*B + v\kappa N\end{aligned}$$

are not orthogonal, we will convert these vectors to a orthonormal base by Gramm Schmidt method and we will compute the matrix of shape operator with respect to this orthonormal base.

For  $(\phi_T)_v = X_1$  and  $(\phi_T)_s = X_2$  ; the vectors of orthonormal base  $\{E_1, E_2\}$  are

$$\begin{aligned}E_1 &= \frac{Y_1}{\|Y_1\|} = T \\ E_2 &= \frac{Y_2}{\|Y_2\|} = \frac{\kappa(-\langle\beta_N, T\rangle + v)N + \kappa^*B}{\sqrt{\kappa^2(-\langle\beta_N, T\rangle + v)^2 + \kappa^{*2}}}\end{aligned}$$

where  $Y_1 = X_1 = T$  and  $Y_2 = \kappa(-\langle\beta_N, T\rangle + v)N + \kappa^*B$ . By doing simple operation we get  $\langle E_1, E_2 \rangle = 0$ .

The normal vector of this ruled surface is given by

$$\begin{aligned}N_T &= E_1 \times E_2 \\ &= \frac{\kappa(-\langle\beta_N, T\rangle + v)B - \kappa^*N}{\sqrt{\kappa^2(-\langle\beta_N, T\rangle + v)^2 + \kappa^{*2}}}.\end{aligned}$$

If the matrix of shape operator is  $S_T$ , then

$$S_T = \begin{bmatrix} \langle S_T(E_1), E_1 \rangle & \langle S_T(E_1), E_2 \rangle \\ \langle S_T(E_2), E_1 \rangle & \langle S_T(E_2), E_2 \rangle \end{bmatrix}$$

can be written. Since  $T$  is director vector, it is an asymptotic line so  $\langle S_T(E_1), E_1 \rangle = \langle S_T(T), T \rangle = 0$  and since the shape operator is symmetric, then  $\langle S_T(E_1), E_2 \rangle = \langle S_T(E_2), E_1 \rangle$ . By above calculations the Gaussian curvature  $K_T$  is determined by

$$\begin{aligned} K_T &= \det S_T \\ &= -(\langle S_T(E_2), E_1 \rangle)^2 \end{aligned}$$

where  $S_T(E_2) = D_{E_2}N_T = \frac{1}{\|Y_2\|} \cdot \frac{dN_T}{ds}$ .

We obtain the Gaussian curvature by doing essential operations as

$$K_T = -\frac{\kappa^2 \kappa^{*2}}{(\kappa^2(-\langle \beta_N, T \rangle + v)^2 + \kappa^{*2})^2} \cdot 1 \quad (1)$$

Also the mean curvature  $H_T$  is given by

$$\begin{aligned} H_T &= Tr(S_T) \\ &= \langle S_T(E_2), E_2 \rangle \\ &= \frac{1}{\|Y_2\|^3} \cdot \left\{ \begin{array}{l} -\kappa^2 \tau \langle \beta_N, T \rangle^2 + \langle \beta_N, T \rangle (\kappa^{*'} \kappa + 2v\kappa^2 \tau - \kappa^* \kappa') \\ -\kappa^* \kappa \langle \beta_N, T \rangle' - v\kappa \kappa^{*'} - v^2 \kappa^2 \tau - \kappa^{*2} \tau + v\kappa^* \kappa' \end{array} \right\} 2 \quad (2) \end{aligned}$$

We obtain Gaussian and mean curvatures of other ruled surfaces below by doing similar calculations.

For ruled surface corresponded to dual spherical principal normal representation curve;

$$K_N = -\frac{(\kappa \kappa^* + \tau \tau^*)^2}{\|Y_{2N}\|^4} 3 \quad (3)$$

where

$$\|Y_{2N}\|^2 = \frac{\kappa^2 \langle \beta_T, N \rangle^2 + (2\kappa \tau^* - 2\kappa^2 v) \langle \beta_T, N \rangle + \kappa^{*2}}{-(2\tau \kappa^* + 2\tau^2 v) \langle \beta_B, N \rangle + 2\kappa^* \tau v + \tau^2 \langle \beta_B, N \rangle^2 + \tau^{*2} - 2\kappa \tau^* v + v^2 (\kappa^2 + \tau^2)}$$

and

$$H_N = \frac{1}{\|Y_{2N}\|^3} \cdot \left\{ \begin{array}{l} (\kappa^{*'} \kappa - \kappa' \kappa^* + \tau^{*'} \tau - \tau' \tau^*) (\langle \beta_T, N \rangle - v) \\ + (\kappa' \tau - \tau' \kappa) (\langle \beta_T, N \rangle^2 - 2v \langle \beta_T, N \rangle + v^2) \\ - \langle \beta_T, N \rangle' (\kappa \kappa^* + \tau \tau^*) + \kappa^{*'} \tau^* - \tau^{*'} \kappa^* \end{array} \right\} 4 \quad (4)$$

are obtained. For ruled surface corresponded to dual spherical binormal representation curve;

$$K_B = -\frac{\tau^2 \tau^{*2}}{\|Y_{2B}\|^4} 5 \quad (5)$$

where

$$\|Y_{2B}\|^2 = \tau^2(\langle \beta_N, B \rangle - v)^2 + \tau^{*2}$$

and

$$H_B = \frac{1}{\|Y_{2B}\|^3} \left\{ \begin{array}{l} -\tau^2 \kappa \langle \beta_N, B \rangle^2 + \langle \beta_N, B \rangle (-\tau' \tau^* + 2v\tau^2 \kappa + \tau^{*\prime} \tau) \\ -\tau \tau^* \langle \beta_N, B \rangle' - \kappa \tau^{*2} + v\tau' \tau^* - v\tau^{*\prime} \tau - v^2 \tau^2 \kappa \end{array} \right\} 6 \quad (6)$$

are obtained.

## 4 Results

In this section we will express theorems and use Gaussian and mean curvatures to prove them.

**Theorem 4.1** *The non-developable ruled surface in  $\mathbb{R}^3$  which is corresponded to dual spherical representation curve (T) is Weingarten and minimal surface if and only if  $\kappa^* = 0$  and  $\tau = 0$ , for  $\kappa \neq 0$  and  $\tau^* \neq 0$ , where  $\widehat{\kappa} = \kappa + \varepsilon \kappa^*$  and  $\widehat{\tau} = \tau + \varepsilon \tau^*$  are curvature and torsion of the dual curve respectively.*

**Proof:** In equations (1) and (2) if we put  $\kappa^* = 0$  and  $\tau = 0$ , for  $\kappa \neq 0$  and  $\tau^* \neq 0$  we find that Gaussian curvature and mean curvature are zero. Since Gaussian curvature is zero then the ruled surface is developable, so it is Weingarten surface because of the theorem given in [7] and the ruled surface is minimal due to the mean curvature which is zero.

The opposite condition can be proved similarly.

**Theorem 4.2** *The non-developable ruled surface in  $\mathbb{R}^3$  which is corresponded to dual spherical representation curve (N) is Weingarten and minimal surface if and only if  $\kappa = 0$  and  $\tau^* = 0$  or  $\kappa^* = 0$  and  $\tau = 0$ , where  $\widehat{\kappa} = \kappa + \varepsilon \kappa^*$  and  $\widehat{\tau} = \tau + \varepsilon \tau^*$  are curvature and torsion of the dual curve respectively.*

**Proof:** The proof is similar to previous proof of theorem, it can be done by taking equations (3) and (4).

**Theorem 4.3** *The non-developable ruled surface in  $\mathbb{R}^3$  which is corresponded to dual spherical representation curve (B) is Weingarten and minimal surface if and only if  $\kappa = 0$  and  $\tau^* = 0$ , for  $\kappa^* \neq 0$  and  $\tau \neq 0$ , where  $\widehat{\kappa} = \kappa + \varepsilon \kappa^*$  and  $\widehat{\tau} = \tau + \varepsilon \tau^*$  are curvature and torsion of the dual curve respectively.*

**Proof:** In this proof, equations (5) and (6) illuminate the solution.

## References

- [1] R.A. Abdel-Baky and H.N. Abd-Ellah, Ruled W-surfaces in Minkowski 3-space  $\mathbb{R}_1^3$ , *Archivum Mathematicum (Brno)*, 44(2008), 251-263.
- [2] F. Dillen and W. Kühnel, Ruled Weingarten surfaces in Minkowski 3-space, *Manuscripta Math.*, 98(1999), 307-320.
- [3] F. Dillen and W. Sodsiri, Ruled surfaces of Weingarten type in Minkowski 3-space, *Journal of Geometry*, 83(2005), 10-21.
- [4] F. Dillen and W. Sodsiri, Ruled surfaces of Weingarten type in Minkowski 3-space II, *Journal of Geometry*, 84(2005), 37-44.
- [5] İ.A. Güven, S. Kaya and H.H. Hacısalihoğlu, On closed ruled surfaces concerned with dual Frenet and Bishop frames, *Journal of Dynamical Systems and Geometric Theories* , 9(1) (2011), 67-74.
- [6] Ö. Köse, Ş. Nizamoğlu and M. Sezer, An explicit characterization of dual spherical curves, *Doğa TU J. Math.*, 12(3) (1988), 105-113.
- [7] W. Kühnel, *Differential Geometry (2nd Edition)*, Am. Math. Soc., (2006).
- [8] W. Sodsiri, Ruled linear Weingarten surfaces in Minkowski 3-space, *Soochow Journal of Math.*, 29(4) (2003), 435-443.