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A Characterization of Central Galois Algebras

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Abstract

Let A be an Azumaya R-algebra over a commutative ring R of a constant rank n for some integer n, G an automorphism group of A of order n, and $J_g = \{a \in A | ax = g(x)a$ for all $x \in A\}$ for $g \in G$. Then A is a central Galois algebra over R with Galois group G if and only if $\sum_{g\in G} RJ_g$ is a separable R-algebra of rank n. In particular, when G is inner induced by $\{U_g$ for $g \in G\}$, A is a central Galois R-algebra if and only if $\sum_g RU_g$ is a separable R-algebra of rank n. Thus all inner Galois groups can be computed from the multiplicative group of units of A.

Keywords: *Azumaya algebras, Central Galois algebras, Inner Galois groups, Rank of a projective module.*

1 Introduction

Let R be a commutative ring with 1 and A an Azumaya R-algebra. Many characterizations of A are given in [1, 2, 7]. Let G be an automorphism group of A of order n for some integer n and $J_g = \{a \in A | ax = g(x)a$ for all $x \in A\}$ and $g \in G$. Then J_g is a rank one projective R-module for each $g \in G$ and $J_g J_h = J_{gh}$ for $g, h \in G$ ([9, Theorem 2]); and so $\sum_g J_g$ is a subalgebra of A. We note that a central Galois algebra is an Azumaya algebra of a constant rank equal to the order of the Galois group and many properties of a central Galois algebra are given in [1, 2, 3, 5, 8, 9]. Central Galois algebras play an important role in the research of Galois cohomology theory of a commutative ring (see [2]) and the Brauer group of a commutative ring ([8]). Assume the rank of A over R is n. We shall show that A is a central Galois R-algebra with Galois group G if and only if $\sum_{g\in G} RJ_g$ is a separable R-algebra of rank n. In particular, when G is inner induced by $\{U_g | g \in G\}$, $J_g = RU_g$, and so A is a central Galois R-algebra with Galois group G if and only if $\sum_g RU_g$ is a separable R-algebra of rank n. Thus all inner Galois groups for A can be computed by the multiplicative group of units of A.

2 Preliminary

Let B be a ring with 1, C the center of B, D a subring of B with the same 1. As given in [1, 2, 5], B is called a separable extension of D if the multiplication map: $B \otimes_D B \longrightarrow B$ splits as a B-bimodule homomorphism. In particular, if $D \subset C$, a separable extension B of D is called a separable D-algebra, and if $D = C$, a separable extension B of D is called an Azumaya C-algebra. Let G be a finite automorphism group of B and $B^G = \{b \in B|\ g(b) = b$ for each $g \in G\}$. If there exist elements $\{a_i, b_i$ in B, $i = 1, 2, \cdots, s$ for some integer $s\}$ such that $\sum_{i=1}^{s} a_i g(b_i) = \delta_{1,g}$ for each $g \in G$, then B is called a Galois extension of B^G with Galois group G, and $\{a_i, b_i\}$ is called a G-Galois system for B. A Galois extension B of B^G is called a Galois algebra if $B^G \subset C$, and a central Galois algebra if $B^G = C$ as studied in [1, 2, 3, 5, 8].

3 A Characterization

In this section, let A be an Azumaya R-algebra with an automorphism group G of order n for some integer n. In [3], it was shown that A is a central Galois R-algebra with Galois group G if and only if $A = \oplus \sum_{g\in G} J_g$. The purpose of the present paper is to show an equivalent condition for a central Galois algebra A in terms of the separability of the subalgebra $\sum_g J_g$ generated by J_g for $g \in G$. We begin with some properties of $\sum_g J_g$.

Lemma 3.1 *Let A be an Azumaya R-algebra with an automorphism group G of order n for some integer n. If $\sum_g J_g$ is a projective R-module of rank n, then $\sum_g J_g = \oplus \sum_g J_g$.*

Proof. Let $\alpha : \oplus \sum_g J_g \longrightarrow \sum_g J_g$ by $\alpha(\oplus \sum_g a_g) = \sum_g a_g$ for $a_g \in J_g$. Then α is an onto module homomorphism over R. Let N be the kernel of α. Then $0 \longrightarrow N \longrightarrow \oplus \sum_g J_g \longrightarrow \sum_g J_g \longrightarrow 0$ is exact. By hypothesis, $\sum_g J_g$ is a projective R-module, so the above exact sequence splits. Hence $\oplus \sum_g J_g \cong N \oplus (\sum_g J_g)$. Since $Rank_R(J_g) = 1$ ([9]), $Rank_R(\oplus \sum_g J_g) = n$. But then $Rank_R(N) = 0$; and so $N = 0$. Thus $\sum_g J_g = \oplus \sum_g J_g$.

Lemma 3.2 *Let A be an Azumaya R-algebra and B a separable subalgebra of A. Then B is a projective R-module.*

Proof. Since B is a separable subalgebra of the Azumaya R-algebra A, B is a direct summand of A as a B-bimodule ([4, Proposition 4]). Hence B is a direct summand of A as an R-module. Noting that A is projective over R, we have that B is a projective R-module.

Theorem 3.3 *Let A be an Azumaya R-algebra of rank n and G an automorphism group of A of order n. Then, A is a central Galois R-algebra with Galois group G if and only if $\sum_g J_g$ is a separable subalgebra of rank n.*

Proof. ($\Rightarrow$) Since A is a central Galois R-algebra with Galois group G of order n, $A = \oplus \sum_g J_g$. Hence $\sum_g J_g = \oplus \sum_g J_g = A$. Also by noting that $Rank_R(J_g) = 1$ for each $g \in G$, the rank of $\sum_g J_g = n$.

($\Leftarrow$) By hypothesis, A is an Azumaya R-algebra and $\sum_g J_g$ is a separable subalgebra of A, so $\sum_g J_g$ is a projective R-module by *Lemma* 3.2. Since the rank of $\sum_g J_g$ is n, $\sum_g J_g \cong \oplus \sum_g J_g$ by *Lemma* 3.1. Moreover, $\sum_g J_g$ is a separable subalgebra of A, so $\sum_g J_g$ is a direct summand of A. But the rank of $\sum_g J_g$ and A are n by hypothesis, so $A = \sum_g J_g = \oplus \sum_g J_g$. Thus A is a central Galois R-algebra with Galois group G ([3, Theorem 1]).

4 The Inner Galois Groups

In [1], a central Galois R-algebra with an inner Galois group G is characterized in terms of the Azumaya projective group algebra RG_f of G over R with a factor set $f : G \times G \longrightarrow R^*$, (= units of R). We shall derive a characterization of a central Galois R-algebra A with an inner Galois group G induced by units $\{U_g \in A \text{ for } g \in G\}$ in terms of the concept of separability, and compute all possible Galois groups from the multiplicative group of units of A.

Theorem 4.1 *Let A be an Azumaya R-algebra of rank n with an inner automorphism group G induced by $\{U_g \in A \text{ for } g \in G\}$. Then A is a central Galois R-algebra with Galois group G if and only if $\sum_g RU_g$ is a separable R-algebra of rank n equal to the order of G.*

Proof. Since G is an inner automorphism group of A induced by $\{U_g \in A$ for $g \in G\}$, $J_g = RU_g$ by the *Corollary* of *Theorem* 1 in [3]. Thus *Theorem* 4.1 is an immediate consequence of *Theorem* 3.3.

By *Theorem* 4.1, we shall compute all inner Galois groups for a central Galois R-algebra A. Let $U(A)$ be the multiplicative group of units of A, and $I(A)$ the inner automorphism group of A induced by the elements of $U(A)$.

Corollary 4.2 *Let A be an Azumaya R-algebra of rank n for some integer n, and H a subgroup of $I(A)$ of order n induced by $\{U_g|g \in H\}$. Then, A is a central Galois R-algebra with Galois group H if and only if $\sum_g RU_g$ is a separable R-algebra of rank n.*

Proof. This is an immediate consequence of *Theorem* 4.1.

Next we compute all inner Galois groups for a central Galois R-algebra A.

Theorem 4.3 *By keeping the notations of Corollary 4.2, let A be an Azumaya R-algebra of rank n for some integer n, and Z the center of the group $U(A)$. Let $\{U_iZ|i = 1, \cdots, n\}$ be in the quotient group $U(A)/Z$ such that $\{U_i\}$ generate A and induce an inner subgroup $\{g_i|i = 1, \cdots, n\}$ of $I(A)$. Then A is a central Galois R-algebra with Galois group $\{g_i|i = 1, \cdots, n\}$ and $\beta : \{U_iZ\} \longrightarrow \{g_i|i = 1, \cdots, n\}$ is a one to one correspondence from the set of $\{U_iZ\}$ to the set of Galois groups of the central Galois R-algebra A.*

Proof. Since $\sum_{i=1}^n RU_i = A$ and the rank of $\sum_{i=1}^n RU_i$ is equal to the rank of A, so $A = \sum_{i=1}^n RU_i = \oplus \sum_{i=1}^n RU_i$ and is a central Galois R-algebra with Galois group $\{g_i \in I(A)\}$ by *Theorem* 3.3. Thus β is well defined. Moreover, β is onto by *Theorem* 4.1. Now let $\beta\{U_iZ\} = \beta\{V_iZ\}$ for some $U_i, V_i \in U(A)$. Then the Galois groups for the central Galois R-algebra A induced by $\{U_i\}$ and $\{V_i\}$ respectively are the same. We have $U_i = V_ia_i$ for some $a_i \in Z$ for each i. Thus $\{U_iZ\} = \{V_iZ\}$; and so β is one-to-one. Therefore β is a one-to-one correspondence.

We conclude the present paper with an example to show a set of generators $\{U_i\}$ of an Azumaya R-algebra A as given in *Theorem* 4.3.

Let A be the algebra of matrices of order 2 over the real field R. Then A is an Azumaya R-algebra of rank 4. Let

$$U_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ U_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ U_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ U_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Then A is generated by $\{U_i|i = 1, 2, 3, 4\}$ in $U(A)$ such that $\{g_i\}$ induced by $\{U_i|i = 1, 2, 3, 4\}$ is a subgroup of $I(A)$. Thus $\{g_i\}$ is a Galois group of the central Galois R-algebra by *Theorem* 4.3.

To obtain more Galois groups for the central Galois R-algebra A, let λ be an automorphism of A. Then $\{\lambda(U_i)|i = 1, 2, 3, 4\}$ is a generating set for A such that the inner automorphisms induced by $\{\lambda(U_i)|i = 1, 2, 3, 4\}$ is also a subgroup of $I(A)$. Thus this group is also a Galois group of A by *Theorem* 4.3.

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