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The Tensor Product of Galois Algebras

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Abstract

Let A and B be R-algebras with automorphism groups G and H respectively. Denote the order of G by n and the order of H by m for some integers n and m. Assume n and m are invertible in R. Then, $A \otimes_R B$ is a Galois R-algebra with Galois group $G \times H$ if and only if A and B are Galois R-algebras with Galois groups G and H respectively. Thus an equivalent condition for a central Galois algebra in terms of the tensor product is obtained.

Keywords: *Azumaya algebras, Galois algebras, Central Galois algebras, Tensor products of Galois algebras.*

1 Introduction

A Galois algebra over its center is called a central Galois algebra. A lot of properties of a central Galois algebra are given in [1, 2, 3, 4, 6, 7]. Central Galois algebras play an important role in the research of Galois cohomology theory of a commutative ring (see [2]) and the Brauer group of a commutative ring ([6]). In [1], a structure theorem is given for a central Galois algebra with an inner Galois group, and in [6], it is known that the tensor product of central Galois algebras is a central Galois algebra. The purpose of the present paper is to show the converse of the above result; that is, if the tensor product of R-algebras A and B is a central Galois algebra with Galois group $G \times H$ where G and H are R-automorphism groups of A and B respectively, then A

and B are central Galois R-algebras with Galois groups G and H respectively. Moreover, we shall give a different proof of the above fact in the case of $B = A^o$ and $H = G^o$ by using the expression of a central Galois algebra as a direct sum of rank one projective modules as shown in [3, 5, 7].

2 Preliminary

Let B be a ring with 1, C the center of B, D a subring of B with the same 1. As given in [1, 2, 4], B is called a separable extension of D if the multiplication map: $B \otimes_D B \longrightarrow B$ splits as a B-bimodule homomorphism. In particular, if $D \subset C$, a separable extension B of D is called a separable D-algebra, and if $D = C$, a separable extension B of D is called an Azumaya C-algebra. Let G be a finite automorphism group of B and $B^G = \{b \in B |\ g(b) = b$ for each $g \in G\ \}$. If there exist elements $\{a_i, b_i$ in B, $i = 1, 2, \cdots, s$ for some integer $s\}$ such that $\sum_{i=1}^{s} a_i g(b_i) = \delta_{1,g}$ for each $g \in G$, then B is called a Galois extension of B^G with Galois group G, and $\{a_i, b_i\}$ is called a G-Galois system for B. A Galois extension B of B^G is called a Galois algebra if $B^G \subset C$, and a central Galois algebra if $B^G = C$ as studied in [1, 2, 4, 6].

3 Galois Algebras

Let A and B be Galois R-algebras over a commutative ring R with Galois groups G and H respectively. Let n be the order of G and m the order of H. Denote the trace of A by $Tr_G(A)$ and the trace of B by $Tr_H(B)$ where $Tr_G(A) = \{\sum_g g(a)$ for $g \in G, a \in A\}$ and $Tr_H(B) = \{\sum_h h(b)$ for $h \in H, b \in B\}$. We shall compute $(A \otimes_R B)^{G\times H}$ when n and m are invertible in R under the above notations, and show that the tensor product of A and B is a Galois algebra if and only if so are A and B. Thus the tensor product of A and B is a central Galois algebra if and only if so are A and B.

Lemma 3.1 *Let A and B be R-algebras with R-automorphism groups G and H respectively. If n and m are invertible in R, then $G \times H$ is an automorphism group of $A \otimes_R B$, $Tr_G(A) = A^G$ ($Tr_H(B) = B^H$) and $(A \otimes_R B)^{G\times H} = Tr_G(A) \otimes_R Tr_H(B)$.*

Proof. Since A and B are R-algebras with R-automorphism groups G and H of orders n and m respectively, $Tr_G(A) \subset A^G$ and $Tr_H(B) \subset B^H$. Also n and m are invertible in R, so $A^G \subset Tr_G(A)$ and $B^H \subset Tr_H(B)$. Thus $Tr_G(A) = A^G$ and $Tr_H(B) = B^H$. Noting that n is invertible in R, the map $n^{-1}Tr_G(\ ) : A \longrightarrow R$ is onto and splits as R-modules, we have that R is a direct summand of A as an R-module; and so $1 \otimes_R B$ is a direct summand of

$A \otimes_R B$. Hence $1 \times H$ is an automorphism group of $A \otimes_R B$. Similarly, by noting that m is invertible in R, $G \times 1$ is an automorphism group of $A \otimes_R B$. Thus $G \times H$ is an automorphism group of $A \otimes_R B$. Moreover, we claim that $(A \otimes_R B)^{G\times H} = Tr_G(A) \otimes_R Tr_H(B)$. Clearly, $Tr_G(A) \otimes_R B \subset (A \otimes_R B)^{G\times 1}$. Conversely, for any $\sum_i x_i \otimes y_i \in (A \otimes B)^{G\times 1}$ for $i = 1, \cdots, k$ for some integer k, $(g \times 1)(\sum_i x_i \otimes y_i) = \sum_i g(x_i) \otimes y_i = \sum_i x_i \otimes y_i$ for each $g \in G$, so $\sum_i Tr_G(x_i) \otimes y_i = n(\sum_i x_i \otimes y_i)$. By hypothesis, n is invertible in R, so $\sum_i x_i \otimes y_i = \sum_i n^{-1} Tr_G(x_i) \otimes y_i = \sum_i Tr_G(n^{-1}x_i) \otimes y_i \in Tr_G(A) \otimes B$. Thus $(A \otimes_R B)^{G\times 1} = Tr_G(A) \otimes_R B$. Similarly, $(Tr_G(A) \otimes_R B)^{1\times H} = Tr_G(A) \otimes_R Tr_H(B)$. But then $(A \otimes_R B)^{G\times H} = ((A \otimes_R B)^{G\times 1})^{1\times H} = Tr_G(A) \otimes_R Tr_H(B)$.

Theorem 3.2 *Let A and B be Galois R-algebras with Galois groups G and H respectively. Denote the order of G by n and the order of H by m for some integers n and m. If n and m are invertible in R, then $A \otimes_R B$ is a Galois R-algebra with Galois group $G \times H$.*

Proof. By *Lemma* 3.1, $G \times H$ is an automorphism group of $A \otimes_R B$ such that $(A \otimes_R B)^{G\times H} = Tr_G(A) \otimes_R Tr_H(B)$. Since $Tr_G(A) = R = Tr_H(B)$, $(A \otimes_R B)^{G\times H} = R$. It suffices to show that $A \otimes_R B$ has an $G \times H$-Galois system. Let $\{a_i, b_i | i = 1, \cdots, s$ for some integer $s\}$ be a G-Galois system for A and $\{c_j, d_j | j = 1, \cdots, k$ for some integer $k\}$ an H-Galois system for B. Then it is straightforward to verify that $\{a_i \otimes c_j, b_i \otimes d_j, | i = 1, \cdots, s,\ j = 1, \cdots, k\}$ is a $G \times H$-Galois system for $A \otimes_R B$.

Corollary 3.3 *Let A and B be central Galois R-algebras with Galois groups G and H respectively. Then $A \otimes_R B$ is a central Galois R-algebra with Galois group $G \times H$.*

Proof. Since A and B are central Galois R-algebras with Galois groups G and H respectively, the orders of G and H are invertible in R. Thus the statement holds by *Theorem* 3.2 and *Proposition* 3.3, *p.* 52 in [2].

Corollary 3.4 *Let A be a central Galois R-algebra with Galois group G. Then $A \otimes_R A^o$ is a central Galois R-algebra with Galois group $G \times G^o$ where A^o is the opposite algebra of A and G^o the opposite group of G.*

Proof. Let $\{a_i, b_i | i = 1, \cdots, s$ for some integer $s\}$ be a G-Galois system for A. Then $\{b_i, a_i | i = 1, \cdots, s\}$ is a G^o-Galois system for A^o. Thus the *Corollary* is immediate by *Theorem* 3.2.

Next we show the converse of *Theorem* 3.2. We need a lemma.

Lemma 3.5 *Let A be a Galois extension of D with Galois group G, and K a normal subgroup of G. If the order of K is invertible in D, then A is a Galois extension of A^K with Galois group K and A^K is a Galois extension of D with Galois group G/K.*

Proof. Denote the order of K by n for some integer n, and let $\{a_i, b_i | i = 1, \cdots, s$ for some integer $s\}$ be a G-Galois system for A. Clearly, the same system is also a K-Galois system for the Galois extension A of A^K with Galois group K. Also it is straightforward to verify that $\{n^{-1}Tr_K(a_i),\ Tr_K(b_i) | i = 1, \cdots, s\}$ is a G/K-Galois system for A^K of D with Galois group G/K.

Theorem 3.6 *Let A and B be R-algebras with automorphism groups G and H respectively. If $A \otimes_R B$ is a Galois R-algebra with Galois group $G \times H$ whose order is invertible in R, then A and B are Galois R-algebras with Galois groups G and H respectively.*

Proof. Since the order of $G \times H$ is invertible in R, the orders of G and H are invertible in R. Then by the proof of *Lemma* 3.1, $(A \otimes_R B)^{G \times H} = Tr_G(A) \otimes_R Tr_H(B)$. By hypothesis, $A \otimes_R B$ is a Galois R-algebra with Galois group $G \times H$, so $Tr_G(A) \otimes_R Tr_H(B) = R$. This implies that $Tr_G(A) = R = Tr_H(B)$. Noting that the orders of G and H are invertible in R, we have that $Tr_G(A) = A^G$ and $Tr_H(B) = B^H$. Thus $A^G = R$ and $B^H = R$. On the other hand, $G \times 1$ is a normal subgroup of $G \times H$ and $(A \otimes_R B)^{G \times 1} = Tr_G(A) \otimes_R B = R \otimes_R B \cong B$, so B is a Galois R-algebra with Galois group $H \cong (G \times H)/(G \times 1)$ by *Lemma* 3.5. Similarly, A is a Galois R-algebra with Galois group G.

By *Theorem* 3.6, we derive a result for central Galois algebras.

Corollary 3.7 *Let A and B be R-algebras with automorphism groups G and H respectively. If $A \otimes_R B$ is a central Galois R-algebra with Galois group $G \times H$, then A and B are central Galois R-algebras with Galois groups G and H respectively.*

Proof. Since $A \otimes_R B$ is a central Galois R-algebra with Galois group $G \times H$, the order of $G \times H$ is invertible in R. Hence A and B are Galois R-algebras with Galois groups G and H respectively by *Theorem* 3.6. Moreover, $A \otimes_R B$ is an Azumaya R-algebra containing $A \times_R R$ and $R \otimes_R B$ as commutator subalgebras, so A and B are Azumaya R-algebras by *Theorem* 4.4, *p.* 58 in [2]. Thus A and B are central Galois R-algebras with Galois groups G and H respectively.

The converse of *Corollary* 3.4 is immediate.

Corollary 3.8 *Let A be an R-algebra with an automorphism group G. If $A \otimes_R A^o$ is a central Galois R-algebra with Galois group $G \times G^o$, then A is a central Galois R-algebra with Galois group G.*

Proof. This is an immediate consequence of *Corollary* 3.7.

4 Central Galois Algebras

Let A be an Azumaya R-algebra with an automorphism group G. In [3], it is shown that A is a central Galois R-algebra with Galois group G if and only if $A = \oplus \sum_g J_g$ where $J_g = \{a \in A | ax = g(x)a$ for all $x \in A\}$ ([3, Theorem 3.8]). By *Corollaries* 3.4 and 3.8, A is a central Galois R-algebra with Galois group G if and only if $A \otimes_R A^o$ is a central Galois R-algebra with Galois group $G \times G^o$. In this section, we shall give a different proof by the above expression of a central Galois algebra. We begin with the property of an Azumaya algebra with a finite automorphism group due to M. Harada in [3] and A. Rosenberg and D. Zelinsky in [7].

Lemma 4.1 ([3, Theorem 1]) and ([7, Theorem 2]) Let A be an Azumaya R-algebra with a finite automorphism group G and $J_g = \{a \in A | ax = g(x)a$ for all $x \in A\}$. Then, (1) $A = \oplus \sum_g J_g$ for all $g \in G$, if and only if A is a central Galois R-algebra with Galois group G, and (2) J_g is a rank one projective R-module.

Theorem 4.2 *Let A be an Azumaya R-algebra with a finite automorphism group G. Then A is a central Galois R-algebra with Galois group G if and only if $A \otimes_R A^o$ is a central Galois R-algebra with Galois group $G \times G^o$.*

Proof. ($\longrightarrow$) Since A is a central Galois R-algebra with Galois group G, A^o is a central Galois R-algebra with Galois group G^o. Hence by *Lemma* 4.1, $A = \oplus \sum_g J_g$ for all $g \in G$ where $Rank_R(J_g) = 1$ for each $g \in G$. Thus $A \otimes_R A^o \cong \oplus \sum_{g,h^o} J_g \otimes J_{h^o}$ for all $g \in G$ and $h^o \in G^o$. We claim that $J_g \otimes J_{h^o} \subset J_{g \times h^o}$. In fact, for any $x \otimes y \in J_g \otimes J_{h^o}$ and $a \otimes b \in A \otimes A^o, (x \otimes y)(a \otimes b) = xa \otimes y \circ b = g(a)x \otimes h^o(b) \circ y = ((g \times h^o)(a \otimes b))(x \otimes y)$. Noting that $A \otimes A^o$ is an Azumaya R-algebra, we have $Rank_R(J_g \otimes J_{h^o}) = 1$ by *Lemma* 4.1 again. Moreover, since $J_g \otimes J_{h^o}$ is a direct summand of $A \otimes A^o$, $J_g \otimes J_{h^o}$ is a direct sumand of $J_{g \times h^o}$. But $Rank_R(J_g \otimes J_{h^o}) = 1 = Rank_R(J_{g \times h^o})$, so $J_g \otimes J_{h^o} = J_{g \times h^o}$. Thus $A \otimes_R A^o = \oplus \sum_{g,h^o} J_{g \times h^o}$ for all $g \in G$ and $h^o \in G^o$. Therefore $A \otimes A^o$ is a central Galois R-algebra with Galois group $G \times G^o$ by *Lemma* 4.1.

($\longleftarrow$) Since $A \otimes_R A^o$ is a central Galois R-algebra with Galois group $G \times G^o, A \otimes_R A^o = \oplus \sum_{g,h^o} J_{g \times h^o}$ for all $g \in G, h^o \in G^o$ by *Lemma* 4.1. Noting that A is an Azumaya R-algebra, we have an isomorphism from A into $A \otimes_R A^o$ by $a \longrightarrow a \otimes 1$ for each $a \in A$. Then $J_g \cong J_{g \times 1}$ for each $g \in G$. Thus $J_{g \times 1} \subset A \otimes_R R$. On the other hand, $Rank_R(A \otimes_R A^o) = n^2$ where n is the order of G, so $Rank_R(A) = Rank_R(A^o) = n$. But $J_{g \times 1}$ is a direct summand of $A \otimes_R A^o$, so it is also a direct summand of $A \otimes_R R$. Therefore, $A \cong A \otimes_R R = \oplus \sum_g J_{g \times 1}$. Consequently, $A = \oplus \sum_g J_g$; and so A is a central Galois R-algebra with Galois group G by *Lemma* 4.1 again.

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