Gen. Math. Notes, Vol. 20, No. 1, January 2014, pp.12-18
ISSN 2219-7184; Copyright ©ICSRS Publication, 2014
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On Almost b-Continuous Functions in Bitopological Spaces

Z. Duszyǹski[1], N. Rajesh[2] and N. Balambigai[3]

[1]Casimirus the Great University,
Department of Mathematics
pl. Weyssenhoffa 11
85072 Bydgoszcz, Poland
E-mail: imath@ukw.edu.pl
[2]Department of Mathematics,
Rajah Serfoji Govt. College
Thanjavur-613005, Tamilnadu, India
E-mail: nrajesh_topology@yahoo.co.in
[3]Department of Mathematics, Prist University
Thanjavur, Tamilnadu, India
E-mail: balatopology@gmail.com

(Received: 1-10-13 / Accepted: 11-11-13)

Abstract

In this paper we introduce and study the concept of almost b-continuous functions in bitopological spaces.

Keywords: *Bitopological spaces, (i, j)-b-open sets, almost (i, j)-b-continuous functions.*

1 Introduction

The concept of bitopological spaces was first introduced by Kelly [4]. After the introduction of the Definition of a bitopological space by Kelly, a large number of topologists have turned their attention to the generalization of different concepts of a single topological space in this space. In this paper, we introduce and study the concept of almost b-continuous functions in bitopological spaces. Throughout this paper, the triple (X, τ_1, τ_2) where X is a set and τ_1 and τ_2

are topologies on X, will always denote a bitopological space. For a subset A of a bitopological space (X, τ_1, τ_2), the closure of A and the interior of A with respect to τ_i are denoted by $iCl(A)$ and $iInt(A)$, respectively, for $i = 1, 2$.

2 Preliminaries

Definition 2.1 *A subset A of a bitopological space (X, τ_1, τ_2) is said to be:*

1. *(i, j)-semiopen [3] if $A \subset jCl(iInt(A))$,*
2. *(i, j)-semi-preopen [6] if $A \subset jCl(iInt(jCl(A)))$,*
3. *(i, j)-b-open [1] if $A \subset jCl(iInt(A)) \cup iInt(jCl(A))$,*
4. *(i, j)-regular open [2] if $A = iInt(jCl(A))$,*

On each definition above, $i, j = 1, 2$ and $i \neq j$.

The complement of an (i, j)-semiopen (resp. (i, j)-semi-preopen, (i, j)-b-open, (i, j)-regular open) set is called an (i, j)-semiclosed (resp. (i, j)-semi-preclosed, (i, j)-b-closed, (i, j)-regular closed) set.

Definition 2.2 *[1] Let A be a subset of a bitopological space (X, τ_1, τ_2). Then the intersection of all (i, j)-b-closed sets of X containing A is called the (i, j)-b-closure of A and is denoted by (i, j)-$bCl(A)$. The union of all (i, j)-b-open sets of X contained in A is called the (i, j)-b-interior of A and is denoted by (i, j)-$bInt(A)$.*

Definition 2.3 *A point x of X is said to be the (i, j)-δ-cluster point [5] of A if $A \cap U \neq \emptyset$ for every (i, j)-regular open set U containing x, the set of all (i, j)-δ-cluster points of A is called the (i, j)-δ-closure of A, a subset A of X is said to be (i, j)-δ-closed if the set of all (i, j)-δ-cluster points of A is a subset of A, the complement of an (i, j)-δ-closed set is an called an (i, j)-δ-open set or a subset A of X is called (i, j)-δ-open if and only if there exist (i, j)-regular open sets $A_k, k \in I$ such that $A = \bigcup_{k \in I} A_k$.*

Lemma 2.4 *[1] Let (X, τ_1, τ_2) be a bitopological space and A a subset of X. Then*

1. *(i, j)-$bInt(A)$ is an (i, j)-b-open set;*
2. *(i, j)-$bCl(A)$ is an (i, j)-b-closed set;*
3. *A is (i, j)-b-open if and only if $A = (i, j)$-$bInt(A)$;*

4. *A is (i,j)-b-closed if and only if $A = (i,j)$-$bCl(A)$;*

5. *(i,j)-$bInt(X \backslash A) = X \backslash (i,j)$-$bCl(A)$;*

6. *(i,j)-$bCl(X \backslash A) = X \backslash (i,j)$-$bInt(A)$.*

Lemma 2.5 *[1] Let (X, τ_1, τ_2) be a bitopological space and $A \subset X$. A point $x \in (i,j)$-$bCl(A)$ if and only if $U \cap A \neq \emptyset$ for every (i,j)-b-open set U containing x.*

Definition 2.6 *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be (i,j)-b-continuous if $f^{-1}(B)$ is (i,j)-b-open in X for each σ_i-open set B of Y.*

3 Almost (i,j)-b-Continuous Functions

Definition 3.1 *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called an almost (i,j)-b-continuous at a point $x \in X$ if for each σ_i-open set V of Y containing $f(x)$, there exists an (i,j)-b-open set U of X containing x such that $f(U) \subset iInt(jCl(V))$. If f is almost (i,j)-b-continuous at every point x of X, then it is called almost (i,j)-b-continuous.*

It is obvious from the definition that (i,j)-b-continuity implies almost (i,j)-b-continuity. However, the converse is not true in general as it is shown in the following example.

Example 3.2 *Let $X = \{a,b,c,d\}$, $\tau_1 = \{\emptyset, \{a\}, X\}$, $\tau_2 = \{\emptyset, \{b,c\}, X\}$, $\sigma_1 = \{\emptyset, \{a\}, X\}$ and $\sigma_2 = \{\emptyset, \{a,b\}, X\}$. Then the function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ defined by $f(a) = d$, $f(b) = b$ $f(c) = c$ and $f(d) = a$ is almost (i,j)-b-continuous but not (i,j)-b-continuous.*

Theorem 3.3 *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following statements are equivalent:*

1. *f is almost (i,j)-b-continuous.*

2. *For each $x \in X$ and each (i,j)-regular open set V of Y containing $f(x)$, there exists an (i,j)-b-open U in X containing x such that $f(U) \subset V$.*

3. *For each $x \in X$ and each (i,j)-δ-open set V of Y containing $f(x)$, there exists an (i,j)-b-open U in X containing x such that $f(U) \subset V$.*

Proof: (1) $\Rightarrow$ (2): Let $x \in X$ and let V be any (i,j)-regular open subset of Y containing $f(x)$. By (1), there exists an (i,j)-b-open set U of X containing x such that $f(U) \subset iInt(jCl(V))$. Since V is (i,j)-regular open, $iInt(jCl(V)) = V$. Therefore, $f(U) \subset V$.

(2) $\Rightarrow$ (3): Let $x \in X$ and let V be any (i,j)-δ-open set of Y containing $f(x)$. Then for each $f(x) \in V$, there exists a σ_i-open set G containing $f(x)$ such that $G \subset iInt(jCl(G)) \subset V$. Since $iInt(jCl(G))$ is (i,j)-regular open set of Y containing $f(x)$. By (2), there exists an (i,j)-b-open set U in X containing x such that $f(U) \subset iInt(jCl(G)) \subset V$.
(3) $\Rightarrow$ (1): Let $x \in X$ and let V be any σ_i-open set of Y containing $f(x)$. Then $iInt(jCl(V))$ is (i,j)-δ-open set of Y containing $f(x)$. By (3), there exists an (i,j)-b-open set U in X containing x such that $f(U) \subset iInt(jCl(V))$. Therefore, f is almost (i,j)-b-continuous.

Theorem 3.4 *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following statements are equivalent:*

1. *f is almost (i,j)-b-continuous.*
2. *$f^{-1}(iInt(jCl(V)))$ is (i,j)-b-open set in X for each σ_i-open set V in Y.*
3. *$f^{-1}(iCl(jInt(F)))$ is (i,j)-b-closed set in X for each σ_i-closed set F in Y.*
4. *$f^{-1}(F)$ is (i,j)-b-closed set in X for each (i,j)-regular closed set F of Y.*
5. *$f^{-1}(V)$ is (i,j)-b-open set in X for each (i,j)-regular open set V of Y.*

Proof: (1) $\Rightarrow$ (2): Let V be any σ_i-open set in Y. We have to show that $f^{-1}(iInt(jCl(V)))$ is (i,j)-b-open set in X. Let $x \in f^{-1}(iInt(jCl(V)))$. Then $f(x) \in iInt(jCl(V))$ and $iInt(jCl(V))$ is an (i,j)-regular open set in Y. Since f is almost (i,j)-b-continuous, by Theorem 3.3, there exists an (i,j)-b-open set U of X containing x such that $f(U) \subset iInt(jCl(V))$. Which implies that $x \in U \subset f^{-1}(iInt(jCl(V)))$. Therefore, $f^{-1}(iInt(jCl(V)))$ is an (i,j)-b-open set in X.
(2) $\Rightarrow$ (3): Let F be any σ_i-closed set of Y. Then $Y \backslash F$ is a σ_i-open set of Y. By (2), $f^{-1}(iInt(jCl(Y \backslash F)))$ is an (i,j)-b-open set in X and $f^{-1}(iInt(jCl(Y \backslash F))) = X \backslash f^{-1}(iCl(jInt(F)))$ is an (i,j)-b-open set in X and hence $f^{-1}(iCl(jInt(F)))$ is an (i,j)-b-closed set in X.
(3) $\Rightarrow$ (4): Let F be any (i,j)-regular closed set of Y. Then F is a σ_i-closed set of Y. By (3), $f^{-1}(iCl(jInt(F)))$ is an (i,j)-b-closed set in X. Since F is (i,j)-regular closed, $f^{-1}(iCl(jInt(F))) = f^{-1}(F)$. Therefore, $f^{-1}(F)$ is an (i,j)-b-closed set in X.
(4) $\Rightarrow$ (5): Let V be any (i,j)-regular open set of Y. Then $Y \backslash V$ is an (i,j)-regular closed set of Y and by (4), we have $f^{-1}(Y \backslash V) = X \backslash f^{-1}(V)$ is (i,j)-b-closed set in X and hence $f^{-1}(V)$ is (i,j)-b-open in X.
(5) $\Rightarrow$ (1): Let $x \in X$ and let V be any (i,j)-regular open set of Y containing

$f(x)$. Then $x \in f^{-1}(V)$. By (5), we have $f^{-1}(V)$ is (i,j)-b-open set in X. Therefore, we obtain $f(f^{-1}(V)) \subset V$. Hence by Theorem 3.3, f is almost (i,j)-b-continuous.

Theorem 3.5 *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following statements are equivalent:*

1. *f is almost (i,j)-b-continuous.*
2. *(i,j)-$bCl(f^{-1}(V)) \subset f^{-1}(iCl(V))$ for each (j,i)-semi-preopen set V of Y.*
3. *$f^{-1}(iInt(F)) \subset (i,j)$-$bInt(f^{-1}(F))$ for each (j,i)-semi-preclosed set F of Y*
4. *$f^{-1}(iInt(F)) \subset (i,j)$-$bInt(f^{-1}(F))$ for each (j,i)-semiclosed set F of Y.*
5. *(i,j)-$bCl(f^{-1}(V)) \subset f^{-1}(iCl(V))$ for each (j,i)-semiopen set V of Y.*

Proof: (1) $\Rightarrow$ (2): Let V be any (j,i)-semi-preopen set of Y. Since $iCl(V)$ is an (i,j)-regular closed set in Y and f is almost (i,j)-b-continuous, by Theorem 3.4, $f^{-1}(V)$ is (i,j)-b-closed set in X.
Therefore, (i,j)-$bCl(f^{-1}(V)) \subset f^{-1}(jCl(V))$. (2) $\Rightarrow$ (3) and (3)$\Rightarrow$ (4) are clear.
(4) $\Rightarrow$ (5): Let V be any (j,i)-semiopen set of Y. Then $Y\backslash V$ is (j,i)-semiclosed set and by (4), we have $f^{-1}(iInt(Y\backslash V)) \subset (i,j)$-$bInt(f^{-1}(Y\backslash V)) \subset X\backslash(i,j)$-$bCl(f^{-1}(V))$. Therefore, (i,j)-$bCl(f^{-1}(V)) \subset f^{-1}(iCl(V))$.
(5) $\Rightarrow$ (1): Let F be any (i,j)-regular closed set of Y. Then F is an (j,i)-semiopen set of Y. By (5), we have (i,j)-$bCl(f^{-1}(F)) \subset f^{-1}(jCl(F)) = f^{-1}(F)$. This shows that $f^{-1}(F)$ is (i,j)-b-closed set in X. Therefore, by Theorem 3.4, f is almost (i,j)-b-continuous.

Theorem 3.6 *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is almost (i,j)-b-continuous if and only if $f^{-1}(V) \subset (i,j)$-$bInt(f^{-1}(iInt(jCl(V))))$ for each σ_i-open set V of Y.*

Proof: Let V be any σ_i-open set of Y. Then $V \subset iInt(jCl(V))$ and $iInt(jCl(V))$ is (i,j)-regular open set in Y. Since f is almost (i,j)-b-continuous, by Theorem 3.4, $f^{-1}(iInt(jCl(V)))$ is (i,j)-b-open set in X and hence we obtain that $f^{-1}(V) \subset f^{-1}(iInt(jCl(V))) = (i,j)$-$bInt(f^{-1}(iInt(jCl(V))))$. Conversely, let V be any (i,j)-regular open set of Y. Then V is σ_i-open set of Y. By hypothesis, we have $f^{-1}(V) \subset (i,j)$-$bInt(f^{-1}(iInt(jCl(V)))) = (i,j)$-$bInt(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is (i,j)-b-open set in X and hence by Theorem 3.4, f is almost (i,j)-b-continuous.

Corollary 3.7 *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is almost (i, j)-b-continuous if and only if (i, j)-$bCl(f^{-1}(jCl(iInt(F)))) \subset f^{-1}(F)$ for each σ_i-closed set F of Y.*

Theorem 3.8 *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be an almost (i, j)-b-continuous function and $V \in \sigma_i \cap \sigma_j$. If $x \in (i, j)$-$bCl(f^{-1}(V)) \backslash f^{-1}(V)$, then $f(x) \in (i, j)$-$bCl(V)$.*

Proof: Let $x \in X$ be such that $x \in (i, j)$-$bCl(f^{-1}(V)) \backslash f^{-1}(V)$ and suppose $f(x) \notin (i, j)$-$bCl(V)$. Then there exists an (i, j)-b-open set H containing $f(x)$ such that $H \cap V = \emptyset$. Then $iInt(jCl(H)) \cap V = \emptyset$ and $iInt(jCl(H))$ is an (i, j)-regular open set. Since f is almost (i, j)-b-continuous, by Theorem 3.4, there exists an (i, j)-b-open set U in X containing x such that $f(U) \subset iInt(jCl(H))$. Therefore, $f(U) \cap V = \emptyset$. However, since $x \in (i, j)$-$bCl(f^{-1}(V)), U \cap f^{-1}(V) \neq \emptyset$ for every (i, j)-b-open set U in X containing x, so that $f(U) \cap V \neq \emptyset$. We have a contradiction. It follows that $f(x) \in (i, j)$-$bCl(V)$.

Definition 3.9 *Let (X, τ_1, τ_2) be a bitopological space and A a subset of X. The (i, j)-b-frontier of A, (i, j)-$bFr(A)$, is defined by (i, j)-$bFr(A) = (i, j)$-$bCl(A) \cap (i, j)$-$bCl(X \backslash A) = (i, j)$-$bCl(A) \backslash (i, j)$-$bInt(A)$*

Theorem 3.10 *The set of all points x of X at which $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is not almost (i, j)-b-continuous is identical with the union of the (i, j)-b-frontiers of the inverse images of (i, j)-regular open subsets of Y containing $f(x)$.*

Proof: If f is not almost (i, j)-b-continuous at $x \in X$, then there exists an (i, j)-regular open set V containing $f(x)$ such that for every (i, j)-b-open set U of X containing x, $f(U) \cap (Y \backslash V) \neq \emptyset$. This means that for every (i, j)-b-open set U of X containing x, we must have $U \cap (X \backslash f^{-1}(V)) \neq \emptyset$. Hence it follows that $x \in (i, j)$-$bCl(X \backslash f^{-1}(V))$. But $x \in f^{-1}(V)$ and hence $x \in (i, j)$-$bCl(f^{-1}(V))$. This means that x belongs to the (i, j)-b-frontier of $f^{-1}(V)$. Conversely, suppose that x belongs to the (i, j)-b-frontier of $f^{-1}(V_1)$ for some (i, j)-regular open subset V_1 of Y such that $f(x) \in V_1$. Suppose that f is almost (i, j)-b-continuous at x. Then by Theorem 3.3, there exists an (i, j)-b-open set U of X containing x such that $f(U) \subset V_1$. Then we have $U \subset f^{-1}(V_1)$. This shows that $x \in (i, j)$-$bInt(f^{-1}(V_1))$. Therefore, we have $x \notin (i, j)$-$bCl(X \backslash f^{-1}(V_1))$ and $x \notin (i, j)$-$bFr(f^{-1}(V_1))$, which is a contradiction. This means that f is not almost (i, j)-b-continuous.

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