Gen. Math. Notes, Vol. 37, No. 2, December 2016, pp.13-19
ISSN 2219-7184; Copyright ©ICSRS Publication, 2016
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Some Properties of Auto-Permutable Subgroups

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(Received: 29-11-16 / Accepted: 30-12-16)

Abstract

The concept of autopermutable subgroups was introduced by Housieni and Moghaddam in 2014. In the present article, we study the properties of auto-permutable subgroups of a given group and among other results, it is shown that how auto-permutability induces characteristic properties. Also we show that all subgroups of a Hamiltonian nilpotent group always enjoy this property.

Keywords: *Permutable subgroup, Auto-permutable subgroup, Central automorphism, Characteristic subgroup.*

1 Introduction and Preliminaries

In this paper we continue the work on auto-permutable subgroups started in [4].

Definition 1.1. *Let H be a subgroup of a given group G. Then we call H to be auto-permutable, if $HH^\alpha = H^\alpha H$ for all $\alpha \in Aut(G)$ and denoted by $H <_{a-p} G$.*

Clearly, if α runs over the inner automorphisms of the group G, we obtain the notion of conjugate-permutability (see [2,3]). To see an example that conjugate-permutability doesn't imply auto-permutability, we may consider the subgroup $\langle b \rangle$ of the Dihedral group $G = D_8 = \langle a, b : \ a^4 = b^2 = 1, \ a^b =$

$a^{-1}\rangle$ and the automorphism α which fixes a and sends b into ab. For more example, one may take the direct product of D_8 by any other group.

In the present article, we study some properties of these new notion. Among other results, we show that every subgroup of a Hamiltonian nilpotent group is auto-permutable.

Proofs of main results are based on the following lemmas.

Lemma 1.2. *Let H be an auto-permutable subgroup of a group G, then*

1. $H^\alpha <_{a-p} G$, *for all* $\alpha \in Aut(G)$;

2. $\bigcap_{i\in I} H_i <_{a-p} G$, *for all* $H_i <_{a-p} G$.

Proof: (1) Since H is an auto-permutable subgroup, we have

$$H^{\alpha\beta\alpha^{-1}}H = HH^{\alpha\beta\alpha^{-1}},\ for\ all\ \alpha, \beta \in Aut(G).$$

This implies that $(H^\alpha)^\beta H^\alpha = H^\alpha(H^\alpha)^\beta$.

(2) The proof of this part follows easily.

Lemma 1.3. *If $H <_{a-p} G$ and $\beta_1, \ldots, \beta_n \in Aut(G)$, then $H^{\beta_1}H^{\beta_2}\ldots H^{\beta_n}$ permutes with any finite product $H^{\alpha_1}H^{\alpha_2}\ldots H^{\alpha_m}$, for all $\alpha_1 \ldots \alpha_m \in Aut(G)$.*

Proof: Note that H is an auto-permutable subgroup of G and $\beta_j^{-1}\alpha_i \in Aut(G)$, for all $1 \leq i \leq m$ and $1 \leq j \leq n$. It follows that $HH^{\beta_j^{-1}\alpha_i} = H^{\beta_j^{-1}\alpha_i}H$. This implies that $H^{\beta_j}H^{\alpha_i} = H^{\alpha_i}H^{\beta_j}$, which gives the result.

Lemma 1.4. *Let H be a Hall subgroup of a finite group G. If H is an auto-permutable subgroup, then it is characteristic in G.*

Proof: For each $\alpha \in Aut(G)$, $HH^\alpha \leq G$. Suppose H is of order m then $|HH^\alpha| = \frac{m^2}{d}$, where $|H \cap H^\alpha| = d$. If $[G : H] = n$, then $(m, n) = 1$. Clearly $\frac{m}{d}|n$, which implies that $m = d$ and hence $H^\alpha = H$. This completes the proof.

The above lemma has the following result.

Corollary 1.5. *If a Sylow p-subgroup P of a finite group G is an auto-permutable subgroup, then it is characteristic.*

2 Auto Permutable Properties of Sylow p - Subgroups of a Group

In this section we give some more properties of auto-permutable subgroups, which have earlier studied for conjugate permutable subgroups (see [5,6,7]). First we prove the following useful lemma.

Lemma 2.1. *Let H be a non characteristic maximal subgroup of a Sylow p-subgroup P of a finite group G, which is an auto-permutable subgroup in G. Then P is characteristic in G.*

Proof: Clearly the finiteness of G implies that $Aut(G)$ is also finite. Assume H is not characteristic in G, and $\mathcal{A} = Aut(G) \backslash N_{Aut(G)}(H) = \{\alpha_1, \ldots, \alpha_n\}$. where $N_{Aut(G)}(H) = \{\alpha \in Aut(G); \quad H^\alpha = H\}$.

Put $H_1 = HH^{\alpha_1} \ldots H^{\alpha_n}$ and using Lemma 1.3, then $HH^{\alpha_1} \ldots H^{\alpha_{n-1}}$ commutes with H^{α_n}. Thus $H_1 = H^{\alpha_n} \ldots H^{\alpha_1} H$, and so H_1 is a subgroup of G containing H properly. Hence the maximality of H in P implies that $|H_1| = |P|$. Thus H_1 is also a Sylow p-subgroup of G and so it is conjugate with P. Therefore, there exists $\varphi_g \in Inn(G)$ such that

$$\begin{aligned}(H_1)^{\varphi_g} &= (HH^{\alpha_1} \ldots H^{\alpha_n})^{\varphi_g} \\ &= H^{\varphi_g} H^{\alpha_1 \varphi_g} \ldots H^{\alpha_n \varphi_g} = P.\end{aligned}$$

Now, we claim that $\alpha_1\varphi_g, \ldots, \alpha_n\varphi_g \in \mathcal{A}$. If $\varphi_g \in N_{Aut(G)}(H)$ then $H^{\varphi_g} = H$ and $H^{\alpha_i} \neq H$, which implies $H^{\alpha_i\varphi_g} \neq H$. Also if $\varphi_g \notin N_{Aut(G)}(H)$ then clearly $H^{\alpha_i\varphi_g} \neq H$ and hence in both cases we have $\alpha_i\varphi_g \in \mathcal{A}$, $1 \leq i \leq n$. Thus,

$$P = (HH^{\alpha_1} \ldots H^{\alpha_n})^{\varphi_g} \subseteq HH^{\alpha_1} \ldots H^{\alpha_n}.$$

By the assumption $\alpha_1 \ldots \alpha_n \notin N_{Aut(G)}(H)$ and so

$$P = HH^{\alpha_1} \ldots H^{\alpha_n}.$$

If β is an arbitrary automorphism of G, then as explained above $\alpha_1\beta, \ldots, \alpha_n\beta \in \mathcal{A}$ and hence

$$P^\beta = H^\beta H^{\alpha_1\beta} \ldots H^{\alpha_n\beta} \subseteq HH^{\alpha_1} \ldots H^{\alpha_n} = P,$$

which implies that P is a characteristic subgroup of G.

In the next theorem we show how the auto-permutability implies characteristic property in Sylow p-subgroups.

Theorem 2.2. *If all the cyclic subgroups of order a power of p in a finite group G are auto-permutable subgroups, for any prime number p dividing the order of G. Then all Sylow p-subgroups of G are characteristic in G.*

Proof: Let $P = \{x_1, \ldots, x_k\}$ be a Sylow p-subgroup of G. By the assumption and using Lemma 1.3,

$$H_i = \langle x_i \rangle \langle x_i \rangle^{\alpha_{i_1}} \ldots \langle x_i \rangle^{\alpha_{i_n}}$$

is a subgroup of G, where $\alpha_{i_1}, \ldots, \alpha_{i_n} \in Aut(G) \setminus N_{Aut(G)} \langle x_i \rangle$. By the same argument as in Lemma 2.1,

$$H_i^{\beta} \subseteq H_i, \quad \text{for all } \beta \in Aut(G).$$

Thus H_i is a characteristic subgroup of G, for $i = 1, \ldots, k$. Therefore $H = H_1 H_2 \ldots H_k$ is a characteristic p-subgroup of G containing P. Hence $P = H$ is characteristic in G.

The following corollary shows that the above theorem holds for all locally finite groups.

Corollary 2.3. *If all cyclic subgroups of a locally finite group G, of order a power of p are auto-permutable subgroups, for some prime p. Then each Sylow p-subgroup of G is characteristic in G.*

Proof: For all p-elements x and y of the group G, the subgroup $H = \langle x, y \rangle$ is finite and, by Theorem 2.2 the Sylow p-subgroups $\langle x \rangle$ and $\langle y \rangle$ of H are characteristic in H. Hence $\langle x \rangle \langle y \rangle$ is a subgroup of H and $|xy| = |\langle xy \rangle|$. Since $\langle xy \rangle \subseteq \langle x \rangle \langle y \rangle$, it implies that $\langle xy \rangle$ is of prime power order p, and so xy is a p-element. Thus the set

$$S = \{x \in G; \;\; x \; is \; a \;\; p\text{-element}\}$$

is a p-subgroup of G, and hence it is contained in a Sylow p-subgroup P, say of G. Also it is clear that $P \subseteq S \subseteq P$, and therefore $S = P$. Clearly the subgroup S is characteristic in G and so is P.

3 Auto-Permutability Properties of Semidirect and Direct Product of Hamiltonian Finite Groups

In this section we study the notion of auto-permutability subgroups of semidirect and direct product of Hamiltonian finite groups.

The following theorems of [8] are needed in proving our results.

Theorem 3.1. *([8], Theorem 2.2.) Let $G = H \rtimes K$ be the semidirect product of its subgroups H and K. Then for all $\theta \in Aut(G)$,*

$Aut(G) = C_{Aut(G)}(H)C_{Aut(G)}(K)$ if and only if $\theta(K) \cap H = 1$ and $K^{-1}\theta(K) \subseteq C_G(H)$.

Theorem 3.2. *([8], Theorem 3.2.) Let $G = H \rtimes K = \langle x, y | \ x^{p^m} = y^{p^n} = 1, \quad x^y = x^{1+p^{m-r}} \rangle$, where $H = \langle x \rangle$, $K = \langle y \rangle$, $m \geq 2$, $n \geq 1$ and $1 \leq r \leq min\{m-1, n\}$. Then $Aut(G) = C_{Aut(G)}(H)C_{Aut(G)}(K)$.*

The following results give the properties of auto-permutability of semidirect product of Hamiltonian groups.

Theorem 3.3. *Let H and K be subgroups of the Hamiltonian group G, with $G = H \rtimes K$. Also, assume that H_1 and K_1 are subgroups of H and K, respectively. If for any automorphism θ of G and $k \in K$, $k^{-1}\theta(k) \in C_G(H)$ and $\theta(K) \cap H = 1$, then H_1K_1 is an auto-permutable subgroup of G.*

Proof: By Theorem 3.1, $Aut(G) = C_{Aut(G)}(H)C_{Aut(G)}(K)$ and hence any automorphism θ of G can be written as $\theta = \alpha\beta$, where $\alpha \in C_{Aut(G)}(H)$ and $\beta \in C_{Aut(G)}(K)$. To prove our claim, it is enough to show that for all $h, h_1 \in H_1$ and $k, k_1 \in K_1$

$$\theta(hk)h_1k_1 \in H_1K_1(H_1K_1)^\theta.$$

Clearly,

$$\begin{aligned}
\theta(hk)h_1k_1 &= \theta(h)\theta(k)(h_1k_1) = \theta(h)\alpha(k)h_1k_1 \\
&= \theta(h)kk^{-1}\alpha(k)h_1k_1 \\
&= \theta(h)kh_1k^{-1}\alpha(k)k_1 \\
&= \theta(h)kh_1k^{-1}\theta(h^{-1})\theta(h)\alpha(k)k_1 \\
&= h_1^{\theta(h)k}\theta(h)\alpha(k)k_1 \\
&= h_1^{\theta(h)k}\theta(h)\alpha(k)k_1\theta((hk)^{-1}))\theta(hk) \\
&= h_1^{\theta(h)k}k_1^{\theta(hk)}\theta(hk) \in (H_1K_1)^\theta.
\end{aligned}$$

Therefore $(H_1K_1)^\theta H_1K_1$ is contained in $H_1K_1(H_1K_1)^\theta$ and the equality holds, by using similar argument.

Theorem 3.4. *Let the Hamiltonian group $G = H \rtimes K = \ \langle x, y | \ x^{p^m} = y^{p^n} = 1, \quad x^y = x^{1+p^{m-r}} \rangle$, where $H = \langle x \rangle$, $K = \langle y \rangle$, $m \geq 2$, $n \geq 1$ and $1 \leq r \leq min\{m-1, n\}$. If H_1 and K_1 are subgroups of H and K, respectively, then H_1K_1 is auto-permutable subgroup of G.*

Proof: Assume that h, h_1 and k, k_1 are arbitrary elements of H_1 and K_1, respectively. By Theorem 3.2, any automorphism $\theta \in Aut(G)$ may be written as $\theta = \alpha\beta$, where $\alpha \in C_{Aut(G)}(H)$ and $\beta \in C_{Aut(G)}(K)$. Then

$$\begin{aligned}
\theta(hk)h_1k_1 &= \theta(h)\alpha(k)h_1k_1 = \theta(h)\alpha(k)h_1\alpha(k^{-1})\alpha(k)k_1 \\
&= \theta(h)x\theta(h^{-1})\theta(h)\alpha(k)k_1
\end{aligned}$$

$$
\begin{aligned}
&= y\theta(h)\alpha(k)k_1 \\
&= y\theta(h)\alpha(k)k_1\alpha(k^{-1})\theta(h^{-1})\theta(h)\alpha(k) \\
&= yk_1^{\theta(h)\alpha(k)}\theta(h)\alpha(k) \\
&= yz\theta(h)\alpha(k) \\
&= yz\theta(hk) \in H_1K_1(H_1K_1)^{\theta},
\end{aligned}
$$

where $x = \alpha(k)h_1\alpha(k^{-1})$, and $y = \theta(h)x\theta(h^{-1})$ are elements of H_1 and $z = k_1^{\theta(h)\alpha(k)}$ is an element of K_1. Therefore, $(H_1K_1)^{\theta}H_1K_1$ is contained in $H_1K_1(H_1K_1)^{\theta}$, which gives the result.

The following result of [1] gives the structure of automorphisms group of the direct product $G = H \times K$, in which H and K have no common direct factor.

Theorem 3.5. *([1], Theorem 1.3.). Let $G = H \times K$, be the direct product of the subgroups H and K with no common direct factors. Then*

$$
Aut(G) \cong \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \; : \; \begin{matrix} \alpha \in Aut(H) & \beta \in Hom(K, Z(H)) \\ \gamma \in Hom(H, Z(K)) & \delta \in Aut(K) \end{matrix} \right\},
$$

where $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} \alpha(h)\beta(k) \\ \gamma(h)\delta(k) \end{pmatrix}$, *for all* $\begin{pmatrix} h \\ k \end{pmatrix}$ *in* $H \times K$.

Now, we are able to state and prove our final result.

Theorem 3.6. *Let the Hamiltonian group $G = H \times K$ be the direct product of its subgroups H and K with no common direct factors. Then any subgroup of G is auto-permutable.*

Proof: Let G_1 be a subgroup of the group G, then we may assume that $G_1 = H_1 \times K_1$ such that H_1 and K_1 are subgroups of H and K, respectively. Since G is a hamiltonian group, H_1 is normal in H and K_1 is normal in K. We must show that $G_1^{\theta}G_1 = G_1G_1^{\theta}$, for any $\theta \in Aut(G)$. Using Theorem 3.5, then for all $h, h_1 \in H_1$ and $k, k_1 \in K_1$, we have

$$
\begin{aligned}
\theta(h,k)(h_1,k_1) &= (\alpha(h)\beta(k), \gamma(h)\delta(k))(h_1,k_1) \\
&= (\alpha(h)\beta(k)h_1, \gamma(h)\delta(k)k_1) \\
&= (\alpha(h)h_1\beta(k), \delta(k)k_1\gamma(h)) \\
&= (\alpha(h)h_1\alpha(h^{-1})\alpha(h)\beta(k), \delta(k)k_1\delta(k^{-1})\delta(k)\gamma(h)) \\
&= (h_1^{\alpha(h)}\alpha(h)\beta(k), k_1^{\delta(k)}\gamma(h)\delta(k)) \\
&= (h_1^{\alpha(h)}, k_1^{\delta(k)})(\alpha(h)\beta(k), \gamma(h)\delta(k)) \\
&= (h_1^{\alpha(h)}, k_1^{\delta(k)})(h,k)^{\theta}.
\end{aligned}
$$

Therefore $G_1^{\theta}G_1$ is contained in $G_1G_1^{\theta}$, which proves our claim.

Corollary 3.7. *If G is a Hamiltonian nilpotent group. Then all of its subgroups are auto-permutable.*

4 Conclusion

In this paper some interesting properties of auto-permutable subgroups are obtained. Moreover, we studied elementary properties of groups with auto-permutable subgroups. It is proved that in certain conditions auto-permutability and characteristic properties are equivalent. In addition we have shown that every subgroup of a Hamiltonian nilpotent group is auto-permutable. We believe that more properties for these subgroups can be found which leads to new classifications for finite groups.

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