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Orbital Graphs for the Small Residue Class of $PSL(2, 5)$

Murat Beşenk[1], Bahadır Özgür Güler[2] and Tuncay Köroğlu[3]

[1,2,3]Karadeniz Technical University
Department of Mathematics, Science Faculty
Trabzon, Turkey
[1]E-mail: mbesenk@ktu.edu.tr
[2]E-mail: boguler@ktu.edu.tr
[3]E-mail: tkor@ktu.edu.tr

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Abstract

Let $q \geq 2$ integer. Then the projective special linear groups $PSL(2, q) \cong SL(2, q)/Z$, where Z is the central subgroup of $SL(2, q)$ consisting of the scalar matrices with determinant equal to 1. $PSL(2, q)$ is the most frequently studied subgroup of the modular group. The elements of this group correspond to matrices of $SL(2, q)$. In this study, for $q = 5$ we find all triangles circuits, loops and paths in blocks.

Keywords: *Suborbital graphs, Orbit, Circuit, Primitive action.*

1 Introduction

Let F be a field. Then the *general linear group* $GL(n, F)$ is the group of invertible $n \times n$ matrices with entries in F under matrix multiplication. In addition that the projective general linear group $PGL(n, F)$ and the projective special linear group $PSL(n, F)$ are the quotients of $GL(n, F)$ and $SL(n, F)$ by their centers they are the induced action on the associated projective space. It is clear that if F is a finite field, $|F| = q$, then $GL(n, F)$ has only finitely many elements. For example let $n = 1$, then $GL(n, F_q) \cong F_q \setminus \{0\}$, which has $q - 1$ elements.

Since the following two lemmas are well known [4, 7], we only give the statements;

Lemma 1.1 *The number of the elements in* $GL(n, F_q)$ *is* $\prod_{k=0}^{n-1}(q^n - q^k)$.

Now we will consider well known subgroup of $GL(n, F)$. The determinant function, $det : GL(n, F) \longrightarrow F \setminus \{0\}$ is a homomorphism; it maps the identity matrix to 1, and it is multiplicative, as desired.

We define the *special linear group* $SL(n, F)$, to be kernel of this homomorphism. That is, $SL(n, F) = \{T \in GL(n, F) \,|\, detT = 1\}$.

Lemma 1.2 *The number of the elements in* $SL(n, F_q)$ *is* $\frac{|GL(n, F_q)|}{|F \setminus \{0\}|} = \frac{\prod_{k=0}^{n-1}(q^n - q^k)}{q-1}$. *And also* $|GL(n, q)| = q^{\frac{n(n-1)}{2}} \prod_{k=1}^{n}(q^k - 1)$.

In here if q is a prime, then $|GL(2, q)| = q(q-1)(q^2-1)$ and $|PGL(2, q)| = (q+1)q(q-1)$.

Now we give brief information about modular group. Let $F = \mathbb{R}$ denote the real number field and let $SL(2, \mathbb{R})$ be the group

$$SL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R}, \;\; ad - bc = 1 \right\}.$$

Actually, dividing $SL(2, \mathbb{R})$ by its center $\{\pm I\}$ we have the group $PSL(2, \mathbb{R})$ correspond to the matrices $PSL(2, \mathbb{R}) \cong SL(2, \mathbb{R})/\{\pm I\}$. It is common to denote an element of $PSL(2, \mathbb{R})$ by 2×2 matrix with real entries and determinant 1 under the convention that the matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$$

represent the same element. The group $PSL(2, \mathbb{R})$ acts on the upper half plane $\mathbb{H} := \{\tau \in \mathbb{C} : \text{Im}(\tau) > 0\}$ in the standart way, that is by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \tau \longrightarrow \frac{a\tau + b}{c\tau + d}$$

One of the most interesting and definitely the most thoroughly studied the modular group is the group $\Gamma := PSL(2, \mathbb{Z})$, is the subgroup of $PSL(2, \mathbb{R})$

with integral coefficients. It is generated by the matrices

$$U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \quad V = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

with defining relationships $U^2 = V^3 = -I$, where I is the identity matrix. Furthermore this is the automorphism group of $\mathbb{H}$. The group $PSL(2,\mathbb{Z})$ acts faithfully on the upper half plane $\mathbb{H}$ by linear fractional transformations, and moreover when equipped with the hyperbolic metric this action is by orientation preserving isometries. As well as, the space $\mathbb{H}$ can be intrinsically characterized as the unique two dimensional simply connected Riemann manifold with constant curvature.The hyperbolic metric and the Euclidean metric on $\mathbb{H}$ are equivalent, inducing the same topology. However, lengths and geodesics are different.

2 Preliminaries

In this section we aim short presentation is to describe the behavior of small classical groups $PGL(2,\mathbb{Z}_2)$, $PGL(2,\mathbb{Z}_3)$ and $PGL(2,\mathbb{Z}_5)$. For each integer $q \geq 2$, let $\mathbb{Z}_q$ denote the ring of integers $mod(q)$; then the 2×2 unimodular matrices with coefficients in $\mathbb{Z}_q$ form a group $SL(2,\mathbb{Z}_q)$ in which the matrices $\pm I$ form a normal subgroup. The natural ring-epimorphism $\mathbb{Z} \to \mathbb{Z}_q$, $a \to [a]$, induces a group-homomorphism $SL(2,\mathbb{Z}) \to SL(2,\mathbb{Z}_q)$ and also this in turn induces a group-homomorphism ϕ_q from $PSL(2,\mathbb{Z})$ to $PSL(2,\mathbb{Z}_q)$. And also we know that $|PSL(2,\mathbb{Z}_p)| = \frac{(p^2-1)p}{2}$, p is a prime. Then $|PSL(2,\mathbb{Z}_2)| = 3$, $|PSL(2,\mathbb{Z}_3)| = 12$ and $|PSL(2,\mathbb{Z}_5)| = 60$ is obviously. If p is a prime then $PSL(2,\mathbb{Z}_p) \cong SL(2,\mathbb{Z}_p)/\{\pm I\}$. Moreover these groups are all isomorphic because they each contain the same matrices. For example, if $p = 2$ then,

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

matrices are congruence according to $mod(2)$ or if $p = 3$ then,

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}, \right.$$

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix}1 & 1\\ 2 & 0\end{pmatrix} \sim \begin{pmatrix}2 & 2\\ 1 & 0\end{pmatrix}, \begin{pmatrix}1 & 2\\ 1 & 0\end{pmatrix} \sim \begin{pmatrix}2 & 1\\ 2 & 0\end{pmatrix}, \begin{pmatrix}0 & 2\\ 1 & 1\end{pmatrix} \sim \begin{pmatrix}0 & 1\\ 2 & 2\end{pmatrix}, \begin{pmatrix}0 & 1\\ 2 & 1\end{pmatrix} \sim \begin{pmatrix}0 & 2\\ 1 & 2\end{pmatrix} \Big\}$$

matrices are congruence according to $mod(3)$. Similarly we can write for $p = 5$.

The projective special linear groups $PSL(n, F_q)$ for a finite field F_q are often written as $PSL(n, q)$.Therefore in this study for abbreviate we will be written $PSL(2, \mathbb{Z}_q)$ instead of $PSL(2, q)$. Recall that a permutation of a finite set S is a bijection $\sigma : S \to S$. And we show that for a set S, $P(S)$ is the set of all permutations of S. If $\sigma, \tau \in P(S)$ then multiplication of two elements is their composition $\sigma\tau = \sigma \circ \tau$. Moreover the symmetric group S_n is the group $P(\{1, 2, ..., n\})$ of all permutations on the first n integers.

Lemma 2.1 *If* $|S| = n$ *then* $P(S) \approx S_n$.

Proof: Because of S has n elements, we may index them $S = \{x_1, ..., x_n\}$. Then isomorphism $\phi : S_n \to P(S)$ operates simply as $\phi(\sigma)(x_i) = x_{\sigma(i)}$. ϕ is obviously a homomorphism and bijective.

Now we give the theorem without proof.

Theorem 2.2 *Every group of order* n *is isomorphic to a subgroup of* S_n.

Additionally we know that an alternating group is the group of even permutations of a finite set. Also the alternating group on a set of n elements is named the alternating group of degree n and denoted by A_n.

All of the groups $PSL(n, q)$ are simple for prime powers q and integers $n > 1$. The exceptions are $PSL(2, 2)$ and $PSL(2, 3)$, which are not simple.The groups $PSL(n, F)$ are also simple when F is an infinite field. In fact these groups are isomorphic to the symmetric groups or alternating groups as follows: $PSL(2, 2) \cong S_3$, $PSL(2, 3) \cong A_4$, $PSL(2, 5) \cong A_5$, $PGL(2, 2) \cong S_3$, $PGL(2, 3) \cong S_4$, $PGL(2, 5) \cong S_5$ and so on. Also from above the theorem $|PGL(2, 2)| = |S_3| = 6$, $|PGL(2, 3)| = |S_4| = 24$ and $|PSL(2, 5)| = |\frac{S_5}{2}| = |A_5| = 60$.

3 Main Calculation

The group $PSL(2, 5)_0 := \left\{ \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in PSL(2, 5) : c \equiv 0\, mod(5) \right\}$ is special congruence subgroup of $PSL(2, 5)$. We proceed to a description of the special subgroup of $PSL(2, 5)$ process and the draw of the orbital graphs in blocks.

3.1 Imprimitive Action

Let $\mathbb{Q}\cup\{\infty\}$ element of the extended rational number set and $\Delta := \{\frac{a}{b}|a, b \in \mathbb{Z}_5\}$. It is clear that $\Delta \subset \mathbb{Q}\cup\{\infty\}$. Then any element of Δ can be given by as a reduced fraction $\dfrac{x}{y}$ with $x, y \in \mathbb{Z}_5$ and $(x, y) = 1$. Besides, ∞ is represented as $\frac{1}{0} = \frac{-1}{0}$. The action of $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in PSL(2, 5)$ on $\dfrac{x}{y}$ is

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \frac{x}{y} \to \frac{\alpha x + \beta y}{\gamma x + \delta y}.$$

Actually we will say Δ is orbit later. The elements of Δ as follows:

$\{\frac{0}{0}, \frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \frac{4}{4}, \frac{0}{1}, \frac{0}{2}, \frac{0}{3}, \frac{0}{4}, \frac{1}{0}, \frac{2}{0}, \frac{3}{0}, \frac{4}{0}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{2}{1}, \frac{3}{1}, \frac{4}{1}, \frac{2}{3}, \frac{2}{4}, \frac{3}{2}, \frac{4}{2}, \frac{3}{4}, \frac{4}{3}\}$. And also by mutual matching we may write according to $mod(5)$; $\{\frac{1}{0}, \frac{2}{0}, \frac{3}{0}, \frac{4}{0}\} \sim \{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\}$, $\{\frac{0}{1}, \frac{0}{2}, \frac{0}{3}, \frac{0}{4}\} \sim \{\frac{5}{1}, \frac{5}{2}, \frac{5}{3}, \frac{5}{4}\}$, $\{\frac{0}{0}\} \sim \{\frac{5}{5}\}$.

Now we consider the imprimitivity of the action of $PSL(2, 5)$ on Δ.

Firstly let (G, Ω) be a transitive permutation group. So, an equivalence relation $\approx$ on Ω is named G invariant if, whenever $\zeta_1, \zeta_2 \in \Omega$ satisfy $\zeta_1 \approx \zeta_2$, then $g(\zeta_1) \approx g(\zeta_2)$ for all $g \in G$. The equivalence classes are called blocks. We call (G, Ω) imprimitive, if Ω admits some G invariant equivalence relation different from

(i) $\zeta_1 \approx \zeta_2 \Leftrightarrow \zeta_1 = \zeta_2$, the identity relation

(ii) $\zeta_1 \approx \zeta_2$ for all $\zeta_1, \zeta_2 \in \Omega$, the universal relation.

Otherwise (G, Ω) is called primitive. These two relations are supposed to be trivial relations. Also $\approx$ relation of equivalence classes are called orbits of action.

Remark 3.1 *The stabilizer of* $x \in \Omega$ *is the* $G_x = \{g \in G : g(x) = x\}$.

Since the following lemma is well known [3], we only give the statement;

Lemma 3.2 *Let* (G, Ω) *be a transitive permutation group.*(G, Ω) *is primitive if and only if* G_σ *is a maximal subgroup of* G *for each* $\sigma \in \Omega$.

Consequently we understand that if $G_\sigma < H < G$ then Ω is imprimitive. Because of the transitivity, every element of Ω has the form $g(\sigma)$ for some $g \in G$. Therefore one of the non trivial G invariant equivalence relation on Ω is given as follows:

$$g_1(\sigma) \approx g_2(\sigma) \text{ if and only if } g_1^{-1}g_2 \in H.$$

The number of the blocks is the index $|G : H|$.

We can apply these ideas to the case where G is the $PSL(2,5)$ and Ω is Δ. We have the following lemmas:

Lemma 3.3 *$PSL(2,5)$ acts transitively on Δ.*

Proof: We will prove that the orbit containing ∞ is Δ. If $\frac{a}{b} \in \Delta$ then as $(a,b) = 1$ there exist $u, v \in \mathbb{Z}_5$ with $ux - vy = 1$. Then the element $\begin{pmatrix} a & u \\ b & v \end{pmatrix}$ of $PSL(2,5)$ sends ∞ to $\frac{a}{b}$.

Lemma 3.4 *The stabilizer of ∞ in Δ is the set of* $\left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} : \lambda \in \mathbb{Z}_5 \right\}$ *denoted by $PSL(2,5)_\infty$.*

Proof: Since the action is transitive, stabilizer of any two points conjugate. Therefore we can only look at the stabilizer of ∞ in $PSL(2,5)$.

Let $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,5)$. Thus, $A(\infty) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then $a = 1$, $c = 0$, $d = 1$ and $b = \lambda$. Therefore $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ is obtained. That is,
$PSL(2,5)_\infty = \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} : \lambda \in \mathbb{Z}_5 \right\}$. It is clear that $PSL(2,5)_\infty < PSL(2,5)_0 < PSL(2,5)$.

Let $\approx$ denote the $PSL(2,5)$ invariant equivalence relation on Δ by $PSL(2,5)_0$, let $v = \frac{r}{s}$ and $w = \frac{x}{y}$ be elements of Δ. Then there are the elements $g_1 := \begin{pmatrix} r & * \\ s & * \end{pmatrix}$ and $g_2 := \begin{pmatrix} x & * \\ y & * \end{pmatrix}$ in $PSL(2,5)$ such that $v = g_1(\infty)$ and $w = g_2(\infty)$. So we have

$$v \approx w = g_1(\infty) \approx g_2(\infty) \Leftrightarrow g_1^{-1} g_2 \in PSL(2,5)_0$$

and so from the above we can easily calculate that $g_1^{-1} g_2 = \begin{pmatrix} * & * \\ ry - sx & * \end{pmatrix} \in PSL(2,5)_0$. Hence $ry - sx \equiv 0 \, mod(5)$ is obtained.

It is known that the number of the blocks is the index $|\Gamma : \Gamma_0(n)| = n\prod_{p|n}(1+\frac{1}{p})$.
In particular, if n is a prime p, then there are $p+1$ blocks, these being $[0],[1],[2],...,[p-1],[\infty]$. Because of the number of blocks is $|PSL(2,5) : PSL(2,5)_0| = 6$. These blocks are

$[\infty] := [\frac{1}{0}] = \{\frac{x}{y} \in \Delta : (x,y) = 1$ and $y \equiv 0\, mod(5)\}$,

$[j] := [\frac{j}{1}] = \{\frac{x}{y} \in \Delta : (x,y) = 1$ and $x - jy \equiv 0\, mod(5)\}$ where $j \neq \infty$. Consequently, blocks for Δ residue class:

$$[\overline{\infty}] = [\frac{1}{0}] = \{\infty, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\}, \quad [\overline{0}] = [\frac{0}{1}] = \{0, \frac{5}{1}, \frac{5}{2}, \frac{5}{3}, \frac{5}{4}\}, \quad [\overline{1}] = [\frac{1}{1}] = \{1\},$$

$$[\overline{2}] = [\frac{2}{1}] = \{2, \frac{1}{3}, \frac{3}{4}\}, \quad [\overline{3}] = [\frac{3}{1}] = \{3, \frac{1}{2}, \frac{4}{3}\}, \quad [\overline{4}] = [\frac{4}{1}] = \{4, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}\}.$$

3.2 Orbital Graphs

Definition 3.5 [9] *Let $\Omega \neq \emptyset$ is a set and $\Sigma \subset \Omega \times \Omega$ is a relation. Then $\mathbb{G} = (\Omega, \Sigma)$ pair is called a graph. Elements of Ω are vertices of graph and elements of Σ are edges of the graph. If $(\alpha, \beta) \in \Sigma$, this is indicated as $\alpha \to \beta$. If $(\alpha, \beta) \in \Sigma$ or $(\beta, \alpha) \in \Sigma$ then α and β are connected to a edge.*

Definition 3.6 [9] *Let $\mathbb{G}$ be a graph and $A \subset \Omega$. $(A, \Sigma \cap A \times A)$ subgraph is named of $\mathbb{G}$, vertex set of which is A.*

Definition 3.7 [9] *Let a sequence $v_1, v_2, \ldots, v_k$ of different vertices. Then the form $v_1 \longrightarrow v_2 \longrightarrow \ldots \longrightarrow v_k \longrightarrow v_1$, where $k \in \mathbb{N}$ and $k \geq 3$, is called a directed circuit in $\mathbb{G}$.*

If $k = 2$, then we will say the configuration $v_1 \longrightarrow v_2 \longrightarrow v_1$ a self paired edge.

If $k = 3$ or $k = 4$, then the circuit, directed or not, is called a triangle or quadrilateral. In a graph is a finite or infinite sequence of edges which connect a sequence of vertices which are all distinct from one another are called a path.

Let (G, Ω) be transitive permutation group. Then G acts on $\Omega \times \Omega$ by

$$\Theta : \underset{(g,(\alpha,\beta))}{G \times (\Omega \times \Omega)} \underset{\to}{\longrightarrow} \underset{(g(\alpha),g(\beta))}{\Omega \times \Omega}$$

where $g \in G$ and $\alpha, \beta \in \Omega$. The orbits of this action are called *suborbitals* of G. The orbit containing (α, β) is denoted by $O(\alpha, \beta)$. From $O(\alpha, \beta)$ we can form a *suborbital graph* $\mathbb{G}$. Its vertices are the elements of Ω, and if $(\gamma, \delta) \in O(\alpha, \beta)$

there is a directed edge from γ to δ. Moreover $O(\alpha,\alpha)$ is diagonal of $\Omega\times\Omega$. The corresponding suborbital graph called the trivial suborbital graph, it consists of a *loop* based at each vertex.

Indeed this theory reveal the relationship between graphs and permutation groups (see [1, 2, 5, 6, 10]). In this paper our investigation concerns $PSL(2,5)$, so we can draw this edge as a hyperbolic geodesic in the upper half-plane $\mathbb{H}$.

Since $PSL(2,5)$ acts transitively on Δ, it permutes the blocks transitively (see [8]). Let $F(\frac{1}{0},\frac{u}{5})$ denote the subgraphs in $\mathbb{G}$ whose vertices form the block $[\overline{\infty}]$. Similarly we may write subgraphs $F(\frac{0}{1},\frac{5}{u})$, $F(\frac{1}{1},\frac{u}{v})$, $F(\frac{2}{1},\frac{u}{v})$, $F(\frac{3}{1},\frac{u}{v})$ and $F(\frac{4}{1},\frac{u}{v})$ are for other blocks.

Theorem 3.8 *There is an edge* $\frac{r}{s}\longrightarrow\frac{x}{y}$ *in* $F(\frac{1}{0},\frac{u}{5})$ *if and only if either*

(i) $x\equiv ur\,mod(5)$, $y\equiv us\,mod(5)$ *and* $ry-sx=5$, *or*
(ii) $x\equiv -ur\,mod(5)$, $y\equiv -us\,mod(5)$ *and* $ry-sx=-5$.

Proof: Assume that $\frac{r}{s}\longrightarrow\frac{x}{y}$ be an edge in $F(\frac{1}{0},\frac{u}{5})$. Then there is some element $T=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in PSL(2,5)$ such that $T(\frac{1}{0})=\frac{a}{c}=\frac{(-1)^m r}{(-1)^m s}$ gives that $r=(-1)^m a$, $s=(-1)^m c$ for $m=0,1$. Again $T(\frac{u}{5})=\frac{au+5b}{cu+5d}=\frac{(-1)^n x}{(-1)^n y}$ for $n=0,1$. Hence $x\equiv(-1)^{m+n}ur\,mod(5)$ and $y\equiv(-1)^{m+n}us\,mod(5)$. So we have the matrix equation $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1 & u\\ 0 & 5\end{pmatrix}=\begin{pmatrix}(-1)^m r & (-1)^n x\\ (-1)^m s & (-1)^n y\end{pmatrix}$ for $m,n=0,1$. If we get determinant it is easily seen that $ry-sx=\pm 5$.

Conversely we do calculations only for (i). Therefore $x\equiv ur\,mod(5)$, $y\equiv us\,mod(5)$ and $ry-sx=5$. Then there exist integers b and d such that $x=ur+5b$, $y=us+5d$. So $\begin{pmatrix}a & ur+5b\\ c & us+5d\end{pmatrix}=\begin{pmatrix}r & x\\ s & y\end{pmatrix}$. As $ad-bc=1$ from determinants we have $ry-sx=5$. Consequently we obtain $\begin{pmatrix}a & ur+5b\\ c & us+5d\end{pmatrix}\in PSL(2,5)$ and $\frac{r}{s}\longrightarrow\frac{x}{y}$ in $F(\frac{1}{0},\frac{u}{5})$.

Theorem 3.9 *The subgraph* $F(\frac{1}{0},\frac{u}{5})$ *contains a hyperbolic triangle if and only if* $u^2\pm u+1\equiv 0\,mod(5)$.

Proof: As $PSL(2,5)$ permutes the vertices transitively of $F(\frac{1}{0},\frac{u}{5})$, then we may suppose that hyperbolic triangle has the form $\frac{1}{0}\longrightarrow\frac{k_0}{5}\longrightarrow\frac{x_0}{5y_0}\longrightarrow\frac{1}{0}$.

In addition to assume that $\dfrac{k_0}{5} < \dfrac{x_0}{5y_0}$. Using Theorem 3.7, from the first edge, we get $k_0 \equiv u\, mod(5)$. The second edge gives $x_0 \equiv -uk_0\, mod(5)$ and $ky_0 - x_0 = -1$. From the last edge we have $1 \equiv -ux_0\, mod(5)$ and $y_0 = 1$. Hence $1 \equiv -u(k_0+1)\, mod(5)$ is obtained. This gives that $u^2+u+1 \equiv 0\, mod(5)$.

If $\dfrac{k_0}{5} > \dfrac{x_0}{5y_0}$ holds then we conclude that this equation $u^2 - u + 1 \equiv 0\, mod(5)$ is achieved.

On the other hand suppose that $u^2 \pm u + 1 \equiv 0\, mod(5)$. Obviously we obtain the hyperbolic triangle $\dfrac{1}{0} \longrightarrow \dfrac{u}{5} \longrightarrow \dfrac{u \pm 1}{5} \longrightarrow \dfrac{1}{0}$ from Theorem 3.7.

Now we will give several examples for different blocks.

Example 3.10 *For some* $u \in \mathbb{Z}_5$ *hyperbolic triangles in subgraph* $F(\frac{1}{0}, \frac{u}{5})$ *figures are as follows:*

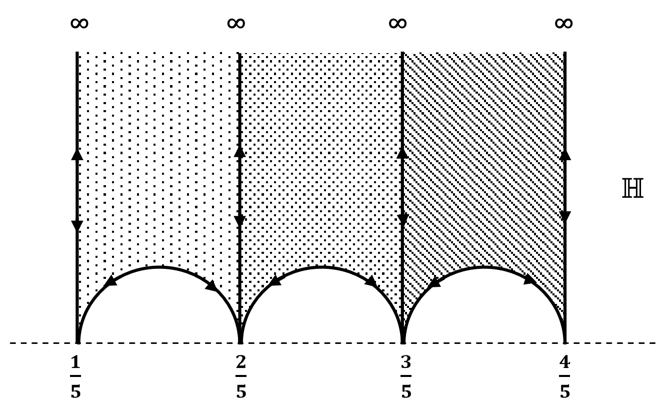

Figure 1: $[\overline{\infty}]$ block

Remark 3.11 *Since* $u \in \mathbb{Z}_5$ *and* $(u,p) = 1$ *there is finite number of hyperbolic triangles in suborbital graph.*

Corollary 3.12 *Actually* $F(\frac{1}{0}, \frac{u}{5})$ *contains hyperbolic triangle if and only if the group* $PSL(2,5)_0$ *contains elliptic element* $\psi = \begin{pmatrix} -u & \frac{u^2+u+1}{5} \\ -5 & u+1 \end{pmatrix}$ *of order 3 in* $PSL(2,5)_0$. *It is clear that* $\psi(\frac{1}{0}) = \frac{u}{5}$, $\psi(\frac{u}{5}) = \frac{u+1}{5}$ *and* $\psi(\frac{u+1}{5}) = \frac{1}{0}$. *That is, by the mapping the* ψ *transform vertices to each other.*

Example 3.13 *Similarly hyperbolic triangles in subgraph* $F(\frac{0}{1}, \frac{5}{u})$ *figures are as follows:*

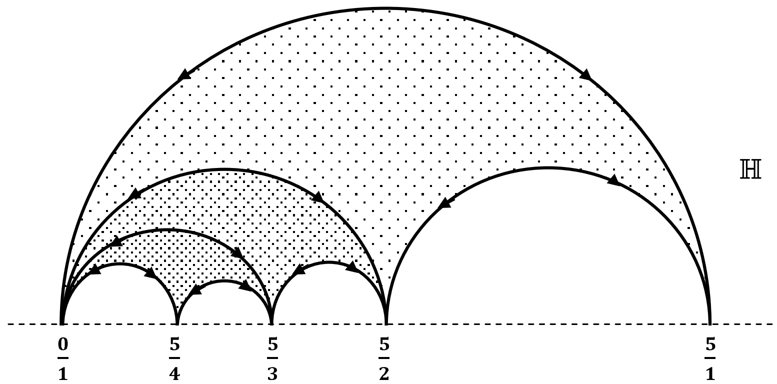

Figure 2: $[\bar{0}]$ block

Example 3.14 *In this example we give a loop in* $F(\frac{1}{1}, \frac{u}{v})$*, hyperbolic triangles in* $F(\frac{2}{1}, \frac{u}{v})$ *and* $F(\frac{3}{1}, \frac{u}{v})$*, and also 3-lengths path in* $F(\frac{4}{1}, \frac{u}{v})$*.*

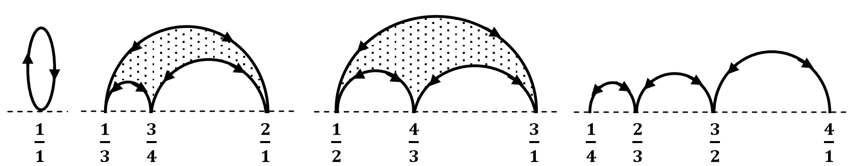

Figure 3: $[\bar{1}]$, $[\bar{2}]$, $[\bar{3}]$, $[\bar{4}]$ blocks

Remark 3.15 *We know that there are self paired edges in suborbital graphs. If using the group* $PSL(2,2)$*, then self paired edges reveal in subgraps. For* $PSL(2,2)$ *there are 3 blocks. Below, we will give a lemma for infinite block.*

Lemma 3.16 *The subgraph* $F(\frac{1}{0}, \frac{u}{2})$ *is self paired if and only if* $u^2 + 1 \equiv 0\,mod(2)$*.*

Proof: Assume that $O(\frac{1}{0}, \frac{u}{2}) = O(\frac{u}{2}, \frac{1}{0})$. Then there exists $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $T(\infty) = \frac{u}{2}$ and $T(\frac{u}{2}) = \infty$. Hence T must be $\begin{pmatrix} u & -\frac{u^2+1}{2} \\ 2 & -u \end{pmatrix}$ and $detT = 1$. So we have $u^2 + 1 \equiv 0\,mod(2)$. Conversely case is obvious.

Example 3.17 *We can easily see that there are self paired edges in* $F(\frac{1}{0}, \frac{u}{2})$ *and* $F(\frac{0}{1}, \frac{2}{u})$*, and also there exists a loop in* $F(\frac{1}{1}, \frac{u}{v})$*. Figures are as follows:*

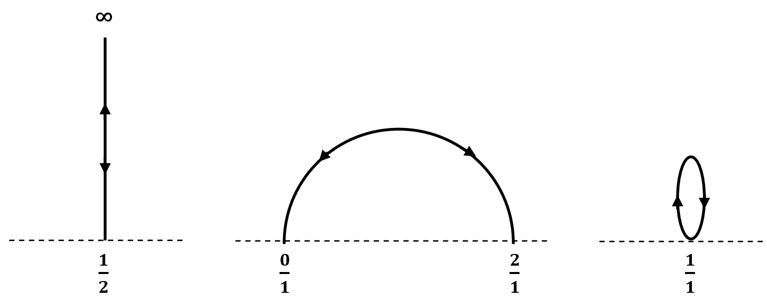

Figure 4: $[\overline{\infty}]$, $[\overline{2}]$, $[\overline{1}]$ blocks

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