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Modification of $\mathcal{H}(\theta)$-Open Sets on Hereditary Generalized Topological Spaces

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Abstract

We introduce the $\mathcal{H}(\theta)$-open sets modification given by a hereditary class and a generalized topology μ and investigate some properties and characterizations. With the help of $\mathcal{H}(\theta)$-open sets, we define new classes of sets $\mathcal{H}(\theta)-preopen$, $\mathcal{H}(\theta)-semiopen$, $\mathcal{H}(\theta)-\alpha-open$, $\mathcal{H}(\theta)-\beta-open$ in hereditary generalized topological spaces and study the relation between such sets and $\theta-open$ sets on generalized topologies.

Keywords: *$\mathcal{H}(\theta)$-open sets,$\mathcal{H}(\theta)-pre-open$, $\mathcal{H}(\theta)-semiopen$, $\mathcal{H}(\theta)-\alpha-open$, $\mathcal{H}(\theta)-\beta-open$, ideal,hereditary class, generalized topology.*

1 Introduction

In 1990, Jankovic and Hamlett [5] obtained a new topology τ^* by using a given ideal $\mathcal{H}$ and topology τ. An *ideal* $\mathcal{H}$ on X is a nonempty family $\mathcal{H} \subseteq expX$ satisfying the following conditions (i) $A \subseteq B$, $B \in \mathcal{H}$ implies $A \in \mathcal{H}$; (ii) $A, B \in \mathcal{H}$ implies $A \cup B \in \mathcal{H}$. In 2002, A.Csaszar [1], introduced the notions of generalized topology. A.Csaszar [3] define a class of subsets of nonempty set called hereditary class and studied modification of generalized topology via hereditary classes. Instead of τ, consider a generalized topology and ideal is replaced by hereditary class, i.e. a class $\emptyset \neq \mathcal{H} \subseteq expX$ satisfying (i) he constructed the generalized topology μ^*. The aim of the paper is to extend further the study of the properties the modifications of generalized topologies via hereditary classes. In 2008, A.Csaszar [4] introduced the class of $\theta(\mu)$-open sets. Quite recently Kim Y.K. and Min W.K. [7] have investigated their properties of $\mathcal{H}(\theta)$-open sets in generalized topologies via hereditary class.

In this paper, with the help of $\mathcal{H}(\theta)$-open sets, we define new classes of sets $\mathcal{H}(\theta) - preopen$, $\mathcal{H}(\theta) - semiopen$, $\mathcal{H}(\theta) - \alpha - open$, $\mathcal{H}(\theta) - \beta - open$ in generalized topologies via hereditary class. Also, we investigate some properties and relationships of these sets.

2 Preliminaries

Let X be a nonempty set and $expX$ be the power of X. The collection μ of subset of X satisfying the following conditions is called generalized topology [2], $\emptyset \in \mu$ and $M_i \in \mu$ for $i \in I \neq \emptyset$ implies $M = \bigcup_{i \in I} M_i \in \mu$. The elements of μ are called μ-open and their compliments are called μ-closed. The pair (X, μ) is called a generalized topological spaces (GTS). For any set $A \subset X$ we define interior of A as the union of all μ-open sets contained A, i.e., $i_\mu(A) = \bigcup\{A \subset X : U \subset A \, and \, U \in \mu\}$ similarly, we can define closure of A as intersection of all μ-closed sets containing A, i.e., $c_\mu(A) = \bigcap\{A \subset X : U \subset A \, and \, U^c \in \mu\}$. Denote M_μ the union of all μ-open sets in a GTS (X, μ). Let X be a nonempty set. A hereditary class $\mathcal{H}$ of X, if $A \in \mathcal{H}$ and $B \subset A$ then $B \in \mathcal{H}$. A generalized topological spaces (X, μ) with a hereditary class $\mathcal{H}$ is call the triple $(X, \mu, \mathcal{H})$ a hereditary Generalized Topological Spaces [3], denoted by HGTS. A.Csaszar [3] defined local function in hereditary generalized topological spaces and constructed a new generalized topology μ^*. For each $A \subset X$, $A^*(\mathcal{H}, \mu) = \{x \in X : A \cap M \notin \mathcal{H}$ for every $M_i \in \mu$ such that $x \in G\}$. If there is no ambiguity then we write A^* in place of $A^*(\mathcal{H}, \mu)$. According to the definition, $x \notin A^*$ if and only if there exists $x \in G \in \mu$ such that $(A \cap M) \in \mathcal{H}$. For each $A \subset X$, define $c^*_\mu(A) = A \cup A^*(\mathcal{H}, \mu)$. $\mathcal{H}$ said to be $\mu - codence$[3] if $\mu \cap \mathcal{H} = \emptyset$ and is said to be strongly $\mu - codence$ if M and $N \in \mu$ and $M \cap N \in \mathcal{H}$, then $M \cap N = \emptyset$. A subset of A of (X, μ) is $\mu - \alpha - open$[2](resp.$\mu - semiopen$, $\mu - preopen$, $\mu - \beta - open$) if $A \subset i_\mu c_\mu i_\mu(A)(resp. A \subset c_\mu i_\mu(A), A \subset i_\mu c_\mu(A), A \subset c_\mu i_\mu c_\mu(A))$.

Theorem 2.1 *[3] Let μ be a GT in X and $\mathcal{H}$ a hereditary class on X.*

(1) $A \subseteq B \subseteq X$ *implies* $A^* \subseteq B^*$.
(2) $A \subset A^*$ *then* $c_\mu(A) = A^* = c^*_\mu(A) = c^*_\mu(A^*)$.
(3) $A^* \subseteq c^*(A) \subseteq cA$.

(4) *If* $H \in \mathcal{H}$*, then* $H^* = X - M_\mu$.
(5) A^* *is* $\mu-$*closed.*
(6) *If* F *is* $\mu-$*closed, then* $F^* \subseteq F$.
(7) F *is* μ^*-*closed iff* $F^* \subseteq F$.
(8) $X = X^*$ *iff* $\mu \cap \mathcal{H} = \emptyset$.
(9) $M \in \mu$ *implies* $M \subseteq M^*$ *iff* $M, M' \in \mu, M \cap M' \in \mathcal{H}$ *implies* $M \cap M' = \emptyset$.

Theorem 2.2 *[7] Let* μ *be a GT in* X *and* $\mathcal{H}$ *a hereditary class on* X*. For* $A \subseteq X$*,*

(1) $i^*(A) = X - c^*(X - A)$.
(2) $c^*(A) = X - i^*(X - A)$.
(3) $i(A) \subseteq i^*(A) \subseteq A$.

Definition 2.3 *[4] Let* (X, μ) *generalized topological spaces and* $\theta = \theta(\mu) \subset expX$ *be defined by* $A \in \theta$ *iff* $A \subset X$ *and* $x \in A$ *implies the existence of* $M \in \mu$ *such that* $x \in M \subset c_\mu M \subset A$*.* θ *is a GT on* X *and* $\theta \subset \mu$*. Let define a set* $c_\theta(A)$ *composed of all points* $x \in X$ *such that* $x \in M \in \mu$ *implies* $c_\mu M \cap A \neq \emptyset$.

Definition 2.4 *[8] Let* $(X, \mu, \mathcal{H})$ *be a HGTS in* X*. Let define the collection* $\mathcal{H}(\theta) \subseteq P(X)$ *by* $A \in \mathcal{H}(\theta)$ *iff for each* $x \in A$*, there is* $M \in \mu$ *such that* $x \in M \subseteq c^*M \subseteq A$*. The elements* $\mathcal{H}(\theta)$ *are called* $\mathcal{H}(\theta)-$*open sets and the complements are called* $\mathcal{H}(\theta)-$*closed sets. Every* $\mathcal{H}(\theta)-$*open set is* $\mu-$*open and* $\mathcal{H}(\theta)$ *also is a GT included in* μ.

Definition 2.5 *[8] Let* $(X, \mu, \mathcal{H})$ *be a HGTS in* X*. The* $\mathcal{H}(\theta)-$*closure of a subset* A *of* X*, denoted by* $c_{\mathcal{H}(\theta)}A$*, is the intersection of* $\mathcal{H}(\theta)-$*closed sets including* A*.* $\mathcal{H}(\theta)-$*interior of* A*, denoted by* $i_{\mathcal{H}(\theta)}A$*, the union of* $\mathcal{H}(\theta)-$*open sets included in* A*.* $c_{\mathcal{H}(\theta)}A = \{x \in X : c^*M \cap A \neq \varnothing$ *for every* $\mu-$*open set* M *containing* $x\}$.

Theorem 2.6 *[8] Let* $(X, \mu, \mathcal{H})$ *be a HGTS in* X*. For* $A \subseteq X$*,*

(1) $c_{\mathcal{H}(\theta)}\emptyset = \emptyset$.
(2) $c_{\mathcal{H}(\theta)}$ *is monotonic.*
(3) $A \subseteq cA \subseteq c_{\mathcal{H}(\theta)}A$.
(4) $c_{\mathcal{H}(\theta)}A$ *is* $\mu-$*closed.*
(5) A *is* $\mathcal{H}(\theta)-$*closed iff* $c_{\mathcal{H}(\theta)}A = A$.

3 $\mathcal{H}(\theta)$-Open Sets Modifications

In this section, we introduce the concepts of generalized topological spaces via hereditary class.

Definition 3.1 *A subset A of an HGTS* $(X, \mu, \mathcal{H})$ *is said to be*

1. $\mathcal{H}(\theta) -$ *preopen if* $A \subseteq i(c_{\mathcal{H}(\theta)}(A))$;
2. $\mathcal{H}(\theta) -$ *semiopen if* $A \subseteq c(i_{\mathcal{H}(\theta)}(A))$;
3. $\mathcal{H}(\theta) - \alpha -$ *open if* $A \subseteq i(c(i_{\mathcal{H}(\theta)}(A)))$;
4. $\mathcal{H}(\theta) - \beta -$ *open if* $A \subseteq c(i(c_{\mathcal{H}(\theta)}(A)))$.

Proposition 3.2 (*i*)*Every* $\mathcal{H}(\theta) -$ *open set is* $\mathcal{H}(\theta) - \alpha -$ *open set.*
(*ii*)*Every* $\mathcal{H}(\theta) - \alpha -$ *open set is* $\mathcal{H}(\theta) -$ *semiopen set.*
(*iii*)*Every* $\mathcal{H}(\theta) -$ *semiopen set is* $\mathcal{H}(\theta) - \beta -$ *open set.*
(*iv*)*Every* $\mathcal{H}(\theta) - \alpha -$ *open set is* $\mathcal{H}(\theta) -$ *preopen set.*
(*v*)*Every* $\mathcal{H}(\theta) -$ *preopen set is* $\mathcal{H}(\theta) - \beta -$ *open set.*

Proof: The proof follows from the definitions.

Remark 3.3 *By this definition, we have the following diagram in which none of the implications is reversible as shown by examples stated below.*

$$\begin{array}{ccccc} \mathcal{H}(\theta) - open & \longrightarrow & \mathcal{H}(\theta) - \alpha - open & \longrightarrow & \mathcal{H}(\theta) - semiopen \\ & & \downarrow & & \downarrow \\ & & \mathcal{H}(\theta) - preopen & \longrightarrow & \mathcal{H}(\theta) - \beta - open \end{array}$$

Example 3.4 *Let* $X = \{a, b, c\}$, $\mu = \{\emptyset, \{a\}, \{a, b\}\}$ *and* $\mathcal{H} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. *Then* $A = \{a, c\}$ *is a* $\mathcal{H}(\theta) - \alpha -$ *open set which is not* $\mathcal{H}(\theta) -$ *open. Since* $i_{\mathcal{H}(\theta)}(A) = \{a, b\}$, $A \subseteq i(c(i_{\mathcal{H}(\theta)}(A))) = X$ *and* $i_{\mathcal{H}(\theta)}(A) = \{a, b\} \neq A$.

Example 3.5 *Let* $X = \{a, b, c, d\}$, $\mu = \{\emptyset, \{d\}, \{b, c, d\}\}$ *and* $\mathcal{H} = \{\emptyset, \{c\}\}$. *Then* $A = \{a, c\}$ *is a* $\mathcal{H}(\theta) -$ *semiopen set which is not* $\mathcal{H}(\theta) - \alpha -$ *open since* $i_{\mathcal{H}(\theta)}(A) = \{b, c, d\}$, $A \subseteq c(i_{\mathcal{H}(\theta)}(A)) = X$ *and* $A \nsubseteq i(c(i_{\mathcal{H}(\theta)}(A))) = \{b, c, d\}$.

Example 3.6 *Let* $X = \{a, b, c, d\}$, $\mu = \{\emptyset, \{a, b\}, \{a, b, c\}\}$ *and* $\mathcal{H} = \{\emptyset, \{c\}\{d\}, \{c, d\}\}$. *Then* $A = \{a, c\}$ *is a* $\mathcal{H}(\theta) -$ *preopen set which is not* $\mathcal{H}(\theta) - \alpha -$ *open since* $c_{\mathcal{H}(\theta)}(A) = \{a, b, c\}$, $A \subseteq i(c_{\mathcal{H}(\theta)}(A)) = \{a, b, c\}$ *and* $A \nsubseteq i(c(i_{\mathcal{H}(\theta)}(A))) = \{b, c, d\}$.

Example 3.7 *Let* $X = \{a, b, c, d\}$, $\mu = \{\emptyset, \{a\}, \{a, c\}, \{a, c, d\}\}$ *and* $\mathcal{H} = \{\emptyset, \{b\}\}$. *Then* $A = \{a, b\}$ *is a* $\mathcal{H}(\theta) - \beta -$*open set which is not* $\mathcal{H}(\theta) -$*preopen. Since* $c_{\mathcal{H}(\theta)}(A) = \{a, c, d\}$, $A \subseteq i(c_{\mathcal{H}(\theta)}(A)) = X$ *and* $A \nsubseteq i(c_{\mathcal{H}(\theta)}(A)) = \{a, c, d\}$.

Example 3.8 *Let* $X = \{a, b, c, d\}$, $\mu = \{\emptyset, \{a, b\}, \{a, b, c\}\}$ *and* $\mathcal{H} = \{\emptyset, \{c\}, \{d\}\}$. *Then* $A = \{a, c\}$ *is a* $\mathcal{H}(\theta) - \beta -$ *open set which is not* $\mathcal{H}(\theta) -$ *semiopen. Since* $c_{\mathcal{H}(\theta)}(A) = \{a, b, c\}$, $A \subseteq i(c_{\mathcal{H}(\theta)}(A)) = X$ *and* $A \nsubseteq c(i_{\mathcal{H}(\theta)}(A)) = \{a, c, d\}$.

Theorem 3.9 *Let $(X, \mu, \mathcal{H})$ be a HGTS in X and $A \subset X$. A is $\mathcal{H}(\theta) - semiopen$ iff $c(A) = c(i_{\mathcal{H}(\theta)}(A))$.*

Proof: Let A is $\mathcal{H}(\theta) - semiopen$. By this definition, we obtain $A \subseteq c(i_{\mathcal{H}(\theta)}(A))$. Therefore $c(A) \subseteq c(i_{\mathcal{H}(\theta)}(A))$ and $i_{\mathcal{H}(\theta)}(A) \subseteq A$ always hold. Hence, $c(A) = c(i_{\mathcal{H}(\theta)}(A))$. Conversely, since $A \subseteq c(A) = c(i_{\mathcal{H}(\theta)}(A))$, A is $\mathcal{H}(\theta) - semiopen$.

Theorem 3.10 *Let $(X, \mu, \mathcal{H})$ be a HGTS in X and $A \subset X$. A is $\mathcal{H}(\theta) - semiopen$ iff for some $U \in \mathcal{H}(\theta)$, $U \subseteq A \subseteq c(U)$.*

Proof: Let A is $\mathcal{H}(\theta) - semiopen$ then $A \subseteq c(i_{\mathcal{H}(\theta)}(A))$. If we take $U = i_{\mathcal{H}(\theta)}(A)$, we can write $c(U) = c(i_{\mathcal{H}(\theta)}(A))$ and $U \subseteq A$. Hence, $U \subseteq A \subseteq c(U)$. Conversely, if we have for some $U \in \mathcal{H}(\theta)$, $U \subseteq A \subseteq c(U)$ then $A \subseteq c(U) = c(i_{\mathcal{H}(\theta)}(U)) \subseteq c(i_{\mathcal{H}(\theta)}(A))$ and hence A is $\mathcal{H}(\theta) - semiopen$.

Theorem 3.11 *Let $(X, \mu, \mathcal{H})$ be a HGTS. Then, the family of $\mathcal{H}(\theta) - \alpha - open$ sets is a GT for X.*

Proof: $\emptyset$ is $\mathcal{H}(\theta) - \alpha - open$ set. Let A_i be a $\mathcal{H}(\theta) - \alpha - open$ set for each $i \in I$. Then, by definition of $\mathcal{H}(\theta) - \alpha - open$, we get

$$A_i \subseteq i(c(i_{\mathcal{H}(\theta)}(A_i))) \subseteq i(c(i_{\mathcal{H}(\theta)}(\bigcup_{i \in I} A_i))).$$

So we have $\bigcup_{i \in I} A_i \subseteq i(c(i_{\mathcal{H}(\theta)}(\bigcup_{i \in I} A_i)))$. This implies that $\bigcup_{i \in I} A_i$ is a $\mathcal{H}(\theta) - \alpha - open$ set.

This GT we denote $\mu^{\mathcal{H}(\theta)\alpha}$. Finite intersection of $\mathcal{H}(\theta) - \alpha - open$ sets is not $\mathcal{H}(\theta) - \alpha - open$ sets shown in the following example.

Example 3.12 *Let $X = \{a, b, c\}$, $\mu = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$ and $\mathcal{H} = \{\emptyset, \{c\}\}$. Let $A = \{a, b\}$ and $B = \{b, c\}$ is a $\mathcal{H}(\theta) - \alpha - open$ sets. Hence, $A \cap B = \{b\}$ which is not $\mathcal{H}(\theta) - \alpha - open$.*

Theorem 3.13 *Let M and N be subsets of an HGTS $(X, \mu, \mathcal{H})$, the following properties hold:*

1. *$M \in \mu^{\mathcal{H}(\theta)\alpha}$ iff $A \subseteq M \subseteq i(c(A))$ for every $\mathcal{H}(\theta) - open$ set A,*
2. *If $M \in \mu^{\mathcal{H}(\theta)\alpha}$ and $M \subseteq N \subseteq i(c(M))$, then $N \in \mu^{\mathcal{H}(\theta)\alpha}$.*

Proof: (1) It is obvious.

(2) Let $M \in \mu^{\mathcal{H}(\theta)\alpha}$ set.Since,

$$N \subseteq i(c(M)) \subseteq i(c(i(c(i_{\mathcal{H}(\theta)}(M))))) \subseteq i(c(i_{\mathcal{H}(\theta)}(M))) \subseteq i(c(i_{\mathcal{H}(\theta)}(N))),$$

we have $N \in \mu^{\mathcal{H}(\theta)\alpha}$.

Theorem 3.14 *Let $(X, \mu, \mathcal{H})$ be a HGTS in X and $A \subseteq X$. A is $\mathcal{H}(\theta)-$ semiopen iff $c(A) = c(i_{\mathcal{H}(\theta)}(A))$.*

Proof: A is be a $\mathcal{H}(\theta) - semiopen$ from here $A \subseteq c(i_{\mathcal{H}(\theta)}(A))$ and $c(A) \subseteq c(i_{\mathcal{H}(\theta)}(A))$. $c(i_{\mathcal{H}(\theta)}(A)) \subseteq c(A)$ is always true. Hence, $c(A) = c(i_{\mathcal{H}(\theta)}(A))$.

Conversely, $A \subseteq c(A) = c(i_{\mathcal{H}(\theta)}(A))$ hence A is $\mathcal{H}(\theta) - semiopen$.

Theorem 3.15 *Let $(X, \mu, \mathcal{H})$ be an HGTS, then the following hold:*

1. *If A is $\mathcal{H}(\theta) - preopen$ and B is $\mathcal{H}(\theta) - \alpha - open$, then $A \cap B$ is a $\mathcal{H}(\theta) - preopen$.*
2. *If A is $\mathcal{H}(\theta) - semiopen$ and B is $\mathcal{H}(\theta) - \alpha - open$, then $A \cap B$ is a $\mathcal{H}(\theta) - semiopen$.*
3. *If A is $\mathcal{H}(\theta) - \beta - open$ and B is $\mathcal{H}(\theta) - \alpha - open$, then $A \cap B$ is a $\mathcal{H}(\theta) - \beta - open$.*
4. *If A is $\mathcal{H}(\theta) - semiopen$ and B is $\mathcal{H}(\theta) - pre - open$, then $A \cap B$ is a $\mathcal{H}(\theta) - \beta - open$.*

Proof: (1) Let A be $\mathcal{H}(\theta) - preopen$ and B be $\mathcal{H}(\theta) - \alpha - open$, then $A \subseteq i(c_{\mathcal{H}(\theta)}(A))$ and $B \subseteq i(c(i_{\mathcal{H}(\theta)}(B)))$. Then,

$$\begin{aligned} A \cap B = i(c_{\mathcal{H}(\theta)}(A)) \cap i(c(i_{\mathcal{H}(\theta)}(B))) &= i(ic_{\mathcal{H}(\theta)}(A) \cap c(i_{\mathcal{H}(\theta)}(B) \\ &\subseteq i(c(c_{\mathcal{H}(\theta)}(A) \cap i_{\mathcal{H}(\theta)}(B))) \\ &\subseteq i(c_{\mathcal{H}(\theta)}(c_{\mathcal{H}(\theta)}(A \cap B))) \\ &= i(c_{\mathcal{H}(\theta)}(A \cap B)). \end{aligned}$$

Thus, $(A \cap B)$ is a $\mathcal{H}(\theta) - preopen$.
(2) Let A be $\mathcal{H}(\theta) - semiopen$ and B be $\mathcal{H}(\theta) - \alpha - open$, then $A \subseteq c(i_{\mathcal{H}(\theta)}(A))$ and $B \subseteq i(c(i_{\mathcal{H}(\theta)}(B)))$.Then,

$$\begin{aligned} A \cap B = c(i_{\mathcal{H}(\theta)}(A)) \cap i(c(i_{\mathcal{H}(\theta)}(B))) &\subseteq c(i_{\mathcal{H}(\theta)}(A)) \cap c(c(i_{\mathcal{H}(\theta)}(B))) \\ &\subseteq c(i_{\mathcal{H}(\theta)}(A \cap B)). \end{aligned}$$

Thus $(A \cap B)$ is a $\mathcal{H}(\theta) - semiopen$.
(3) Let A be $\mathcal{H}(\theta) - \beta - open$ and B be $\mathcal{H}(\theta) - \alpha - open$, then $A \subseteq c(i(c_{\mathcal{H}(\theta)}(A)))$ and $A \subseteq i(c(i_{\mathcal{H}(\theta)}(A)))$. This implies that

$$\begin{aligned} A \cap B \subseteq c(i(c_{\mathcal{H}(\theta)}(A))) \cap i(c(i_{\mathcal{H}(\theta)}(B))) &\subseteq c(i(ic_{\mathcal{H}(\theta)}(A) \cap ci_{\mathcal{H}(\theta)}(B))) \\ &\subseteq c(i(c(c_{\mathcal{H}(\theta)}(A) \cap i_{\mathcal{H}(\theta)}(B)))) \\ &\subseteq c(i(c(c_{\mathcal{H}(\theta)}(A \cap B)))) \\ &\subseteq c(i(c_{\mathcal{H}(\theta)}(c_{\mathcal{H}(\theta)}(A \cap B)))) \\ &\subseteq c(i(c_{\mathcal{H}(\theta)}(A \cap B))). \end{aligned}$$

Thus, $(A \cap B)$ is a $\mathcal{H}(\theta) - \beta - open$.

(4) A is $\mathcal{H}(\theta) - semiopen$ then $A \subseteq c(i_{\mathcal{H}(\theta)}(A))$ and B is $\mathcal{H}(\theta) - preopen$ then$\mathcal{H}(\theta) - preopen$ if $A \subseteq i(c_{\mathcal{H}(\theta)}(A))$.

$$\begin{aligned} A \cap B \subseteq c(i_{\mathcal{H}(\theta)}(A)) \cap i(c_{\mathcal{H}(\theta)}(B)) &= c(i(i_{\mathcal{H}(\theta)}(A))) \cap i(c_{\mathcal{H}(\theta)}(B)) \\ &\subseteq c(i(i_{\mathcal{H}(\theta)}(A)) \cap i(c_{\mathcal{H}(\theta)}(B))) \\ &\subseteq c(i(i_{\mathcal{H}(\theta)}(A)) \cap c_{\mathcal{H}(\theta)}(B)) \\ &\subseteq c(i(c(c_{\mathcal{H}(\theta)}(A \cap B)))) \\ &\subseteq c(i(c_{\mathcal{H}(\theta)}(A \cap B))). \end{aligned}$$

Thus, $(A \cap B)$ is a $\mathcal{H}(\theta) - \beta - open$.

Definition 3.16 *[6] A spaces (X, μ) is μ-extremally disconnected if the closure of every μ-open set in X is μ-open.*

Theorem 3.17 *If a generalized topological spaces (X, μ) is extremally disconnected and $A, B \in \mathcal{H}(\theta)SO(X)$ then $A \cap B \in \mathcal{H}(\theta)SO(X)$.*

Proof: Let $A, B \in \mathcal{H}(\theta)SO(X)$ then $A \cap B \subseteq c(i_{\mathcal{H}(\theta)}(A)) \cap c(i_{\mathcal{H}(\theta)}(B))$ and (X, μ) is extremally disconnected. Hence,

$$\begin{aligned} \mathrm{A} \cap B \subseteq c(i_{\mathcal{H}(\theta)}(A)) \cap c(i_{\mathcal{H}(\theta)}(B)) &\subseteq c(i_{\mathcal{H}(\theta)}(A) \cap c(i_{\mathcal{H}(\theta)}(B))) \\ &\subseteq c(c(i_{\mathcal{H}(\theta)}(A) \cap i_{\mathcal{H}(\theta)}(B))) \\ &= c(i_{\mathcal{H}(\theta)}(A \cap B). \end{aligned} \tag{2}$$

Thus, $A \cap B \in \mathcal{H}(\theta)SO(X)$.

Theorem 3.18 *Let $(X, \mu, \mathcal{H})$ be a HGTS in X and $A \subset X$. If A is $\mathcal{H}(\theta) - \alpha - open$ and $\mathcal{H}(\theta) - preclosed$ then $c(i_{\mathcal{H}(\theta)}(A)) = i(c(i_{\mathcal{H}(\theta)}(A)))$.*

Proof: Since, A is $\mathcal{H}(\theta) - \alpha - open$ and $\mathcal{H}(\theta) - preclosed$ we have $c(i_{\mathcal{H}(\theta)}(A)) \subseteq A \subseteq i(c(i_{\mathcal{H}(\theta)}(A)))$. $i(c(i_{\mathcal{H}(\theta)}(A))) \subseteq c(i_{\mathcal{H}(\theta)}(A))$ obvious. Therefore, $c(i_{\mathcal{H}(\theta)}(A)) = i(c(i_{\mathcal{H}(\theta)}(A)))$.

Theorem 3.19 *Let $(X, \mu, \mathcal{H})$ be a HGTS, the following are equivalent;*

1. *The $\mathcal{H}(\theta) - closure$ of every $\mathcal{H}(\theta) - open$ subset of X is $\mathcal{H}(\theta) - open$,*
2. *$c(i_{\mathcal{H}(\theta)}(A)) \subseteq i(c_{\mathcal{H}(\theta)}(A))$ for every subset A of X,*
3. *$\mathcal{H}(\theta)SO(X) \subseteq \mathcal{H}(\theta)PO(X)$,*
4. *The $\mathcal{H}(\theta) - closure$ of every $\mathcal{H}(\theta) - \beta - open$ set is $\mathcal{H}(\theta) - open$,*
5. *$\mathcal{H}(\theta) - \beta - open \subseteq H(\theta) - preopen$.*

Proof: (1) $\Rightarrow$ (2) Assume that the $\mathcal{H}(\theta)-closure$ of every $\mathcal{H}(\theta)-open$ subset of X is $\mathcal{H}(\theta)-open$. Then, we have $c_{\mathcal{H}(\theta)}(i_{\mathcal{H}(\theta)}(A)) = c(i_{\mathcal{H}(\theta)}(A))$. Since, $c(i_{\mathcal{H}(\theta)}(A))$ is $\mathcal{H}(\theta)-open$ we have $c(i_{\mathcal{H}(\theta)}(A)) = i_{\mathcal{H}(\theta)}(c(i_{\mathcal{H}(\theta)}(A))) = i(c(i_{\mathcal{H}(\theta)}(A))) = i(c(A)) \subseteq i(c_{\mathcal{H}(\theta)}(A))$. Thus, $c(i_{\mathcal{H}(\theta)}(A)) \subseteq i(c_{\mathcal{H}(\theta)}(A))$.
(2) $\Rightarrow$ (3) Let $A \in \mathcal{H}(\theta)SO(X)$ then we have by(2) $A \subseteq c(i_{\mathcal{H}(\theta)}(A)) \subseteq i(c_{\mathcal{H}(\theta)}(A))$. Therefore, $A \in \mathcal{H}(\theta)PO(X)$.
(3) $\Rightarrow$ (4) Let $A \in \mathcal{H}(\theta)BO(X)$ then $A \subseteq c(i(c_{\mathcal{H}(\theta)}(A))) \subseteq c(i_{\mathcal{H}(\theta)}(c_{\mathcal{H}(\theta)}(A)))$ and $c_{\mathcal{H}(\theta)}(A) \subseteq c_{\mathcal{H}(\theta)}(c(i(c_{\mathcal{H}(\theta)}(A))))$. Thus, $c_{\mathcal{H}(\theta)}(A) \subseteq c(i_{\mathcal{H}(\theta)}(c_{\mathcal{H}(\theta)}(A)))$ and $c_{\mathcal{H}(\theta)}(A) \in \mathcal{H}(\theta)SO(X)$. By (3), $c_{\mathcal{H}(\theta)}(A) \in \mathcal{H}(\theta)PO(X)$. Hence, $c_{\mathcal{H}(\theta)}(A) \subseteq i(c_{\mathcal{H}(\theta)}(c_{\mathcal{H}(\theta)}(A)))$ and $c_{\mathcal{H}(\theta)}(A) \subseteq i(c_{\mathcal{H}(\theta)}(A)) = i_{\mathcal{H}(\theta)}(c_{\mathcal{H}(\theta)}(A))$. Therefore, $c_{\mathcal{H}(\theta)}(A)$ is $\mathcal{H}(\theta)-open$.
(4) $\Rightarrow$ (5) Let $A \in \mathcal{H}(\theta)BO(X)$. By (4), we get $c_{\mathcal{H}(\theta)}(A) = i(c_{\mathcal{H}(\theta)}(A))$. Hence, $A \subseteq c_{\mathcal{H}(\theta)}(A) = i(c_{\mathcal{H}(\theta)}(A))$ and therefore $A \in \mathcal{H}(\theta)-preopen$.
(5) $\Rightarrow$ (1) Let $A \in \mathcal{H}(\theta)PO(X)$. Then $c_{\mathcal{H}(\theta)}(A)$ is $\mathcal{H}(\theta)BO(X)$. By (5), we deduce $c_{\mathcal{H}(\theta)}(A)$ is $\mathcal{H}(\theta)PO(X)$. Since, $c_{\mathcal{H}(\theta)}(A) \subseteq i(c_{\mathcal{H}(\theta)}(c_{\mathcal{H}(\theta)}(A)))$ and $c_{\mathcal{H}(\theta)}(A) \subseteq i(c_{\mathcal{H}(\theta)}(A)) = i_{\mathcal{H}(\theta)}(c_{\mathcal{H}(\theta)}(A))$. Thus, $c_{\mathcal{H}(\theta)}(A)$ is $\mathcal{H}(\theta)-open$.

Definition 3.20 *Let $(X, \mu, \mathcal{H})$ be a HGTS. A subset A of X is said to be $\mathcal{H}(\theta)-pre-t-set(resp.\mathcal{H}(\theta)-\beta-t-set)$ if $i(c_{\mathcal{H}(\theta)}(A)) = i(A)(resp. c(i(c_{\mathcal{H}(\theta)}(A))) = i(A))$.*

Theorem 3.21 *Let $(X, \mu, \mathcal{H})$ be a HGTS in X and $A \subseteq X$. A is $regularopen$ iff $\mathcal{H}(\theta)-preopen$ and $\mathcal{H}(\theta)-pre-t-set$.*

Proof: Let A is $regularopen$. Then, A is $open$ and $c(A) = c_{\mathcal{H}(\theta)}(A)$. Hence, we have $i(c_{\mathcal{H}(\theta)}(A)) = i(c(A)) = A$. Consequently $A \subseteq i(c_{\mathcal{H}(\theta)}(A))$, A is $\mathcal{H}(\theta)-preopen$ and A is $open$ $i(c_{\mathcal{H}(\theta)}(A)) = i(A)$. Then, A is $\mathcal{H}(\theta)-pre-t-set$. Conversely, let A is $\mathcal{H}(\theta)-preopen$, $\mathcal{H}(\theta)-pre-t-set$. Then, we get $A \subseteq i(c_{\mathcal{H}(\theta)}(A)) = i(A) = A$ then, A is $open$. This implies that, $i(c(A)) = A$. So we have A is $regularopen$.

Theorem 3.22 *Let A and B be subsets of an HGTS $(X, \mu, \mathcal{H})$. If A and B are $\mathcal{H}(\theta)-pre-t-set$, then $A \cap B$ is a $H(\theta)-pre-t-set$.*

Proof: We know that $A \cap B \subseteq c_{\mathcal{H}(\theta)}(A \cap B)$. Hence $i(A \cap B) \subseteq i(c_{\mathcal{H}(\theta)}(A \cap B)) \subseteq i(c_{\mathcal{H}(\theta)}(A) \cap c_{\mathcal{H}(\theta)}(B)) = i(c_{\mathcal{H}(\theta)}(A)) \cap i(c_{\mathcal{H}(\theta)}(B)) = i(A) \cap i(B) = i(A \cap B)$. Thus , $A \cap B$ is a H$(\theta)-pre-t-set$.

Proposition 3.23 *Let $(X, \mu, \mathcal{H})$ be a HGTS in X and $A \subseteq X$. A is $\mathcal{H}(\theta)-closed$ then $\mathcal{H}(\theta)-pre-t-set$.*

Proof: Since A is $\mathcal{H}(\theta)-closed$, we have $A = c_{\mathcal{H}(\theta)}(A)$. Then, we can write $i(A) = i(c_{\mathcal{H}(\theta)}(A))$. Thus, A is $\mathcal{H}(\theta)-pre-t-set$.

Definition 3.24 *A subset A of a GTS $(X, \mu, \mathcal{H})$ is said to be*

(i) $\mathcal{H}(\theta) - pre - B - set$ if there exist $M \in \mu$ and a $\mathcal{H}(\theta) - pre - t - set$ V in X such that $A = M \cap V$,
(ii) $\mathcal{H}(\theta) - \beta - B - set$ if there exist $M \in \mu$ and a $\mathcal{H}(\theta) - \beta - t - set$ V in X such that $A = M$.

Definition 3.25 *Let $(X, \mu, \mathcal{H})$ be an HGTS. A subset A of X is called a $\mathcal{H}(\theta) - semiclosed$ set if $i(c_{\mathcal{H}(\theta)}(A)) \subseteq A$.*

Definition 3.26 *A subset A of a GTS $(X, \mu, \mathcal{H})$ is called a $\mathcal{H}^* - set$ if $A = M \cap V$, where M is μ-open, V is $\mathcal{H}(\theta) - semiclosed$ and $i(c_{\mathcal{H}(\theta)}(V)) = c(i_{\mathcal{H}(\theta)}(V))$.*

Theorem 3.27 *For a subset A of an HGTS $(X, \mu, \mathcal{H})$, the following properties are equivalent:*

1. *A is open set,*
2. *A is α-open and a $\mathcal{H}^* - set$,*
3. *A is preopen and a $\mathcal{H}^* - set$,*
4. *A is $\mathcal{H}(\theta) - preopen$ and a $\mathcal{H}^* - set$,*
5. *A is $\mathcal{H}(\theta) - \beta - open$ and a $\mathcal{H}^* - set$,*
6. *A is $\mathcal{H}(\theta) - preopen$ and a $\mathcal{H}(\theta) - pre - B - set$,*
7. *A is $\mathcal{H}(\theta) - \beta - open$ and a $\mathcal{H}(\theta) - \beta - B - set$.*

Proof: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5): Since every open is a α-open, every α-open is a *preopen*, every *preopen* is a $\mathcal{H}(\theta) - preopen$, every $\mathcal{H}(\theta) - preopen$ is a $\mathcal{H}(\theta) - \beta - open$, proof is obvious.
(5) $\Rightarrow$ (1): Let A be $\mathcal{H}(\theta) - \beta - open$ and a $\mathcal{H}^* - set$. Then we have $A \subseteq c(i(c_{\mathcal{H}(\theta)}(A)))$ and $A = M \cap V$, where M is μ-open, V is $\mathcal{H}(\theta) - semiclosed$ and $i(c_{\mathcal{H}(\theta)}(V)) = c(i_{\mathcal{H}(\theta)}(V))$. Therefore,

$$\begin{aligned} \text{A}=\text{A}\cap M \subseteq c(i(c_{\mathcal{H}(\theta)}(A))) \cap M &= c(i(c_{\mathcal{H}(\theta)}(A \cap M))) \cap M \\ &\subseteq c(i(c_{\mathcal{H}(\theta)}(M))) \cap c(i(c_{\mathcal{H}(\theta)}(V))) \cap M \\ &= c(i(c_{\mathcal{H}(\theta)}(V))) \cap M \\ &= c(c(i_{\mathcal{H}(\theta)}(V))) \cap M = c(i_{\mathcal{H}(\theta)}(V)) \cap M \\ &= i(c_{\mathcal{H}(\theta)}(V)) \cap M = i(V) \cap M \subseteq V \cap M = A. \end{aligned}$$

Hence, A is open set.
(1) $\Rightarrow$ (6): is clear.
(6) $\Rightarrow$ (1): Let A is $\mathcal{H}(\theta) - pre - open$ and a $\mathcal{H}(\theta) - pre - B - set$. Clearly, then there exist $M \in \mu$ and a $\mathcal{H}(\theta) - pre - t - set$ V in X such that $A = M \cap V$.

Hence, V is a $\mathcal{H}(\theta)-pre-t-set$ then $i(V)=i(c_{\mathcal{H}(\theta)}(V))$. Since, A is $\mathcal{H}(\theta)-pre-open$,

$$\begin{aligned} A\subseteq i(c_{\mathcal{H}(\theta)}(A)) &= i(c_{\mathcal{H}(\theta)}(M\cap V)) \\ &\subseteq i(c_{\mathcal{H}(\theta)}(M)\cap c_{\mathcal{H}(\theta)}(v)) \\ &= i(c_{\mathcal{H}(\theta)}(M))\cap i(c_{\mathcal{H}(\theta)}(V) \\ &= i(c_{\mathcal{H}(\theta)}(M))\cap i(V). \end{aligned}$$

Thus, $A=M\cap V\subseteq(M\cap V)\cap M\subseteq i(c_{\mathcal{H}(\theta)}(M))\cap i(V)\cap M=M\cap i(V)$ and $A\subseteq M\cap i(V)\subseteq M\cap V=A$. Hence, $A=M\cap i(H)$ and A is *open*.
(1) ⇒ (7): It is obvious.
(7) ⇒ (1): Let A is $\mathcal{H}(\theta)-\beta-open$ and $\mathcal{H}(\theta)-\beta-B-set$. Then one can write following;

$$\begin{aligned} A\subseteq c(i(c_{\mathcal{H}(\theta)}(A))) &= c(i(c_{\mathcal{H}(\theta)}(M\cap V))) \\ &\subseteq c(i(c_{\mathcal{H}(\theta)}(M)\cap c_{\mathcal{H}(\theta)}(V))) \\ &= c(i(c_{\mathcal{H}(\theta)}(M))\cap i(c_{\mathcal{H}(\theta)}(V))) \\ &= c(i(c_{\mathcal{H}(\theta)}(M)))\cap i(V). \end{aligned}$$

Thus, $A=M\cap V\subseteq(M\cap V)\cap M\subseteq c(i(c_{\mathcal{H}(\theta)}(M)))\cap i(V)\cap M=M\cap i(V)$ and $A\subseteq M\cap i(V)\subseteq M\cap V=A$. Hence, $A=M\cap i(H)$ and A is *open*.

4 Decomposition of Continuity

Definition 4.1 *Let $(X,\mu,\mathcal{H})$ be an HGTS and a space (Y,λ) is GT. A function $f:(X,\mu,\mathcal{H})\to(Y,\lambda)$ is said to be;*

1. *$\alpha-continuous$ if $f^{-1}(G)$ is α-open for each $G\in\lambda$*
2. *Precontinuous if $f^{-1}(G)$ is pre open for each $G\in\lambda$*
3. *$\mathcal{H}(\theta)-precontinuous$ if $f^{-1}(G)$ is $\mathcal{H}(\theta)-preopen$ for each $G\in\lambda$*
4. *$\mathcal{H}(\theta)-\beta-continuous$ if $f^{-1}(G)$ is $\mathcal{H}(\theta)-\beta-open$ for each $G\in\lambda$*
5. *$\mathcal{H}^*-continuous$ if $f^{-1}(G)$ is $\mathcal{H}^*-set$ for each $G\in\lambda$*
6. *$\mathcal{H}(\theta)-pre-B-continuous$ if $f^{-1}(G)$ is $\mathcal{H}(\theta)-pre-B-set$ for each $G\in\lambda$*
7. *$\mathcal{H}(\theta)-\beta-B-continuous$ if $f^{-1}(G)$ is $\mathcal{H}(\theta)-pre-B-set$ for each $G\in\lambda$*

Theorem 4.2 *For a function $f:(X,\mu,\mathcal{H})\to(Y,\lambda)$ the following properties are equivalent:*

1. f *is continuous,*
2. f *is* $\alpha -$ *continuous and* $\mathcal{H}^* -$ *continuous,*
3. f *is Precontinuous and* $\mathcal{H}^* -$ *continuous,*
4. f *is* $\mathcal{H}(\theta) -$ *precontinuous and* $\mathcal{H}^* -$ *continuous,*
5. f *is* $\mathcal{H}(\theta) - \beta -$ *continuous and* $\mathcal{H}^* -$ *continuous,*
6. f *is* $\mathcal{H}(\theta) -$ *precontinuous and* $\mathcal{H}(\theta) - pre - B -$ *continuous,*
7. f *is* $\mathcal{H}(\theta) - \beta -$ *continuousand* $\mathcal{H}(\theta) - \beta - B -$ *continuous.*

Proof: It follows from Theorem 3.27.

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