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The Traveling Solution and Tanh-Solution for a Burgers Equation

Jumei Zhang[1] **and Li Yin**[2]

[1,2]Department of Mathematics, Binzhou University
Binzhou City, Shandong Province, 256603, China
[1]E-mail: wdzjm02@163.com
[2]E-mail: yinli7979@163.com

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Abstract

In this paper, we show that nonlinear Burgers equation $u_t + 2uu_x - u_{xx} = 0$ *has traveling solution and tanh-solution, and the two solutions are consistent under some suitable conditions.*

Keywords: *Burgers equation, Traveling solutions, Tanh-solutions.*

1 Introduction

Consider the nonlinear evolution equation,

$$H(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \cdots) = 0. \tag{1}$$

where x is the space variable, t is the time variable, u is a function of x and t, and H is a function of u and its derivatives. If $\phi(\zeta)$ is the solution of (1) and $\zeta = x - ct + \xi_0$ is only dependent on x and t, then $\phi(\zeta)$ is called the travelling solution of (1). In order to get the traveling solution of the evolution equation, the general method is converted partial differential equation into differential equations, but the directed integral method can also be used to get the traveling solutions of some classical equations [2].

In 1992, Malfiet [11] proposed that the hyperbolic tangent function expanded method can be used to get solutions of some nonlinear equations. Many researchers have paid their attention to this field, such as [3, 4, 9, 10]. It is divided into four steps:

Firstly, a new variable ϑ is introduced, such as $\vartheta = k\zeta + \zeta_0$, $\zeta = x - \alpha t$, etc. Bring ϑ to (1), then (1) can be written in the form

$$F(u, u', u'', \cdots) = 0.$$

It is a differential equation.

Secondly, the unknown function u is expanded for the hyperbolic tangent function in finite series form, namely

$$u = \sum_{j=0}^{n} a_j T^j, \; T \equiv \tanh(\vartheta). \tag{2}$$

The arbitrary order derivation of hyperbolic tangent function can be expressed in terms of its own, for example,

$$\frac{\mathrm{d}}{\mathrm{d}\vartheta} = \left(1 - T^2\right) \frac{\mathrm{d}}{\mathrm{d}T}, \tag{3}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\vartheta^2} = -2T\left(1 - T^2\right) \frac{\mathrm{d}}{\mathrm{d}T} + \left(1 - T^2\right)^2 \frac{\mathrm{d}^2}{\mathrm{d}T^2}, \tag{4}$$

$$\frac{\mathrm{d}^3}{\mathrm{d}\vartheta^3} = 2(1 - T^2)(3T^2 - 1)\frac{\mathrm{d}}{\mathrm{d}T} - 6T\left(1 - T^2\right)^2 \frac{\mathrm{d}^2}{\mathrm{d}T^2} + \left(1 - T^2\right)^3 \frac{\mathrm{d}^3}{\mathrm{d}T^3}, \tag{5}$$

so the hyperbolic tangent function expanded method has great advantages in studying the solutions of nonlinear evolution equations.

Thirdly, suppose the highest derivative order of u is n, that is

$$O(u) = n, \tag{6}$$

and by simply calculating we can get

$$O(u^p) = pn, \; O\left(\frac{\mathrm{d}^q}{\mathrm{d}T^q}\right) = n + q, \; O\left(u^p \frac{\mathrm{d}^q}{\mathrm{d}T^q}\right) = (p+1)\,n + q, \tag{7}$$

then balance between the highest derivative order of the linear term and the highest derivative order of nonlinear term to determine n.

The last step is to substitute (2) into (1) to determine the expansion coefficient a_j and the solutions u.

The Burgers equation is the easiest nonlinear physical model various flows problems consisting of sound and shock wave theory, wave vorticity transportation, hydrodynamic turbulence, processes in thermo-elastic medium, dispersion in porous media, continuous stochastic processes, mathematical modeling of turbulent fluid. The equation was first introduced by Bateman [5]. Later, many researchers have been deeply studied it, for example [1, 6, 7, 8, 12]. In this paper, we give the traveling solution by directing integral method and the tanh-solution of a kind of Burgers equation

$$u_t + 2uu_x - u_{xx} = 0, \tag{8}$$

and prove that the two solutions are consistent, under some suitable conditions.

2 The Travelling Solution of (1)

Suppose the travelling solution of (1) has the following form:

$$u = \varphi(\zeta), \zeta = x - \alpha t + \xi_0, \tag{9}$$

where α is a constant and ξ_0 is an arbitrary constant, respectively. Then substituting (9) into (8), we get

$$-\alpha\frac{\mathrm{d}\varphi}{\mathrm{d}\zeta} + 2\varphi\frac{\mathrm{d}\varphi}{\mathrm{d}\zeta} - \frac{\mathrm{d}^2\varphi}{\mathrm{d}\zeta^2} = 0. \tag{10}$$

Integrating the above equation once, we obtain

$$-\alpha\varphi - \frac{\mathrm{d}\varphi}{\mathrm{d}\zeta} + \varphi^2 = -C, \tag{11}$$

where C is an integration constant. From the above equation, we can obtain

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\zeta} = \varphi^2 - \alpha\varphi + C. \tag{12}$$

Assuming $\alpha^2 - 4C > 0$, then $\varphi^2 - \alpha\varphi + C = 0$ has two real roots

$$\varphi_1 = \frac{\alpha + \sqrt{\alpha^2 - 4C}}{2}$$

and

$$\varphi_2 = \frac{\alpha - \sqrt{\alpha^2 - 4C}}{2},$$

satisfying $\varphi_1 + \varphi_2 = \alpha$, $\varphi_1 - \varphi_2 = \sqrt{\alpha^2 - 4C}$. Then (12) can be rewritten into

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\zeta} = (\varphi - \varphi_1)(\varphi - \varphi_2).$$

Integrating the above equation, we get

$$\varphi = \frac{\varphi_1 + \varphi_2}{2} - \frac{\varphi_1 - \varphi_2}{2} \tanh\left[\frac{\varphi_1 - \varphi_2}{2}(\zeta + \xi_1)\right] \tag{13}$$

$$= \frac{\alpha}{2} - \frac{\sqrt{\alpha^2 - 4C}}{2} \tanh\left[\frac{\sqrt{\alpha^2 - 4C}}{2}(\zeta + \xi_1)\right], \tag{14}$$

where ξ_1 is an arbitrary constant.

Returning to the original variable, we can get the travelling solution of the equation (1)

$$u(x,t) = \frac{\alpha}{2} - \frac{\sqrt{\alpha^2 - 4C}}{2} \tanh\left[\frac{\sqrt{\alpha^2 - 4C}}{2}(x - \alpha t + \xi^*)\right], \tag{15}$$

where $\xi^* = \xi_1 + \xi_0$ is an arbitrary constant.

3 The Tanh-Solution of (1)

Set

$$u = \varphi(\vartheta),\ \vartheta = k\zeta + \zeta_0,\ \zeta = k(x - \alpha t). \tag{16}$$

Substituting (16) into (8), we get

$$-\alpha\frac{\mathrm{d}u}{\mathrm{d}\vartheta} + 2u\frac{\mathrm{d}u}{\mathrm{d}\vartheta} - k\frac{\mathrm{d}^2u}{\mathrm{d}\vartheta^2} = 0. \tag{17}$$

We study the tanh-solution of the above equation in the following two cases.

Case 1: By the assumption (6) and (7), the highest derivative order of the linear term of (17) is

$$O(\frac{\mathrm{d}^2u}{\mathrm{d}\vartheta^2}) = n + 2, \tag{18}$$

and the highest derivative order of nonlinear term is

$$O(u\frac{\mathrm{d}u}{\mathrm{d}\vartheta}) = 2n + 1. \tag{19}$$

Balancing between (18) and (19), we get $n = 1$. So, the hyperbolic tangent function solution of (17) is expanded as

$$u = b_0 + b_1 T,\ T \equiv \tanh(\vartheta) = \tanh(k(x - \alpha t) + \zeta_0), \tag{20}$$

and

$$\frac{\mathrm{d}u}{\mathrm{d}T} = b_1,\ \frac{\mathrm{d}^2u}{\mathrm{d}T^2} = 0 \tag{21}$$

where ζ_0 is an arbitrary constant. Bringing (5), (20) and (21) into (17), then (17) can be written in

$$(-\alpha+2b_0)b_1+2(b_1+k)b_1T+(\alpha-2b_0)b_1T^2-2(b_1+k)b_1T^3=0.$$

Let the coefficients of T^0, T^1, T^2, T^3 be zero, we get

$$b_0=\frac{\alpha}{2},\ b_1=-k.$$

So, the tanh-solution of (1) is

$$u=\frac{\alpha}{2}-k\tanh(k(x-\alpha t)+\zeta_0). \tag{22}$$

Let $k=\frac{\sqrt{\alpha^2-4C}}{2}$, then (22) is consistent with (15).

Case 2: Integrating the equation (17) once, we get

$$-\alpha u+u^2-k\frac{\mathrm{d}u}{\mathrm{d}\vartheta}=D, \tag{23}$$

where D is an integration constant.

By the assumption (6) and (7), the highest derivative order of the linear term of (23) is

$$O\left(\frac{\mathrm{d}u}{\mathrm{d}\vartheta}\right)=n+1, \tag{24}$$

and the highest derivative order of nonlinear term is

$$O(u^2)=2n. \tag{25}$$

Balancing between (24) and (25), we get $n=1$. So, the hyperbolic tangent function solution of (23) is expanded as

$$u=d_0+d_1T,\ T\equiv\tanh(\vartheta)=\tanh(k(x-\alpha t)+\zeta_0), \tag{26}$$

and

$$\frac{\mathrm{d}u}{\mathrm{d}T}=d_1, \tag{27}$$

where ζ_0 is an arbitrary constant. Bringing (5), (26) and (27) into (23), then (23) can be written in

$$(-\alpha d_0+d_0^2-kd_1-D)+(2d_0-\alpha)d_1T+(d_1+k)d_1T^2=0.$$

Let the coefficients of T^0, T^1, T^2 be zero, we get

$$d_0=\frac{\alpha}{2},\ d_1=-k,$$

with D satisfying

$$k^2 - \frac{\alpha^2}{4} = D.$$

So, the tanh-solution of (1) is

$$u = \frac{\alpha}{2} - k \tanh(k(x - \alpha t) + \zeta_0). \tag{28}$$

4 Conclusion

In section 2 and section 3, we give the traveling solution and the tanh-solution of the equation (8), respectively.

Let $k = \frac{\sqrt{\alpha^2-4C}}{2}$ and $D = -C$, then we can get that the tanh-solutions of (28) are consistent with the travelling solutions (15) under some suitable conditions.

In the future, we will discuss the the tanh-solutions and the travelling solutions of some other equations.

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