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Odd Graceful Labeling on Two Classes of Graphs

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Abstract

A labeling or numbering of a graph G is an assignment of labels to the vertices of G that induces for each edge uv a labeling depending on the vertex labels f(u) and f(v). In this paper, we study the odd graceful labeling on two classes of graphs such as Roman rings having cycle with 6 vertices and hanging complete bipartite graphs.

Keywords: *Odd graceful labeling, Roman rings, Hanging Complete Bipartite graphs.*

1 Introduction

All graphs in this paper are finite, simple, undirected and without having isolated vertices. For all terminology and notations in graph theory, we follow Harary [2] and for all terminology regarding odd graceful labeling, we follow [3]. A connected graph G with p vertices and q edges is called odd graceful if it is possible to label the vertices of G with pairwise distinct integers in $\{0, 1, 2, 3, ..., 2q - 1\}$ so that each edge, xy, is labeled $|f(x) - f(y)|$, the resulting edge labels are the entire set $\{1, 3, 5, ..., 2q - 1\}$. In this paper we develop two different class of graphs and discuss the odd graceful labeling of these graphs.

The field of graph theory plays vital role in various fields. Graph labeling is one of the important area in graph theory. Graph labelings where the vertices are assigned values subject to certain conditions have been motivated by practical problems. Labeled graphs serves as useful mathematical models for a broad range of applications such as communication network addressing system, data base management, circuit designs, coding theory, X-ray crystallography, the design of good radar type codes, synch-set codes, missile guidance codes and radio astronomy problems etc. The detailed discription of the applications of graph labelings can be seen in [1]. In [4] proved the Gracefulness of union of paths and cycles. The odd graceful labeling of union of paths and cycles were discussed in [5] and [6]. Odd graceful labeling of disjoint union of generalized combs with paths were discussed in [7]. The detailed discussion of odd gracefulness of complete bipartite graphs in [8].

2 Roman Rings having Cycle with 6 Vertices

In this section, we consider the roman rings having cycle with 6 vertices and discuss the odd graceful labeling of this graph.

Definition 2.1. *Consider the comb graph $P_n \bigodot L_1$ and the cycle C_6 with 6 vertices. The n teeth of comb graph is merged with the n copies of cycle C_6. The resulting graph is called the Roman rings with cycle C_6 and denoted by $R(6, n)$.*

Here we called n copies of cycles are n rings. Let $X_1, X_2, ..., X_n$ be n copies of cycle C_6 and the supporting points in the n rings(cycles) are $S_1, S_2, ..., S_n$. The supporting points are merged respectively to n teeth of comb graph. Let $b_1, b_2, ..., b_n$ be base points of the comb graph from which the cycle C_6 are hanging, each of which at a tooth of n teeth respectively. Let the points of i^{th} ring be $S_i, Y_1^i, Y_2^i, Y_3^i, Y_4^i and Y_5^i$. In R(6,n), the total number of vertices are 7n and the total number of edges are 8n-1.

Definition 2.2. *A graph G = (V(G),E(G)) is said to admit odd graceful labeling if $f : V(G) \rightarrow \{0, 1, 2, 3, ..., 2q - 1\}$ is injective and the induced function $f^* : E(G) \rightarrow \{1, 3, 5, ..., 2q - 1\}$ defined as $f^*(uv) = |f(u) - f(v)|$ is surjective. A graph which admits odd graceful labeling is called an odd graceful graph.*

Theorem 2.3. *The Roman rings R(6,n) is odd graceful.*

Proof: The procedure for labeling the Roman rings is first we label the (n-1) rings and then label the last ring with the remaining vertices.

Define the labeling $f : V(R(6,n)) \rightarrow \{0, 1, 2, ..., 16n - 3\}$ as follows. The labeling of first two rings in R(6,n) as $f(b_1) = 0$

$f(S_1) = 2q - 1$

$f(Y_1^1) = 2$

$f(Y_2^1) = 4$

$f(Y_3^1) = (n - 2)4 + 23$

$f(Y_4^1) = f(Y_3^1) + 4$

$f(b_2) = 2q - 7$

$f(S_2) = 6$

$f(Y_1^2) = 2q - 3$

$f(Y_2^2) = 2q - 5$

$f(Y_3^2) = (n - 3)12 + 14$

$f(Y_4^2) = f(Y_3^2) - 4$

For labeling the remaining rings consider the following two cases

Case(1): when n is odd, take $n = 2m + 1$

$f(b_{2i+1}) = f(b_{2i-1}) + 8, 1 \leq i \leq \lambda - 1$

$f(b_{2i+2}) = f(b_{2i}) - 8, 1 \leq i \leq \lambda - 1$

$f(S_{2i+1}) = 2q - 1 - f(b_{2i+1}), 1 \leq i \leq \lambda - 1$

$f(S_{2i+2}) = f(S_{2i}) + 8, 1 \leq i \leq \lambda - 1$

$f(Y_1^{2i+1}) = f(Y_1^{2i-1}) + 8, 1 \leq i \leq \lambda - 1$

$f(Y_1^{2i+2}) = f(Y_1^{2i}) - 8, 1 \leq i \leq \lambda - 1$

$f(Y_2^{2i+1}) = f(Y_2^{2i-1}) + 8, 1 \leq i \leq \lambda - 1$

$f(Y_2^{2i+2}) = f(Y_2^{2i}) - 8, 1 \leq i \leq \lambda - 1$

$f(Y_3^{2i+1}) = f(Y_3^{2i-1}) + 16, 1 \leq i \leq \lambda - 1$

$f(Y_3^{2i+2}) = f(Y_3^{2i}) - 16, 1 \leq i \leq \lambda - 1$

$f(Y_4^{2i+1}) = f(Y_4^{2i-1}) + 16, 1 \leq i \leq \lambda - 1$

$f(Y_4^{2i+2}) = f(Y_4^{2i}) - 16, 1 \leq i \leq \lambda - 1$

Case(2): when n is even, take $n = 2m$

$f(b_{2i+1}) = f(b_{2i-1}) + 8, 1 \leq i \leq \lambda - 1$

$f(b_{2i+2}) = f(b_{2i}) - 8, 1 \leq i \leq \lambda - 2$

$f(S_{2i+1}) = 2q - 1 - f(b_{2i+1}), 1 \leq i \leq \lambda - 1$

$f(S_{2i+2}) = f(S_{2i}) + 8, 1 \leq i \leq \lambda - 2$

$f(Y_1^{2i+1}) = f(Y_1^{2i-1}) + 8, 1 \leq i \leq \lambda - 1$

$f(Y_1^{2i+2}) = f(Y_1^{2i}) - 8, 1 \leq i \leq \lambda - 2$

$f(Y_2^{2i+1}) = f(Y_2^{2i-1}) + 8, 1 \leq i \leq \lambda - 1$

$f(Y_2^{2i+2}) = f(Y_2^{2i}) - 8, 1 \leq i \leq \lambda - 2$

$f(Y_3^{2i+1}) = f(Y_3^{2i-1}) + 16, 1 \leq i \leq \lambda - 1$

$f(Y_3^{2i+2}) = f(Y_3^{2i}) - 16, 1 \leq i \leq \lambda - 2$

$f(Y_4^{2i+1}) = f(Y_4^{2i-1}) + 16, 1 \leq i \leq \lambda - 1$

$f(Y_4^{2i+2}) = f(Y_4^{2i}) - 16, 1 \leq i \leq \lambda - 2$

The remaining labeling are for the vertices connected by 2(n-1)+8 edges and the edge values are 1,3,5,..........,4(n-3)+23.

Let $e_i = 4(n-3)+23$

Now consider the n edge pairs $(e_i, e_i-4)(e_i-2, e_i-6)(e_i-8, e_i-12)(e_i-10, e_i-14)(e_16, e_i-20)(e_i-18, e_i-22)$.

The labeling of $f(Y_5)$ are as follows:

(1) choosing the maximum pair $(f(Y_3^i), f(Y_4^i)), i = 1, 2, ..., n-1$. and substitute the maximum edge value pair.The vertex that join the edge value pair lies between 8(n-2)+1 to 2q-1.

(2) choosing the next maximum pair in $(f(Y_3^i), f(Y_4^i))$ and substitute the next maximum edge value pair and so on.

(3) for the $(n-1)^{th}$ pair $(f(Y_3^i), f(Y_4^i))$ substitute the n^{th} edge pair.

The remaining 8 edges consists of $(n-1)^{th}$ pair in the above n pairs and substitute the maximum edge label in the pair to the edge connecting b_{n-1} and b_n. There are only seven edges remains including the last ring consists of 6 edges. Now label the last ring vertices with the remaining edge values.
Then the labeling on the edges in R(6,n) are $\{1, 3, ..., 2q-1\}$.
The Roman rings R(6,n) is odd graceful.

3 Hanging Complete Bipatite Graphs

In this section,we consider hanging complete bipartite graph and discuss the odd graceful labeling of this graph.

Definition 3.1. *Consider the comb graph $P_l \bigodot L_1$ and the complete bipartite graph $K_{m,n}$ on m and n vertices. The l teeth of comb graph is merged with the one vertex of l copies of $K_{m,n}$. The resulting graph is called hanging complete bipartite graph and denoted by $HCB(l, m, n)$.*

Let $b_1, b_2, ..., b_l$ be the path vertices of P_l.The path vertices are called base points. Make l copies of complete bipartite graph $K_{m,n}$ with vertex set $V_1 = v(d,1), v(d,2), ..., v(d,m)$ and $V_2 = u(d,1), u(d,2), ..., u(d,n)$ where $1 \leq d \leq l$. Let $V(G) = V_1UV_2$.Let $v(d,m)$ denote the d^{th} branch contains m vertices and $u(d,n)$ denote the d^{th} branch contains n vertices of bipartite graph. Construct an edge adjoint b_d to $v(d,1), 1 \leq d \leq l$. The resulting graph is $HCB(l, m, n)$.

Theorem 3.2. *The family of hanging complete bipartite graphs $F(HCB(l, m, n) : l, m, n \geq 1)$ are odd graceful.*

Proof: The labeling of vertices of first two branches as follows

First branch
$f(b_1) = 0$
$f(v(1,i)) = 2q + 1 - 2i, 1 \leq i \leq m$
$f(u(1,1)) = 2$
$f(u(1,j)) = 2 + 2m(j-1), 2 \leq j \leq n$

Second branch
$f(b_2) = 2q - 2mn - 3$
$f(v(2,1)) = 2q - 1 - f(b_2)$
$f(v(2,i)) = f(v(2,1)) - 2i - 2, 2 \leq i \leq m$
$f(u(2,1)) = f(b_2) + 2$
$f(u(2,j)) = f(u(2,1)) + 2m(j-1), 2 \leq j \leq n$

Consider the following two cases for labeling the remaining branches.

Case(1): If l is odd, take l = 2k+1.

The labeling on the odd branches are
$f(b_{2d+1}) = f(b_{2d-1}) + 2mn + 4, 1 \leq d \leq k$
$f(v(2d+1,1)) = 2q - 1 - f(b_{2d+1}), 1 \leq d \leq k$
$f(v(2d+1,i)) = f(v(2d+1,1)) + 2 - 2i, 2 \leq i \leq m, 1 \leq d \leq k$
$f(u(2d+1,1)) = f(b_{2d+1}) + 2, 1 \leq d \leq k$
$f(u(2d+1,j)) = f(u(2d+1,1)) + 2m(j-1), 2 \leq j \leq n, 1 \leq d \leq k$

The labeling on the even branches are
$f(b_{2d+2}) = f(b_{2d}) - 2mn - 4, 1 \leq d \leq k-1$
$f(v(2d+2,1)) = 2q - 1 - f(b_{2d+2}), 1 \leq d \leq k-1$
$f(v(2d+2,i)) = f(v(2d+2,1)) + 2 - 2i, 2 \leq i \leq m, 1 \leq d \leq k-1$
$f(u(2d+2,1)) = f(b_{2d+2}) + 2, 1 \leq d \leq k-1$
$f(u(2d+2,j)) = f(u(2d+2,1)) + 2m(j-1), 2 \leq j \leq n, 1 \leq d \leq k-1$

Case(2): If l is even, take l = 2k.

The labeling on the odd branches are
$f(b_{2d+1}) = f(b_{2d-1}) + 2mn + 4, 1 \leq d \leq k$
$f(v(2d+1,1)) = 2q - 1 - f(b_{2d+1}), 1 \leq d \leq k$
$f(v(2d+1,i)) = f(v(2d+1,1)) + 2 - 2i, 2 \leq i \leq m, 1 \leq d \leq k$
$f(u(2d+1,1)) = f(b_{2d+1}) + 2, 1 \leq d \leq k$
$f(u(2d+1,j)) = f(u(2d+1,1)) + 2m(j-1), 2 \leq j \leq n, 1 \leq d \leq k$

The labeling on the even branches are
$f(b_{2d+2}) = f(b_{2d-2}) - 2mn - 4, 1 \leq d \leq k-1$
$f(v(2d+2,1)) = 2q - 1 - f(b_{2d+2}), 1 \leq d \leq k-1$
$f(v(2d+2,i)) = f(v(2d+2,1)) + 2 - 2i, 2 \leq i \leq m, 1 \leq d \leq k-1$
$f(u(2d+2,1)) = f(b_{2d+2}) + 2, 1 \leq d \leq k$
$f(u(2d+2,j)) = f(u(2d+2,1)) + 2m(j-1), 2 \leq j \leq n, 1 \leq d \leq k$

Then the labeling on the edges in H(l,m,n) are $\{1, 3, ..., 2q-1\}$. The Hanging complete bipartite graph is odd graceful.

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