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Characterization of the Parallel Curve of the Adjoint Curve in $\mathbb{E}^3$

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Abstract

In this paper, we obtain some characterizations of parallel curves of adjoint curves. In three-dimensional Euclidean space, the relationship between β adjoint of α curve and γ parallel of β curve are examined. Then, arc-length on γ curve is calculated. Curvature of parallel curve and curvature of the adjoint curve are calculated. Additionally, some results and theorems are presented with special cases. Moreover, α and γ curves Involute-Evolute, Bertrand and Mannheim curve pair situations are examined.

Keywords: *Adjoint curves, Bertrand curve pairs, Involute-Evolute curve pairs, Mannheim curve pairs, Parallel curves.*

1 Introduction

The adjoint curves play an important role in various areas of mathematics such as number theory, coding theory, algebraic geometry, etc. Additionally, Bertrand curves are one of the associated curve pairs for which at the corresponding points of the curves one of the Frenet vectors of a curve coincides with the one of the Frenet vectors of the other curve. In 1997, Mnuk [10] described an algebraic approach to computing the system of adjoint curves to a given absolutely irreducible plane algebraic curve. In 2003, Matsuda and Yorozu [11] showed that every circular helix in $\mathbb{E}^3$ is a typical example of the Bertrand curve. The circular helix is one in a family of special Frenet curves. They proved that no special Frenet curve in $\mathbb{E}^n$ $(n \geq 4)$ is a Bertrand curve. They also generalized the notion of the Bertrand curve in $\mathbb{E}^4$.

In 2007, Huy Thai Ha [3] studied adjoint line bundles. Wang and Liu [16] are concerned with another kind of the associated curves, called Mannheim curve and Mannheim mate (partner curve) in history of differential geometry. They called them simply as Mannheim pair. Chrastinova [2] studied parallel curves in $\mathbb{E}^3$. To a curve P in the plane, there exist two curves P_+, P_- at a given distance r (except for some degenerate cases). The curves P_+, P_- can be alternatively obtained as the envelopes of circles of radius r with centers moving along the curve P. This construction was carried over the three-dimensional space and as a result, two parallel curves are obtained as well. Also, J.Monterde [7] studied curves in $\mathbb{R}^n$ for which the ratios between two consecutive curvatures are constant are characterized by the fact that their tangent indicatrix is a geodesic in a flat torus. In 2009, J.Monterde [8] characterized a family of curves with constant curvature but non-constant torsion as space curves with constant curvature.

In 2011, Tunçer and Unal [14] studied the properties of the spherical indicatrices of a Bertrand curve and its mate curve and presented some characteristic properties in the cases that Bertrand curve and its mate curve and slant helices, spherical indicatries made new curve pairs in the means of Mannheim, Involute-Evolute and Bertrand pairs. Furthermore, they investigated the relations between the spherical images and introduced new representations of spherical indicatrices. Also, Sendra and Sevilla [13] presented algorithms for parametrizing by radicals an irreducible curve, not necessarily plane, when the genus is less than or equal to 4 and the curve is defined over an algebraically closed field of the characteristic zero. They also obtained the radical parametrization with the help of adjoint curves.

In 2014, Yüksel, Karacan, Bükçü and Ikiz [15] adapted Chrastinova's work at Minkowski 3-space. In 2015, İkiz, Keskin, Yüksel and Karacan [5], the ruled surfaces generated by the adjoint curve of the base curve have been investigated in Euclidean 3-space. We obtained the distrubition parameters. It is shown that the ruled surface is developable if and only if the base curve is helix. In addition to, some results and theorems are presented with special cases. We obtained some characterizations on the ruled surfaces by using curvature. Moreover, some relationships among a symtotic curve and striction line of the base curve of the ruled surface have been found.

In this paper, we obtain a characterization of parallel curves of adjoint curves. In three-dimensional Euclidean space, the relationship between β adjoint of α curve and γ parallel of β curve are examined. Then, arc-length on γ curve are calculated. Curvature of parallel curve and curvature of the adjoint curve are calculated. Additionally, some results and theorems are presented with special cases. Moreover, α and γ curves Involute-Evolute, Bertrand and Mannheim curve pair situations are examined.

2 Preliminaries

In this section, we investigate α, β, γ curves. In three-dimensional Euclidean space, the relationship between β adjoint of α curve and γ parallel of β curve were examined. In addition, we give some properties of each curve. Also, we obtain some results and theorems about such special curve pairs as Bertrand, Involute-Evolute, Mannheim. We calculate the curvature and torsion of the curves.

Let $\mathbb{E}^3$ be a 3-dimensional Euclidean space provided with the metric given by

$$\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{E}^3$.

Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ is a unit speed curve with $\tau \neq 0$. Denote by $\{T_\alpha, N_\alpha, B_\alpha\}$ the moving Frenet frame along the curve $\alpha(s)$ in the space $\mathbb{E}^3$. The Frenet equations are

$$\begin{bmatrix} T'_\alpha(s) \\ N'_\alpha(s) \\ B'_\alpha(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_\alpha(s) & 0 \\ -\kappa_\alpha(s) & 0 & \tau_\alpha(s) \\ 0 & -\tau_\alpha(s) & 0 \end{bmatrix} \begin{bmatrix} T_\alpha(s) \\ N_\alpha(s) \\ B_\alpha(s) \end{bmatrix} \tag{1}$$

Adjoint curve of α is defined by

$$\beta(s) = \int_{s_0}^{s} B_\alpha(s) ds \tag{2}$$

Let $\beta : I \subset \mathbb{R} \to \mathbb{E}^3$ is a unit speed curve with $\tau > 0$. Denote by $\{T_\beta, N_\beta, B_\beta\}$ the moving Frenet frame along the curve $\beta(s)$ in the space $\mathbb{E}^3$. Differentiating the Equation (2), we have

$$\begin{aligned} T_\beta(s) &= B_\alpha(s) \\ N_\beta(s) &= -N_\alpha(s) \\ B_\beta(s) &= T_\alpha(s) \end{aligned} \tag{3}$$

and

$$\begin{aligned} \kappa_\beta(s) &= \tau_\alpha(s) \\ \tau_\beta(s) &= \kappa_\alpha(s) \end{aligned} \tag{4}$$

Let $\gamma = \gamma(s^*)$ be a unit speed curve with $\tau \neq 0$. Let β be parallel curve at distance r from a given curve γ. Then, we have the following equations

$$\langle \beta(s) - \gamma, \beta(s) - \gamma \rangle = r^2, \tag{5}$$

$$\langle T_\beta(s), \beta(s) - \gamma \rangle = 0, \tag{6}$$

$$\left\langle (T'_\beta(s), \beta(s) - \gamma) \right\rangle + \langle (T_\beta(s), T_\beta(s)) \rangle = 0. \tag{7}$$

Definition 2.1 *Let $M, N \subset \mathbb{E}^3$ are curves with (I, α), (I, β) coordinate neighbourhoods, respectively. $\lambda(s) \epsilon M$ and $\beta(s) \epsilon N$ are points corresponding to $s \epsilon I$. $\{T, N, B\}$ and $\{T^*, N^*, B^*\}$ are the Frenet frames of M and N curves, respectively. If $\{N, N^*\}$ is linear dependent for $\forall s \epsilon I$, (M, N) curve pair is called a Bertrand curve pair [4].*

Theorem 2.2 *Let (M, N) is a Bertrand curve pair with (I, α), (I, β) coordinate neighbourhoods, respectively. $d(\alpha(s), \beta(s))$ is constant for $\forall s \epsilon I$ [4].*

Theorem 2.3 *A angle between the unit tangent vectors in mutual points of Bertrand curves is constant [4].*

Definition 2.4 *Let $M \subset \mathbb{E}^3$ is curve with (I, α) coordinate neighbourhood and $N \subset \mathbb{E}^3$ curve with (I, β) coordinate neighbourhood. Let be $\{T, N, B\}$ and $\{T^*, N^*, B^*\}$ are Frenet frames of M and N curves, respectively. If tangent of α coincides normal of β, that is to say, if $\langle T(s), T^*(s) \rangle = 0$, β is involute of α and α is evolute of β [4].*

Theorem 2.5 *Let be $M \subset \mathbb{E}^3$ curve with (I, α) coordinate neighbourhood and $N \subset \mathbb{E}^3$ curve with (I, β) coordinate neighbourhood. Let be M and N are involute-evolute curve pair in $\mathbb{E}^3$. Then, $d(\alpha(s), \beta(s)) = |c - s|$, $\forall s \epsilon I$, c constant [4].*

Definition 2.6 *Let $\mathbb{E}^3$ be the 3-dimensional Eulidean space with the standart inner product $\langle , \rangle$. If there exists a corresponding relationship between the space curves M and N such that, at the corresponding points of curves, the principal normal lines of M coincides with the binormal lines of N, then M is called a Mannheim curve, and N is a Mannheim partner curve of M. The pair (M, N) is said to be a Mannheim pair [4].*

Theorem 2.7 *A space curve in $\mathbb{E}^3$ is a Mannheim curve if and only if its curvature κ and torsion τ satisfy the formula $\kappa = \lambda(\kappa^2 + \tau^2)$, where λ is a nonzero constant [4].*

3 Characterization of the Parallel Curve of the Adjoint Curve

Let $\beta(s)$ be a unit speed curve with $\tau > 0$. Explicit formulae for the point β depending on the parameter s can be obtained by using the Equation (1). Then, the Equation (6) with $\beta'=T_\beta$ implies

$$\beta(s) - \gamma = m_2 N_\beta + m_3 B_\beta, \tag{8}$$

for appropriate coefficients m_2, m_3.

Theorem 3.1 *Let $\beta=\beta(s)$ be a unit speed curve with $\tau \neq 0$. Then, we have the following parallel curve γ at the distance r from a given curve β.*

$$\gamma = \int_{s_0}^{s} T_\beta(s)\, ds \pm \sqrt{r^2 - \frac{1}{\kappa_\beta^2}} B_\beta + \frac{1}{\kappa_\beta} N_\beta, \tag{9}$$

where $B_\beta = B_\beta(s)$, $N_\beta = N_\beta(s)$, $\kappa_\beta = \kappa_\beta(s)$ can be calculated from Frenet formulae.

Proof: Evaluating the Equation (8) in the Equation (7) with $\beta' = \kappa_\beta N_\beta$, $\langle T_\beta(s), T_\beta(s)\rangle = 1$, we obtain $m_2\kappa_\beta + 1 = 0$. Moreover, $m_2^2 + m_3^2 = r^2$ follows from the Equation (5). If we replace $m_2 = -\frac{1}{\kappa_\beta}$ in this equality, we obtain $m_3 = \pm\sqrt{r^2 - \frac{1}{\tau_\alpha^2}}$. In the Equation (8), we replace m_2, m_3 and $\beta(s)$. Thus, we obtain characterization of γ.

Corollary 3.2 *Let $\beta = \beta(s)$ be a unit speed curve with $\tau \neq 0$. Then, we have the following parallel curve γ at the distance r from a given curve β which is the adjoint curve of α. Hence,*

$$\gamma = \int_{s_0}^{s} B_\alpha(s)\, ds \pm \sqrt{r^2 - \frac{1}{\tau_\alpha^2}} T_\alpha - \frac{1}{\tau_\alpha} N_\alpha, \tag{10}$$

where $B_\alpha = B_\alpha(s)$, $N_\alpha = N_\alpha(s)$, $\kappa_\alpha = \kappa_\alpha(s)$ can be calculated from Frenet formulae.

Proof: If we replace the Equation (3) and the Equation (4) in the Equation (9).

Theorem 3.3 *Let β be a unit speed curve parametrized by its arc-length s. The arc-length s^* along the parallel curve γ is given by differential equation.*

$$\frac{ds^*}{ds} = \sqrt{\left(m_3\tau_\beta - m_2'\right)^2 + \left(m_2\tau_\beta + m_3'\right)^2}, \tag{11}$$

where $m_2=m_2(s)$, $m_3=m_3(s)$, $\tau_\beta=\tau_\beta(s)$.

Proof: Since β is the parallel curve at the distance r for a given curve γ, the curves (β and γ) satisfy the Equation (5), the Equation (6) and the Equation (7). From the Equation (9),

$$\frac{d\gamma}{ds^*}\cdot\frac{ds^*}{ds} = (1+m_2\kappa_\beta)\,T_\beta + (m_3\tau_\beta - m_2')\,N_\beta - \,(m_2\tau_\beta + m_3')\,B_\beta$$

Then,

$$\left\|\frac{d\gamma}{ds^*}.\frac{ds^*}{ds}\right\| = \frac{ds^*}{ds} = \sqrt{(1+m_2\kappa_\beta)^2 + (m_3\tau_\beta - m_2')^2 + (m_2\tau_\beta + m_3')^2}.$$

We substitute $m_2{=}-\frac{1}{\kappa_\beta}$ in this norm.

Corollary 3.4 *Let β be a unit speed curve parametrized by its arc-length s. The arc-length s^* along the parallel curve γ is given by from the Equation (11). β is the adjoint curve of α. Hence,*

$$\frac{ds^*}{ds} = \sqrt{(m_3\kappa\alpha - m_2')^2 + (m_2\kappa\alpha - m_3')^2}, \tag{12}$$

where $m_2{=}m_2(s)$, $m_3{=}m_3(s)$ $\tau_\alpha = \tau_\alpha(s)$.

Proof: If we replace the Equation (3) and the Equation (4) in the Equation (11).

Theorem 3.5 *Let β be a unit speed adjoint curve of α. If β is helix, then γ is helix as well. In this case, the following equations given for $\kappa_\gamma = \kappa_\gamma(s^*)$ and $\tau_\gamma = \tau_\gamma(s^*)$ are true.*

$$\kappa_\gamma = \frac{1}{r^2\kappa_\alpha^2}\sqrt{\tau_\alpha^2 m_3^2\kappa_\alpha^2 + (m_3\kappa_\alpha' + m_2\kappa_\alpha^2)^2 + (m_3\kappa_\alpha^2 - m_2\kappa_\alpha')^2}, \tag{13}$$

$$\tau_\gamma = \frac{1}{r\kappa_\alpha\begin{pmatrix}(m_3^2\tau_\alpha^2\kappa_\alpha^2)+\\(m_3\kappa_\alpha' + m_2\kappa_\alpha^2)^2+\\(m_3\kappa_\alpha^2 - m_2\kappa_\alpha')^2\end{pmatrix}}\begin{bmatrix}(-\tau_\alpha' m_3\kappa_\alpha^2 - 2\tau_\alpha m_3\kappa_\alpha' -\\ \tau_\alpha m_2\kappa_\alpha^2)r^2\kappa_\alpha^3 +\\ (-\tau - \alpha^2 m_3\kappa_\alpha + m_3\kappa_\alpha'' +\\ 3m_2\kappa_\alpha\kappa - \alpha' -\\ m_3\kappa_\alpha^3)m_2 m_3\kappa_\alpha^2\tau_\alpha +\\ (3m_3\kappa_\alpha\kappa_\alpha' + m_2\kappa_\alpha^3\\ m_2\kappa_\alpha'')m_3^2\kappa_\alpha^2\tau_\alpha\end{bmatrix} \tag{14}$$

Proof: Since α, β and γ are arc parameter curves, $\kappa_\gamma = \|\gamma''\|$, $\tau_\gamma = \langle N_\gamma', B_\gamma\rangle$.

If the necessary actions are done,

$$\gamma'' = (\frac{ds}{ds^*})^2 \begin{bmatrix} (-\kappa_\beta m_3 \tau_\beta - m_2')T_\beta + \\ (2m_3'\tau_\beta + m_3\tau_\beta' - m_2'' + m_2\tau_\beta^2)N_\beta + \\ (m_3\tau_\beta^2 - 2m_2'\tau_\beta - m_2\tau_\beta' - m_3'')B_\beta \end{bmatrix}$$

and

$$\kappa_\gamma = \|\gamma''\| = (\frac{ds}{ds^*})^2 \sqrt{\begin{matrix} \kappa_\beta^2(m_3\tau_\beta - m_2')^2 + \\ (2m_3'\tau_\beta + m_3\tau_\beta' - m_2'' + m_2\tau_\beta^2)^2 + \\ (m_3\tau_\beta^2 - 2m_2'\tau_\beta - m_2\tau_\beta' - m_3'')^2 \end{matrix}}$$

Since β is the adjoint of α, the Equation (3) and the Equation (4) are satisfied. We substitute the Equation (3), the Equation (4) and $\frac{ds}{ds^*} = \frac{1}{r\tau_\beta}$ in the norm of γ'' and find $\kappa_\gamma = \|\gamma''\|$. In order to compute τ_γ, we find $N_\gamma = \frac{\gamma''}{\|\gamma''\|}$ and compute inner product of N_γ' and B_γ. This inner product gives τ_γ. Here, for convenience, m_2 and m_3 functions are constant functions. Hence, $m_2' = 0$ and $m_3' = 0$.

Corollary 3.6 *Let β is the adjoint of α and γ the parallel of β. γ and α are a involute-evolute curve pair if and only if $r = \mp\frac{1}{\tau_\alpha}$ or $\kappa_\alpha = 0$, where $\tau_\alpha \neq 0$.*

Proof: We denote that T_γ, N_γ, B_γ, κ_γ, τ_γ and T_α, N_α, B_α, κ_α, τ_α are the Frenet equipments of γ and α, respectively. If γ and α are a involute-evolute curve pair, $\langle T_\gamma, T_\alpha \rangle = 0$.

$$T_\gamma = \gamma' = ((1 + m_2\kappa_\beta)\, T_\beta + m_3\tau_\beta N_\beta - m_2\tau_\beta B_\beta)\, \tfrac{ds^*}{ds}.$$

Since β is the adjoint of α, the Equation (3) and the Equation (4) are satisfied. Thus,

$$T_\gamma = \gamma' = ((1 + m_2\tau_\alpha)\, B_\alpha - m_3\kappa_\alpha N_\alpha - m_2\kappa_\alpha T_\alpha)\, \tfrac{ds^*}{ds}.$$

If we replace T_γ in the inner product of T_γ and T_α, we find out $r = \mp\frac{1}{\tau_\alpha}$ or $\kappa_\alpha = 0$. Here, for convenience, m_2 and m_3 functions are constant functions. As a result, $m_2' = 0$ and $m_3' = 0$.

Corollary 3.7 *Let β is the adjoint of α and γ be the parallel of β. γ and α are a Bertrand curve pair if and only if $\frac{\kappa_\alpha^2}{\tau_\alpha} \pm \sqrt{r^2 - \frac{1}{\tau_\alpha^2}}\kappa_\alpha' = \lambda$, where λ is constant number.*

Proof: We denote that T_γ, N_γ, B_γ, κ_γ, τ_γ and T_α, N_α, B_α, κ_α, τ_α are the Frenet equipments of γ and α, respectively. If γ and α are a Bertrand curve pair, there $\langle N_\gamma, N_\alpha \rangle = \lambda$ and these curves are linearly dependent. We compute $T_\gamma = \gamma'$ and $N_\gamma = \gamma''$ and the inner product of N_γ and N_α. If this inner product equal λ, γ and α are a Bertrand curve pair.

$$T_\gamma = \gamma' = \left((1 + m_2\kappa_\beta) T_\beta + m_3\tau_\beta N_\beta - m_2\tau_\beta B_\beta \right) \tfrac{ds^*}{ds}$$

and

$$N_\gamma = \gamma'' = \begin{pmatrix} -m_2\kappa'_\alpha T_\alpha - m_2\kappa_\alpha (\kappa_\alpha N_\alpha) - m_3\kappa'_\alpha N_\alpha - \\ m_3\kappa_\alpha (-\kappa_\alpha T_\alpha + \tau_\alpha B_\alpha) + m_2\tau'_\alpha B_\alpha + (1 + m_2\tau_\alpha)(-\tau_\alpha N_\alpha) \end{pmatrix} \tfrac{ds^*}{ds}.$$

Here, for convenience, m_2 and m_3 functions are constant functions. Hence, $m'_2 = 0$ and $m'_3 = 0$.

Corollary 3.8 *Let β is the adjoint of α and γ be the parallel of β. γ and α are a Mannheim curve pair if and only if $\frac{\kappa_\alpha\tau'_\alpha}{\tau_\alpha^2} \mp \sqrt{r^2 - \frac{1}{\tau_\alpha^2}}\kappa_\alpha^2 = \lambda$, where λ is constant number.*

Proof: We denote that T_γ, N_γ, B_γ, κ_γ, τ_γ and T_α, N_α, B_α, κ_α, τ_α are the Frenet equipments of γ and α, respectively. If γ and α are a Mannheim curve pair, there $\langle N_\alpha, B_\gamma \rangle = \lambda$ and these curves are linearly dependent. We compute $T_\gamma = \gamma', N_\gamma = \gamma'', B_\gamma = T_\gamma \times N_\gamma$ and the inner product of N_α and B_γ. If the inner product equal λ, γ and α are a Mannheim curve pair.

$\langle N_\alpha, B_\gamma \rangle = \lambda \Rightarrow m_2^2\kappa_\alpha\tau'_\alpha - m_2\kappa'_\alpha + m_3\kappa_\alpha^2 - m_2^2\kappa'_\alpha\tau_\alpha = \lambda$ where we substitute $m_2 = -\frac{1}{\kappa_\beta}$ and $m_3 = \pm\sqrt{r^2 - \frac{1}{\tau_\alpha^2}}$.

Here, for convenience, m_2 and m_3 functions are fixed functions. Hence, $m'_2 = 0$ and $m'_3 = 0$.

4 Conclusion

In this study, we show that is obtained some characterizations of parallel curves of adjoint curves. Also, we examined Involute-Evolute, Bertrand and Mannheim curve pair situations of α and γ curves. The relationship between β adjoint curve of α curve and γ parallel curve of β curve are examined. Moreover, we calculate curvature and torsion of γ parallel curve in terms of α and β curves.

In consideration of this knowledge, one can showed different characterizations of parallel curves of the other curves.

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