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A Single Formula for Integer Powers of Certain Real Circulant Matrix of Odd and Even Order

Ahmet Öteleş[1] and Mehmet Akbulak[2]

[1]Department of Mathematics, Faculty of Education
Dicle University, TR-21280, Diyarbakir, Turkey
E-mail: aoteles85@gmail.com
[2]Department of Mathematics, Faculty of Art and Science
Siirt University, TR-56100, Siirt, Turkey
E-mail: makbulak@gmail.com

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Abstract

In this paper, we derive a single formula for the entries of the rth ($r \in \mathbb{N}$) power of a certain real circulant matrix of odd and even order, in terms of the Chebyshev polynomials of the first and second kind. In addition, we give two Maple 13 procedures along with some numerical examples in order to verify our calculation.

Keywords: *Circulant matrix, Chebyshev polynomial, Eigenvalue, Eigenvector, Jordan's form.*

1 Introduction

A certain type of transformation of a set of numbers can be represented as the multiplication of a vector by a square matrix. Repetition of the operation is equivalent to multiplying the original vector by a power of the matrix. Solving some difference equations, differential and delay differential equations and boundary value problems, we need to compute the arbitrary integer powers of a square matrix. Properties of powers of matrices are thus of considerable importance [1-3].

One can find the rth power ($r \in \mathbb{N}$) of an $n \times n$ matrix A using the well-known expression

$$A^r = PJ^rP^{-1} \text{ [4]}, \tag{1}$$

where J is the Jordan's form of A, and P is the transforming matrix. Matrices J and P can be found by the help of eigenvalues and eigenvectors of the matrix A.

An $n \times n$ circulant matrix $C_n := circ_n(c_0, c_1, \dots, c_{n-1})$ is a square matrix having the form

$$C_n := \begin{bmatrix} c_0 & c_1 & c_2 & \dots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \ddots & & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & c_2 \\ c_2 & & \ddots & \ddots & \ddots & c_1 \\ c_1 & c_2 & \dots & c_{n-2} & c_{n-1} & c_0 \end{bmatrix},$$

where each row is a cyclic shift of the row above it.

Circulant matrices have a wide range of applications; for examples, in graph theory, signal processing, coding theory and image processing, etc. Numerical solutions of certain types of elliptic and parabolic partial differential equations with periodic boundary conditions often involve linear systems associated with circulant matrices [5-7].

In recent years, computing the integer powers of circulant matrices has been a very popular problem by using equation (1). For instance, Rimas derived a general expression for the entries of the rth power ($r \in \mathbb{N}$) of the $n \times n$ real symmetric circulant $circ_n(0, 1, 0, \dots, 0, 1)$ depending on the Chebyshev polynomials (see [8] or [9] for the odd case and [10] or [11] for the even case).

In [12], Gutiérrez derived a single formula by generalizing the results obtained [8] and [9] for the entries of the positive integer powers of complex symmetric circulant matrix of odd and even order given as

$$\begin{array}{ll} circ_n\left(b_0, b_1, \dots, b_{\frac{n-1}{2}}, b_{\frac{n-1}{2}}, \dots, b_1\right)^T & \text{if } n \text{ is odd}, \\ circ_n\left(b_0, b_1, \dots, b_{\frac{n}{2}-1}, b_{\frac{n}{2}-1}, \dots, b_1\right)^T & \text{if } n \text{ is even}, \end{array}$$

and in [13], he also derived two separate formulas for the entries of the positive integer powers of complex skew-symmetric circulant matrix of odd and even order given as

$$\begin{array}{ll} circ_n\left(0, b_1, \dots, b_{\frac{n-1}{2}}, -b_{\frac{n-1}{2}}, \dots, -b_1\right)^T & \text{if } n \text{ is odd}, \\ circ_n\left(0, b_1, \dots, b_{\frac{n}{2}-1}, -b_{\frac{n}{2}-1}, \dots, -b_1\right)^T & \text{if } n \text{ is even}. \end{array}$$

In [14], Köken and Bozkurt derived a general expression for the entries of the rth power ($r \in \mathbb{N}$) of odd order complex circulant matrices of the type $circ_n(0, a, 0, \ldots, 0, b)$ depending on the Chebyshev polynomials.

In this paper, we present a single formula for the entries of the rth ($r \in \mathbb{N}$) power of the real circulant matrix $A_n := circ_n(a_0, a_1, 0, \ldots, 0, a_{-1})$ of odd and even order depending on the Chebyshev polynomials. Namely,

$$A_n := \begin{bmatrix} a_0 & a_1 & 0 & \ldots & 0 & a_{-1} \\ a_{-1} & a_0 & a_1 & \ddots & & 0 \\ 0 & a_{-1} & a_0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & \ddots & a_1 \\ a_1 & 0 & \ldots & 0 & a_{-1} & a_0 \end{bmatrix}. \tag{2}$$

This study is an extension of the one obtained in [14] for the powers of the matrix $circ_n(0, b, 0, \ldots, 0, a)$ of odd order ($n \in \mathbb{N}$).

2 General Expression for the Entries of A_n^r

The well-known eigenvalue decomposition of an $n \times n$ circulant matrix (see [15]) is that

$$circ_n(c_0, c_1, \ldots, c_{n-1}) = F_n D_n F_n^*, \tag{3}$$

where * denotes conjugate transpose (i.e $F_n^* = \overline{F}_n^T$), F_n is the $n \times n$ Fourier matrix:

$$[F_n]_{u,v} = \frac{1}{\sqrt{n}} e^{-\frac{2\pi(u-1)(v-1)}{n}\mathbf{i}}, \quad 1 \le u, v \le n$$

and $D_n = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with

$$\lambda_k = \sum_{r=1}^{n} c_{r-1} e^{-\frac{2\pi(k-1)(r-1)}{n}\mathbf{i}}, \quad 1 \le k \le n. \tag{4}$$

It is also known that the matrix F_n is unitary (see [15]).

Let $U_k(x)$ be the kth degree Chebyshev polynomial of the second kind:

$$U_k(x) = \frac{\sin((k+1)\arccos x)}{\sin \arccos x}, \quad -1 \le x \le 1 \tag{5}$$

and $T_m(x)$ is the kth degree Chebyshev polynomial of the first kind, with $k \in \mathbb{N} \cup \{0\}$:

$$T_k(x) = \cos(k \arccos x), \quad -1 \le x \le 1. \tag{6}$$

Theorem 2.1. *Let $A_n = circ_n(a_0, a_1, 0, \ldots, 0, a_{-1})$ be an $n\times n$ real circulant matrix and $\alpha_k = \cos\frac{2\pi(k-1)}{n}$, $1 \le k \le n$, $3 \le n \in \mathbb{N}$. Then (u,v)th entry of A_n^r is given by:*

$$[A_n^r]_{u,v} = \frac{1}{n}(S_1 + S_2)$$

for all $r \in \mathbb{N}$ and $1 \le u, v \le n$, where S_1 and S_2 are respectively that

$$\begin{aligned} S_1 &= \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor + 1} \left(a_0 + (a_{-1} + a_1) T_1(\alpha_k) + (a_{-1} - a_1) \,\mathbf{sign}\left(\frac{n}{2} + 1 - k\right) \sqrt{1-\alpha_k^2}\,\mathbf{i} \lim_{\mathbf{j}\to\mathbf{k}} U_0(\alpha_j) \right)^r \\ &\quad \times \left(T_{|v-u|}(\alpha_k) + \mathbf{sign}(v-u)\,\mathbf{sign}\left(\frac{n}{2} + 1 - k\right) \sqrt{1-\alpha_k^2}\,\mathbf{i} \lim_{\mathbf{j}\to\mathbf{k}} U_{|v-u|-1}(\alpha_j) \right), \end{aligned}$$

and

$$\begin{aligned} S_2 &= \sum_{k=2}^{\lfloor \frac{n+1}{2} \rfloor} \left(a_0 + (a_{-1} + a_1) T_1(\alpha_k) - (a_{-1} - a_1) \,\mathbf{sign}\left(\frac{n}{2} + 1 - k\right) \sqrt{1-\alpha_k^2}\,\mathbf{i} \lim_{\mathbf{j}\to\mathbf{k}} U_0(\alpha_j) \right)^r \\ &\quad \times \left(T_{|v-u|}(\alpha_k) - \mathbf{sign}(v-u)\,\mathbf{sign}\left(\frac{n}{2} + 1 - k\right) \sqrt{1-\alpha_k^2}\,\mathbf{i} U_{|v-u|-1}(\alpha_k) \right). \end{aligned}$$

Here $\lfloor x \rfloor$ is the largest integer less than or equal to x.

Proof: By using (3), we get

$$\begin{aligned} [A_n^r]_{u,v} &= [(F_n D_n F_n^*)^r]_{u,v} = [F_n D_n^r F_n^*]_{u,v} \\ &= \sum_{k=1}^{n} [F_n]_{u,k} [D_n^r F_n^*]_{k,v} = \sum_{k=1}^{n} [F_n]_{u,k} \lambda_k^r \overline{[F_n]_{v,k}}. \end{aligned}$$

Hence,

$$[A_n^r]_{u,v} = \frac{1}{n} \sum_{k=1}^{n} \lambda_k^r e^{-\frac{2\pi(u-1)(k-1)}{n}\mathbf{i}} e^{\frac{2\pi(v-1)(k-1)}{n}\mathbf{i}} = \frac{1}{n} \sum_{k=1}^{n} \lambda_k^r e^{\frac{2\pi(k-1)(v-u)}{n}\mathbf{i}}. \quad (7)$$

From (4), and $e^{-\frac{2\pi(k-1)(r-1)}{n}} = e^{\frac{2\pi(k-1)(n+2-r-1)}{n}\mathbf{i}}$ for all $2 \le r \le n$ (see [13]), we can write λ_k as

$$\begin{aligned} \lambda_k &= a_0 + a_1 e^{-\frac{2\pi(k-1)}{n}\mathbf{i}} + a_{-1} e^{\frac{2\pi(k-1)}{n}\mathbf{i}} \\ &= a_0 + (a_{-1} + a_1) \cos\frac{2\pi(k-1)}{n} + (a_{-1} - a_1)\,\mathbf{i} \sin\frac{2\pi(k-1)}{n}. \end{aligned} \quad (8)$$

Observe that from (5) and (6), we have

$$T_m\left(\cos\frac{2\pi(k-1)}{n}\right) = \cos\frac{2\pi(k-1)m}{n},$$

and

$$U_{m-1}\left(\cos\frac{2\pi(k-1)}{n}\right)=\frac{\sin\frac{2\pi(k-1)m}{n}}{\sin\frac{2\pi(k-1)}{n}}. \tag{9}$$

In (9), we have the indeterminate form 0/0 for $k=1$ and $k=\frac{n}{2}+1$. So we can write (8) as

$$\lambda_k=a_0+(a_{-1}+a_1)\,T_1\left(\cos\frac{2\pi(k-1)}{n}\right)+(a_{-1}-a_1)\sin\frac{2\pi(k-1)}{n}\mathbf{i}\lim_{\mathbf{j}\to\mathbf{k}}U_0\left(\cos\frac{2\pi(j-1)}{n}\right). \tag{10}$$

Since $\alpha_k=\cos\frac{2\pi(k-1)}{n}$ and

$$\sin\frac{2\pi(k-1)}{n}=\begin{cases}\sqrt{1-\alpha_k^2} & \text{if } \frac{n}{2}+1-k>0,\\ -\sqrt{1-\alpha_k^2} & \text{if } \frac{n}{2}+1-k<0,\end{cases} \tag{11}$$

λ_k is obtained as

$$\lambda_k=a_0+(a_{-1}+a_1)\,T_1(\alpha_k)+(a_{-1}-a_1)\,\mathbf{sign}\left(\frac{n}{2}+1-k\right)\sqrt{1-\alpha_k^2}\mathbf{i}\lim_{\mathbf{j}\to\mathbf{k}}U_0(\alpha_j). \tag{12}$$

From (12), we can write $\overline{\lambda_k}=\lambda_{n+2-k}$ for $2\le k\le n$. Namely,

$$\begin{aligned} D_n &= diag\left(\lambda_1,\lambda_2,\ldots,\lambda_{\frac{n+1}{2}},\overline{\lambda_{\frac{n+1}{2}}},\ldots,\overline{\lambda_2}\right) && \text{if } n \text{ is odd},\\ D_n &= diag\left(\lambda_1,\lambda_2,\ldots,\lambda_{\frac{n}{2}},\lambda_{\frac{n}{2}+1},\overline{\lambda_{\frac{n}{2}}},\ldots,\overline{\lambda_2}\right) && \text{if } n \text{ is even}.\end{aligned}$$

By using (7), $[A_n^r]_{u,v}$ can be expressed as

$$[A_n^r]_{u,v}=\begin{cases}\frac{1}{n}\left[\lambda_1^r+\sum\limits_{k=2}^{\frac{n+1}{2}}\lambda_k^r e^{\frac{2\pi(k-1)(v-u)}{n}\mathbf{i}}+\sum\limits_{k=\frac{n+1}{2}+1}^{n}\lambda_k^r e^{\frac{2\pi(k-1)(v-u)}{n}\mathbf{i}}\right] & \text{if } n \text{ is odd},\\ \frac{1}{n}\left[\lambda_1^r+\sum\limits_{k=2}^{\frac{n}{2}+1}\lambda_k^r e^{\frac{2\pi(k-1)(v-u)}{n}\mathbf{i}}+\sum\limits_{k=\frac{n}{2}+2}^{n}\lambda_k^r e^{\frac{2\pi(k-1)(v-u)}{n}\mathbf{i}}\right] & \text{if } n \text{ is even},\end{cases}$$

and so,

$$[A_n^r]_{u,v}=\begin{cases}\frac{1}{n}\left[\lambda_1^r+\sum\limits_{k=2}^{\frac{n+1}{2}}\lambda_k^r e^{\frac{2\pi(k-1)(v-u)}{n}\mathbf{i}}+\sum\limits_{k=2}^{\frac{n+1}{2}}\lambda_{n+2-k}^r e^{\frac{2\pi(n+2-k-1)(v-u)}{n}\mathbf{i}}\right] & \text{if } n \text{ is odd},\\ \frac{1}{n}\left[\lambda_1^r+\sum\limits_{k=2}^{\frac{n}{2}}\lambda_k^r e^{\frac{2\pi(k-1)(v-u)}{n}\mathbf{i}}+\sum\limits_{k=2}^{\frac{n}{2}}\lambda_{n+2-k}^r e^{\frac{2\pi(n+2-k-1)(v-u)}{n}\mathbf{i}}+\lambda_{\frac{n}{2}+1}^r e^{\pi\mathbf{i}(v-u)}\right] & \text{if } n \text{ is even}.\end{cases} \tag{13}$$

Observe that $\overline{\lambda_k}=\lambda_{n+2-k},\ 2\le k\le n$ and $e^{\frac{2\pi(n+2-k-1)(v-u)}{n}\mathbf{i}}=e^{-\frac{2\pi(k-1)(v-u)}{n}\mathbf{i}}$ for all $2\le k\le n$. Consequently, from (13) we deduce that

$$[A_n^r]_{u,v}=\begin{cases}\frac{1}{n}\left[\sum\limits_{k=1}^{\frac{n+1}{2}}\lambda_k^r e^{\frac{2\pi(k-1)(v-u)}{n}\mathbf{i}}+\sum\limits_{k=2}^{\frac{n+1}{2}}\overline{\lambda_k}^r e^{-\frac{2\pi(k-1)(v-u)}{n}\mathbf{i}}\right] & \text{if } n \text{ is odd},\\ \frac{1}{n}\left[\sum\limits_{k=1}^{\frac{n}{2}+1}\lambda_k^r e^{\frac{2\pi(k-1)(v-u)}{n}\mathbf{i}}+\sum\limits_{k=2}^{\frac{n}{2}}\overline{\lambda_k^r} e^{-\frac{2\pi(k-1)(v-u)}{n}\mathbf{i}}\right] & \text{if } n \text{ is even}.\end{cases}$$

Thus,

$$[A_n^r]_{u,v} = \frac{1}{n}\left[\sum_{k=1}^{\left\lfloor \frac{n}{2}\right\rfloor+1} \lambda_k^r e^{\frac{2\pi(k-1)(v-u)}{n}\mathbf{i}} + \sum_{k=2}^{\left\lfloor \frac{n+1}{2}\right\rfloor} \overline{\lambda_k}^r e^{-\frac{2\pi(k-1)(v-u)}{n}\mathbf{i}}\right] = \frac{1}{n}\left(S_1+S_2\right),$$

where

$$S_1 = \sum_{k=1}^{\left\lfloor \frac{n}{2}\right\rfloor+1} \lambda_k^r e^{\frac{2\pi(k-1)(v-u)}{n}\mathbf{i}},$$

and

$$S_2 = \sum_{k=2}^{\left\lfloor \frac{n+1}{2}\right\rfloor} \overline{\lambda_k}^r e^{-\frac{2\pi(k-1)(v-u)}{n}\mathbf{i}}.$$

Consequently,

$$S_1 = \sum_{k=1}^{\left\lfloor \frac{n}{2}\right\rfloor+1} \lambda_k^r \left(\cos\frac{2\pi(k-1)(v-u)}{n} + \mathbf{i}\sin\frac{2\pi(k-1)(v-u)}{n}\right),$$

and

$$S_2 = \sum_{k=2}^{\left\lfloor \frac{n+1}{2}\right\rfloor} \overline{\lambda_k}^r \left(\cos\frac{2\pi(k-1)(v-u)}{n} - \mathbf{i}\sin\frac{2\pi(k-1)(v-u)}{n}\right).$$

Notice that (see [13])

$$U_{|v-u|-1}\left(\cos\frac{2\pi(k-1)}{n}\right) = sign\,(v-u)\,\frac{\sin\frac{2\pi(k-1)(v-u)}{n}}{\sin\frac{2\pi(k-1)}{n}} \tag{14}$$

and

$$T_{|v-u|}\left(\cos\frac{2\pi(k-1)}{n}\right) = \cos\frac{2\pi(k-1)(v-u)}{n}.$$

In (14), we have the indeterminate form 0/0 for $k=1$ and $k=\frac{n}{2}+1$ again. Then we can write S_1 as

$$S_1 = \sum_{k=1}^{\left\lfloor \frac{n}{2}\right\rfloor+1} \lambda_k^r \left(T_{|v-u|}\left(\cos\frac{2\pi(k-1)}{n}\right) + \mathbf{sign}\,(v-u)\sin\frac{2\pi(k-1)}{n}\mathbf{i}\lim_{\mathbf{j}\to\mathbf{k}} U_{|v-u|-1}\left(\cos\frac{2\pi(j-1)}{n}\right)\right),$$

and S_2 as

$$S_2 = \sum_{k=2}^{\left\lfloor \frac{n+1}{2}\right\rfloor} \overline{\lambda_k}^r \left(T_{|v-u|}\left(\cos\frac{2\pi(k-1)}{n}\right) - \mathbf{sign}\,(v-u)\sin\frac{2\pi(k-1)}{n}\mathbf{i}U_{|v-u|-1}\left(\cos\frac{2\pi(k-1)}{n}\right)\right).$$

The theorem follows taken (11), (12) and $\alpha_k = \cos\frac{2\pi(k-1)}{n}$ into account.

Theorem 2.2. *Let* $A_n = circ_n(a_0, a_1, 0, \dots, 0, a_1)$ *be an* $n \times n$ *real symmetric circulant matrix and* $\alpha_k = \cos\frac{2\pi(k-1)}{n}$, $1 \le k \le n$, $3 \le n \in \mathbb{N}$. *Then (u,v)th entry of* A_n^r *is given by:*

$$[A_n^r]_{u,v} = \frac{1}{n}\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor + 1} l_{n-2k+2}\left(a_0 + 2a_1 T_1(\alpha_k)\right)^r T_{|v-u|}(\alpha_k).$$

where $\lfloor x \rfloor$ *is the largest integer less than or equal to* x, *and*

$$l_s = \begin{cases} 1 & \text{if } s = 0, n \\ 2 & \text{otherwise.} \end{cases}$$

Proof: From Theorem 2.1, we obtain that

$$S_1 = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor + 1} \left(a_0 + 2a_1 T_1(\alpha_k)\right)^r \left(T_{|v-u|}(\alpha_k) + \mathbf{sign}(v-u)\,\mathbf{sign}\left(\frac{n}{2} + 1 - k\right)\sqrt{1-\alpha_k^2}\,\mathbf{i}\lim_{\mathbf{j}\to\mathbf{k}} U_{|v-u|-1}(\alpha_j)\right),$$

and

$$S_2 = \sum_{k=2}^{\lfloor \frac{n+1}{2} \rfloor} \left(a_0 + 2a_1 T_1(\alpha_k)\right)^r \left(T_{|v-u|}(\alpha_k) - \mathbf{sign}(v-u)\,\mathbf{sign}\left(\frac{n}{2} + 1 - k\right)\sqrt{1-\alpha_k^2}\,\mathbf{i}U_{|v-u|-1}(\alpha_k)\right).$$

Since

$$\left\lfloor \frac{n+1}{2} \right\rfloor = \frac{n+1}{2} \text{ and } \left\lfloor \frac{n}{2} \right\rfloor + 1 = \frac{n+1}{2} \qquad \text{if } n \text{ is odd,}$$

and

$$\left\lfloor \frac{n+1}{2} \right\rfloor = \frac{n}{2} \text{ and } \left\lfloor \frac{n}{2} \right\rfloor + 1 = \frac{n}{2} + 1 \qquad \text{if } n \text{ is even,}$$

then

$$\begin{aligned}
[A_n^r]_{u,v} &= \frac{1}{n}(S_1 + S_2) \\
&= \begin{cases} \frac{1}{n}\left[(a_0 + 2a_1)^r + 2\sum_{k=2}^{\frac{n+1}{2}} \left(a_0 + 2a_1 T_1(\alpha_k)\right)^r T_{|v-u|}(\alpha_k)\right] & \text{if } n \text{ is odd,} \\ \frac{1}{n}\Big[(a_0 + 2a_1)^r + 2\sum_{k=2}^{\frac{n}{2}} \left(a_0 + 2a_1 T_1(\alpha_k)\right)^r T_{|v-u|}(\alpha_k) + & \\ \quad \left(a_0 + 2a_1 T_1\left(\alpha_{\frac{n}{2}+1}\right)\right)^r T_{|v-u|}\left(\alpha_{\frac{n}{2}+1}\right)\Big] & \text{if } n \text{ is even.} \end{cases}
\end{aligned}$$

Therefore,

$$[A_n^r]_{u,v} = \frac{1}{n}\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor + 1} l_{n-2k+2}\left(a_0 + 2a_1 T_1(\alpha_k)\right)^r T_{|v-u|}(\alpha_k),$$

and the proof is completed.

By using Theorem 2.2, we can easily obtain the expression given by Rimas for the entries of the powers of the matrix $A_n = circ_n(0, 1, 0, \dots, 0, 1)$ (see [8] or [9] for the odd case and [10] or [11] for the even case).

3 Numerical Considerations

In this section, we give some examples related to Theorem 2.1 and Theorem 2.2. These examples can be verified by using Maple 13 procedures given in Appendix A and Appendix B.

Example 3.1. *Let A_5 be a 5×5 circulant matrix given in (2)*

$$A_5 = \begin{bmatrix} -1 & 2 & 0 & 0 & 5 \\ 5 & -1 & 2 & 0 & 0 \\ 0 & 5 & -1 & 2 & 0 \\ 0 & 0 & 5 & -1 & 2 \\ 2 & 0 & 0 & 5 & -1 \end{bmatrix},$$

where $a_0 = -1,\ a_1 = 2,\ a_{-1} = 5$ and $\alpha_k = \cos\frac{2\pi(k-1)}{5}$. By using (12), we have

$$\begin{aligned} \lambda_1 &= 6, \\ \lambda_2 &= 1.16311 + 2.85316\mathbf{i}, \\ \lambda_3 &= -6.66311 + 1.76335\mathbf{i}, \\ \lambda_4 = \overline{\lambda_3} &= -6.66311 - 1.76335\mathbf{i}, \\ \lambda_5 = \overline{\lambda_2} &= 1.16311 - 2.85316\mathbf{i}, \end{aligned}$$

From Theorem 2.1, (u,v)th entry of A_5^4 is that

$$\left[A_5^4\right]_{u,v} = \frac{1}{5}\left(S_1 + S_2\right),$$

where

$$S_1 = \sum_{k=1}^{3} \lambda_k^4 \left(T_{|v-u|}\left(\alpha_k\right) + \mathbf{sign}\left(v-u\right)\mathbf{sign}\left(\frac{7}{2}-k\right)\sqrt{1-\alpha_k^2}\,\mathbf{i}\lim_{\mathbf{j}\to\mathbf{k}} U_{|v-u|-1}\left(\alpha_j\right)\right),$$

and

$$S_2 = \sum_{k=2}^{3} \overline{\lambda_k}^4 \left(T_{|v-u|}\left(\alpha_k\right) - \mathbf{sign}\left(v-u\right)\mathbf{sign}\left(\frac{7}{2}-k\right)\sqrt{1-\alpha_k^2}\,\mathbf{i}\, U_{|v-u|-1}\left(\alpha_k\right)\right).$$

Therefore,

$$A_5^4 = \begin{bmatrix} 721 & 377 & -316 & 1118 & -604 \\ -604 & 721 & 377 & -316 & 1118 \\ 1118 & -604 & 721 & 377 & -316 \\ -316 & 1118 & -604 & 721 & 377 \\ 377 & -316 & 1118 & -604 & 721 \end{bmatrix}.$$

Example 3.2. *Let A_6 be a 6×6 circulant matrix given in (2)*

$$A_6 = \begin{bmatrix} 2 & 4 & 0 & 0 & 0 & -1 \\ -1 & 2 & 4 & 0 & 0 & 0 \\ 0 & -1 & 2 & 4 & 0 & 0 \\ 0 & 0 & -1 & 2 & 4 & 0 \\ 0 & 0 & 0 & -1 & 2 & 4 \\ 4 & 0 & 0 & 0 & -1 & 2 \end{bmatrix},$$

where $a_0 = 2,\ a_1 = 4,\ a_{-1} = -1,$ and $\alpha_k = \cos\frac{\pi(k-1)}{3}$. By using (12), we have

$$\begin{aligned} \lambda_1 &= 5, \\ \lambda_2 &= 3.5000 - 4.33012\mathbf{i}, \\ \lambda_3 &= 0.5000 - 4.33012\mathbf{i}, \\ \lambda_4 &= -1, \\ \lambda_5 = \overline{\lambda_3} &= 0.5000 + 4.33012\mathbf{i}, \\ \lambda_6 = \overline{\lambda_2} &= 3.5000 + 4.33012\mathbf{i}, \end{aligned}$$

From Theorem 2.1, (u,v)th entry of A_6^3 is that

$$\left[A_6^3\right]_{u,v} = \frac{1}{6}\left(S_1 + S_2\right),$$

where

$$S_1 = \sum_{k=1}^{4} \lambda_k^3 \left(T_{|v-u|}\left(\alpha_k\right) + \mathbf{sign}\left(v-u\right)\mathbf{sign}\left(4-k\right)\sqrt{1-\alpha_k^2}\mathbf{i}\lim_{\mathbf{j}\to\mathbf{k}} U_{|v-u|-1}\left(\alpha_j\right) \right),$$

and

$$S_2 = \sum_{k=2}^{3} \overline{\lambda_k}^3 \left(T_{|v-u|}\left(\alpha_k\right) - \mathbf{sign}\left(v-u\right)\mathbf{sign}\left(4-k\right)\sqrt{1-\alpha_k^2}\mathbf{i} U_{|v-u|-1}\left(\alpha_k\right) \right).$$

Therefore,

$$A_6^3 = \begin{bmatrix} -40 & 0 & 96 & 63 & 6 & 0 \\ 0 & -40 & 0 & 96 & 63 & 6 \\ 6 & 0 & -40 & 0 & 96 & 63 \\ 63 & 6 & 0 & -40 & 0 & 96 \\ 96 & 63 & 6 & 0 & -40 & 0 \\ 0 & 96 & 63 & 6 & 0 & -40 \end{bmatrix}.$$

Example 3.3. *Let A_4 be a 4×4 symmetric circulant matrix as following*

$$A_4 = \begin{bmatrix} -1 & 3 & 0 & 3 \\ 3 & -1 & 3 & 0 \\ 0 & 3 & -1 & 3 \\ 3 & 0 & 3 & -1 \end{bmatrix},$$

where $a_0 = -1,\ a_1 = a_{-1} = 3$ *and* $\alpha_k = \cos\frac{\pi(k-1)}{2}$. *From Theorem 2.2, (u,v)th entry of* A_4^5 *is that*

$$\left[A_4^5\right]_{u,v} = \frac{1}{4}\sum_{k=1}^{3} l_{6-2k}\left(-1+6T_1\left(\alpha_k\right)\right)^5 T_{|v-u|}\left(\alpha_k\right),$$

where

$$l_{6-2k} = \begin{cases} 1 & \text{if } k = 0,3 \\ 2 & \text{otherwise.} \end{cases}$$

Therefore,

$$A_4^5 = \begin{bmatrix} -3421 & 4983 & -3420 & 4983 \\ 4983 & 39354 & 4983 & -3420 \\ -3420 & 4983 & 39354 & 4983 \\ 4983 & -3420 & 4983 & 39354 \end{bmatrix}.$$

Example 3.4. *Let* A_7 *be a* 7×7 *symmetric circulant matrix as following*

$$A_7 = \begin{bmatrix} 3 & -2 & 0 & 0 & 0 & 0 & -2 \\ -2 & 3 & -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 3 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 3 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 & 3 & -2 \\ -2 & 0 & 0 & 0 & 0 & -2 & 3 \end{bmatrix},$$

where $a_0 = 3,\ a_1 = a_{-1} = -2$ *and* $\alpha_k = \cos\frac{2\pi(k-1)}{7}$. *From Theorem 2.2, (u,v)th entry of* A_7^3 *is that*

$$\left[A_7^3\right]_{u,v} = \frac{1}{7}\sum_{k=1}^{4} l_{9-2k}\left(-1-4T_1\left(\alpha_k\right)\right)^3 T_{|v-u|}\left(\alpha_k\right),$$

where

$$l_{9-2k} = \begin{cases} 1 & \text{if } k = 0 \\ 2 & \text{otherwise.} \end{cases}$$

Therefore,

$$A_7^3 = \begin{bmatrix} 99 & -78 & 36 & -8 & -8 & 36 & -78 \\ -78 & 99 & -78 & 36 & -8 & -8 & 36 \\ 36 & -78 & 99 & -78 & 36 & -8 & -8 \\ -8 & 36 & -78 & 99 & -78 & 36 & -8 \\ -8 & -8 & 36 & -78 & 99 & -78 & 36 \\ 36 & -8 & -8 & 36 & -78 & 99 & -78 \\ -78 & 36 & -8 & -8 & 36 & -78 & 99 \end{bmatrix}.$$

Let's give an example for inverse of the circulant matrix A_n given in (2).

Example 3.5. *Let A_5 be a 5×5 circulant matrix given in (2)*

$$A_5 = \begin{bmatrix} 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & -1 & 0 & 0 \\ 0 & 2 & 1 & -1 & 0 \\ 0 & 0 & 2 & 1 & -1 \\ -1 & 0 & 0 & 2 & 1 \end{bmatrix},$$

where $a_0 = 3,\ a_1 = -1,\ a_{-1} = 2$ *and*$\alpha_k = \cos\frac{2\pi(k-1)}{5}$. *By using (12), we have*

$$\begin{aligned} \lambda_1 &= 2, \\ \lambda_2 &= 1.30901 + 2.85316\mathbf{i}, \\ \lambda_3 &= 0.19098 + 1.76335\mathbf{i}, \\ \lambda_4 = \overline{\lambda_3} &= 0.19098 + 1.76335\mathbf{i}, \\ \lambda_5 = \overline{\lambda_2} &= 1.30901 + 2.85316\mathbf{i}, \end{aligned}$$

From Theorem 2.1, (u,v)th entry of inverse of A_5 is that

$$\left[A_5^{-1}\right]_{u,v} = \frac{1}{5}\left(S_1 + S_2\right),$$

where

$$S_1 = \sum_{k=1}^{3} \lambda_k^4 \left(T_{|v-u|}\left(\alpha_k\right) + \mathbf{sign}\left(v-u\right)\mathbf{sign}\left(\frac{7}{2} - k\right)\sqrt{1-\alpha_k^2}\mathbf{i}\lim_{\mathbf{j}\to\mathbf{k}} U_{|v-u|-1}\left(\alpha_j\right)\right),$$

and

$$S_2 = \sum_{k=2}^{3} \overline{\lambda_k}^4 \left(T_{|v-u|}\left(\alpha_k\right) - \mathbf{sign}\left(v-u\right)\mathbf{sign}\left(\frac{7}{2} - k\right)\sqrt{1-\alpha_k^2}\mathbf{i}U_{|v-u|-1}\left(\alpha_k\right)\right).$$

Therefore,

$$A_5^{-1} = \frac{1}{62}\begin{bmatrix} 11 & 21 & -5 & 13 & -9 \\ -9 & 11 & 21 & -5 & 13 \\ 13 & -9 & 11 & 21 & -5 \\ -5 & 13 & -9 & 11 & 21 \\ 21 & -5 & 13 & -9 & 11 \end{bmatrix}.$$

Appendix A: Following Maple 13 procedure calculates the rth power of the real circulant matrix $A_n = circ_n\left(a_0, a_1, 0, \ldots, 0, a_{-1}\right)$.

```
restart:
with(LinearAlgebra):
```

```
a[0]:='a[0]': a[1]:='a[1]': a[-1]:='a[-1]':
n:='n': r:='r':
f:=(i,j)-> piecewise(j-i=0,a[0],j-i=1,a[1],j-i=n-1,a[-1],i-j=1,a[-1],i-j=n-1,
a[1],0):
alpha:=k->evalf(cos(2*Pi*(k-1)/n)):
lambda:=k->evalf(a[0]+(a[-1]+a[1])*ChebyshevT(1,alpha(k))+(a[-1]-a[1])
*I*sign((n/2)+1-k)*(1-(alpha(k))^2)^(1/2)*limit(ChebyshevU(0,alpha(j)),
j=k)):
g:=(u,v)->evalf((1/n)*(sum((lambda(k)^r)*(ChebyshevT(abs(v-u),alpha
(k))+sign(v-u)*I*sign((n/2)+1-k)*sqrt(1-(alpha(k))^2)*limit(ChebyshevU
(abs(v-u)-1,alpha(j)),j=k)),k=1..floor(n/2)+1)+sum(conjugate(lambda(k))
^r*(ChebyshevT(abs(v-u),alpha(k))-sign(v-u)*I*sign((n/2)+1-k)*sqrt(1-
(alpha(k))^2)*ChebyshevU(abs(v-u)-1,alpha(k))),k=2..floor((n+1)/2)))):
A_n:=Matrix(n,n,f);
rth_power_of_A_n:=Matrix(n,n,g);
```

Appendix B: Following Maple 13 procedure calculates the rth power of the real symmetric circulant matrix $A_n = circ_n\,(a_0, a_1, 0, \ldots, 0, a_1)\,.$

```
restart:
with(LinearAlgebra):
a[0]:='a[0]': a[1]:='a[1]':
n:='n': r:='r':
f:=(i,j)->piecewise(j-i=0,a[0],j-i=1,a[1],j-i=n-1,a[1],i-j=1,a[1],i-j=n-1,a[1],0):
alpha:=k->evalf(cos(2*Pi*(k-1)/n)):
l:=(s)->piecewise(s=n,1,s=0,1,2):
g:=(u,v)->evalf((1/n)*sum(((l(n-2*k+2)*(a[0]+2*a[1]*ChebyshevT(1,alpha
(k)))^r)*(ChebyshevT(abs(v-u),alpha(k))),k=1..floor(n/2)+1))):
A_n:=Matrix(n,n,f);
rth_power_of_A_n:=Matrix(n,n,g);
```

4 Conclusion

Circulant matrices have a wide range of applications in modern science. We meet linear systems associated with circulant matrices in numerical solutions of some equations. In such cases, we need the powers of these matrices. In this paper, we derived a formula for the powers of a certain type of circulant matrix.

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