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Weingarten and Linear Weingarten Type Tubes with Darboux Frame in E^3

Ayşe Zeynep Azak[1] **and Murat Tosun**[2]

[1]Department of Primary Education, Faculty of Education
Sakarya University, Hendek-Sakarya, Turkey
E-mail: apirdal@sakarya.edu.tr
[2]Faculty of Arts and Science, Department of Mathematics
Sakarya University, Sakarya–54187, Turkey
E-mail: tosun@sakarya.edu.tr

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Abstract

In this study, we have found that the tube surface with Darboux frame is also Weingarten surface in the Euclidean 3-space E^3*. Then we get the necessary and sufficient conditions for being* (H, K_{II}), (K, K_{II}) *types Weingarten tube surfaces with Darboux frame. In addition to that we have proved that there are no* (H, K_{II}) *and* (K, K_{II})*-types linear Weingarten surfaces with Darboux frame. Here* K, K_{II} *and* H *are Gauss curvature, second Gauss curvature and mean curvature of tube surface, respectively.*

Keywords: *Darboux frame, Tube, Weingarten surface.*

1 Introduction

One of the important topics in differential geometry of surfaces in E^3 is to investigate surfaces with particular conditions on the curvatures, e.g. minimal surfaces, surfaces of constant Gauss curvature and surfaces of constant mean curvature, etc. Researchers study these surfaces to find special (geometric) coordinates. For this reason in 1861, J. Weingarten called a surface as Weingarten surface if the Gauss curvature K and mean curvature H satisfy a nontrivial relation $\psi(H, K) = 0$ or alternatively a nontrivial relation between the two principal curvatures [14]. On the other hand, if a surface satisfies a linear equation $aK+bH=c$ for some real numbers a,b,c with $(a,b,c)\neq(0,0,0)$, then it is said to be a linear Weingarten surface.

Weingarten and linear Weingarten surfaces have been of interest for geometers [3]-[9], [11]. One can see the applications of Weingarten surfaces on computer aided design and shape investigation [13].

A canal surface is the envelope of a family of one parameter spheres which were first investigated by Monge in 1850. Canal surface around the center curve $\gamma(s)$ is parameterized as

$$C(s,\theta)=\gamma(s)-r(s)r'(s)t(s)\mp r(s)\sqrt{1-r'(s)^2}\left(\cos\theta n(s)+\sin\theta b(s)\right);0\leq\theta\leq 2\pi,$$

where s is arclength parameter and t,n,b are Frenet vectors of $\gamma(s)$. If the radius function $r(s)=r$ is constant then the Canal surface is called tube (pipe, tubular) surface and it is parameterized as

$$Tube(s,\theta)=\gamma(s)+r\left(\cos\theta n(s)+\sin\theta b(s)\right).$$

J. S. Ro and D. W. Yoon studied tubes in Euclidean 3-space which are (K,H), (H,K_{II}), (K,K_{II})-Weingarten and linear Weingarten tubes [10]. Then Y. Tunçer, D. W. Yoon and M. K. Karacan extended the results of this paper and obtained some theorems for $(K_{II},H_{II}),(H,H_{II}),(K,H_{II})$ types Weingarten and linear Weingarten tubes [12]. Also, F. Doğan and Y. Yaylı defined tubes with respect to Darboux frame, computed Gauss and mean curvatures and obtained some characterizations for special curves lying on tube [2]. Taking into account these papers, we obtained the second mean curvature of tube with respect to Darboux frame, proved that tube with Darboux frame is a Weingarten tube and gave some characterizations in terms of the Gauss, mean and second mean curvatures for being $(K,H),(H,K_{II}),(K,K_{II})$ types Weingarten tubes with Darboux frame.

2 Preliminaries

Let M be a surface in R^3 and consider $\varphi=\varphi(s,\beta)$ is the local parametrization of M. $\wedge$ stands the cross product in R^3, so if we denote the unit normal vector field on M by N, then N is given by

$$N=\frac{\varphi_s\wedge\varphi_\beta}{\|\varphi_s\wedge\varphi_\beta\|}\tag{1}$$

where $\varphi_s=\frac{\partial\varphi}{\partial s}$ and $\varphi_\beta=\frac{\partial\varphi}{\partial\beta}$. Then the coefficients of the first fundamental form I and the second fundamental form II of surface M are defined by

$$E=\langle\varphi_s,\varphi_s\rangle\,,\quad F=\langle\varphi_s,\varphi_\beta\rangle\,,\quad G=\langle\varphi_\beta,\varphi_\beta\rangle\,,\tag{2}$$

and

$$e=\langle\varphi_{ss},N\rangle\,,\quad f=\langle\varphi_{s\beta},N\rangle\,,\quad g=\langle\varphi_{\beta\beta},N\rangle\,,\tag{3}$$

respectively.
So, the fundamental forms can be written as follows

$$I=E\,ds^2+2F\,ds\,d\beta+G\,d\beta^2,$$

$$II=e\,ds^2+2f\,ds\,d\beta+g\,d\beta^2.$$

On the other hand, the Gauss curvature K and the mean curvature H are given by

$$K=\frac{eg-f^2}{EG-F^2},\tag{4}$$

and

$$H=\frac{eG-2fF+gE}{2\left(EG-F^2\right)},\tag{5}$$

respectively.

With respect to Brioschi's Formula in the Euclidean 3-space, we are able to compute K_{II} of a surface by replacing the components of the first fundamental form E,F,G by the components of the second fundamental form e,f,g respectively. Consequently, the second Gauss curvature K_{II} of a surface is defined by [1],

$$K_{II}=\frac{1}{(eg-f^2)^2}\left\{\begin{vmatrix}-\frac{1}{2}e_{\beta\beta}+f_{s\beta}-\frac{1}{2}g_{ss} & \frac{1}{2}e_s & f_s-\frac{1}{2}e_\beta\\ f_\beta-\frac{1}{2}g_s & e & f\\ \frac{1}{2}g_\beta & f & g\end{vmatrix}-\begin{vmatrix}0 & \frac{1}{2}e_\beta & \frac{1}{2}g_s\\ \frac{1}{2}e_\beta & e & f\\ \frac{1}{2}g_s & f & g\end{vmatrix}\right\}.\tag{6}$$

Let M be a regular surface and $\gamma:I\subset R\rightarrow M$ be a unit speed curve on the surface. Then, Darboux frame $\{T,Y=N\times T,N\}$ is well-defined along the curve γ and N is the unit normal of M. Darboux equations for this frame are given by

$$\begin{aligned}T'&=k_gY+k_nN,\\ Y'&=-k_gT+\tau_gN,\\ N'&=-k_nT-\tau_gY,\end{aligned}\tag{7}$$

where k_n is the normal curvature, k_g is the geodesic curvature (such that $k_g=\kappa\cos\beta,\quad k_n=\kappa\sin\beta$) and τ_g is the geodesic torsion of γ and defined by $\tau_g=\tau-\beta'$ where β is the angle from osculating plane to the tangent plane.

The curve $\gamma\left(s\right)$ lying on a surface M satisfies the following:

(i) $\gamma(s)$ is a geodesic curve if and only if $k_g = 0$.
(ii) $\gamma(s)$ is an asymptotic curve if and only if $k_n = 0$.
(iii) $\gamma(s)$ is a principal line if and only if $\tau_g = 0$.

Let $\gamma(s)$ lie on the surface M. Since the characteristic circles of a canal surface lie on a plane which is perpendicular to the tangent of the center curve $\gamma(s)$, the tube with the Darboux frame can be written as

$$\varphi(s,\beta) = \gamma(s) + r(\cos\beta\, Y(s) + \sin\beta\, U(s)), \tag{8}$$

where $r > 0$ and U is the unit normal of the surface M along the curve $\gamma(s)$. Therefore, if we consider the equations (1), (2), (3) and use formulas (7), we have

$$N = -\cos\beta\, Y - \sin\beta\, U. \tag{9}$$

The components of the first and the second fundamental form of φ are found as

$$E = (1 - r\cos\beta\, k_g - r\sin\beta\, k_n)^2 + r^2\,\tau_g, \quad F = r^2\tau_g, \quad G = r^2, \tag{10}$$

and

$$\begin{aligned} e &= (k_g\cos\beta + k_n\sin\beta)\left[r(k_g\cos\beta + k_n\sin\beta) - 1\right] + r\,\tau_g^2, \\ f &= r\tau_g, \qquad g = r. \end{aligned} \tag{11}$$

If the second Gauss curvature is non-degenerate,

$$eg - f^2 = (k_g\cos\beta + k_n\sin\beta)\left[r(k_g\cos\beta + k_n\sin\beta) - 1\right] \neq 0,$$

that is, k_g and k_n are nowhere vanishing at the same time. In this case we can define formally the second Gauss curvature K_{II} on M. From the equations (4), (5) and (7) the Gauss and the mean curvatures are given by [2]

$$K = \frac{k_g\cos\beta + k_n\sin\beta}{r(1 - r\cos\beta\, k_g - r\sin\beta\, k_n)} \tag{12}$$

and

$$H = \frac{2r(k_g\cos\beta + k_n\sin\beta) - 1}{2r(rk_g\cos\beta + rk_n\sin\beta - 1)}, \tag{13}$$

3 Linear Weingarten and Weingarten Tube Surfaces

In this section, we will prove that the tube surface with the Darboux frame is also Weingarten surface and investigate the conditions for being (H, K_{II}) and (K, K_{II}) types Weingarten tube surfaces with the Darboux frame.

Theorem 3.1 *Let M be a tube surface with the Darboux frame in E^3. M is also a Weingarten surface.*

Proof: The partial differentiations of the equations (12) and (13) with respect to s and β parameters are, respectively,

$$K_\beta = \frac{\left(-k_g \sin\beta + k_n \cos\beta\right)\left[r\left(1 - rk_g \cos\beta - rk_n \sin\beta\right)\right] - \left[r\left(rk_g \sin\beta - rk_n \cos\beta\right)\right]\left(k_g \cos\beta + k_n \sin\beta\right)}{r^2\left(1 - rk_g \cos\beta - rk_n \sin\beta\right)^2}, \quad (14)$$

$$K_s = \frac{\left(k_g' \cos\beta + k_n' \sin\beta\right)\left[r\left(1 - rk_g \cos\beta - rk_n \sin\beta\right)\right] - \left[r\left(-rk_g' \cos\beta - rk_n' \sin\beta\right)\right]\left(k_g \cos\beta + k_n \sin\beta\right)}{r^2\left(1 - rk_g \cos\beta - rk_n \sin\beta\right)^2}, \quad (15)$$

$$H_\beta = \frac{2r\left(-k_g \sin\beta + k_n \cos\beta\right)\left[2r\left(rk_g \cos\beta + rk_n \sin\beta - 1\right)\right] - 2r\left(-rk_g \sin\beta + rk_n \cos\beta\right)\left[2r\left(k_g \cos\beta + k_n \sin\beta\right) - 1\right]}{4r^2\left(rk_g \cos\beta + rk_n \sin\beta - 1\right)^2}, \quad (16)$$

$$H_s = \frac{2r\left(k_g' \cos\beta + k_n' \sin\beta\right)\left[2r\left(rk_g \cos\beta + rk_n \sin\beta - 1\right)\right] - 2r\left(rk_g' \cos\beta + rk_n' \sin\beta\right)\left[2r\left(k_g \cos\beta + k_n \sin\beta\right) - 1\right]}{4r^2\left(rk_g \cos\beta + rk_n \sin\beta - 1\right)^2} \quad (17)$$

where $A = -k_g \sin\beta + k_n \cos\beta$, $B = k_g \cos\beta + k_n \sin\beta$, $S = k_g' \cos\beta + k_n' \sin\beta$. If we take the above equations, then the reduced forms of the equations (14), (15), (16), (17) are obtained as follows;

$$\begin{aligned} K_\beta &= \tfrac{A}{r(1-rB)^2}, \qquad K_S = \tfrac{S}{r(1-rB)^2}, \\ H_\beta &= -\tfrac{A}{2(1-rB)^2}, \quad H_s = -\tfrac{S}{2(1-rB)^2}. \end{aligned} \quad (18)$$

Now it is obvious that Jacobi identity satisfies

$$J(H, K) = K_s H_\beta - H_S K_\beta = 0.$$

Now we consider a tube surface M defined by (8) is (K, K_{II}) or (H, K_{II})-Weingarten surface with non-degenerate second fundamental form in E^3.

Theorem 3.2 *Let M be a tube surface defined by (8) with non-degenerate second fundamental form in E^3. Then, M is a (K, K_{II})-Weingarten surface if and only if $k_g = constant$ and $k_n = constant$.*

Proof: One can obtain the following equalities related to the coefficients of the second fundamental form

$$\begin{aligned} &eg - f^2 = rB\left(rB - 1\right), \qquad e_\beta = 2rAB - A, \\ &e_s = S\left(rB - 1\right) + rS^2 + 2r\tau_g\tau_g', \\ &f_s = r\tau_g', \qquad f_\beta = 0, \qquad g_s = 0, \qquad g_\beta = 0, \\ &e_{\beta\beta} = 2r\left(A^2 - B^2\right) + B, \qquad f_{s\beta} = 0, \qquad g_{ss} = 0. \end{aligned} \quad (19)$$

Substituting the last equations into the Brioschi's Formula given in (6), we find

$$K_{II} = \frac{4r^3B^4 - 6r^2B^3 + 2rB^2 + rA^2}{4r^2B^2\left(rB-1\right)^2}. \tag{20}$$

Also, partially differentiating of the last equation with respect to s and β gives

$$\left(K_{II}\right)_s = \frac{-8r^6SB^6 + 16r^5SB^5 - 16r^5SA^2B^3 + 24r^4SA^2B^2 - 16r^4AB^3A_s + 8r^4SB^4A_s + 8r^3AB^2A_s - 8r^3SA^2B}{16r^4B^4\left(rB-1\right)^4} \tag{21}$$

and

$$\left(K_{II}\right)_\beta = -\frac{8r^6AB^6 + 136r^5AB^5 + 120r^4AB^4 + 8r^3AB^3 + 16r^5A^3B^3 - 24r^4A^3B^2 + 8r^3A^3B}{16r^4B^4\left(rB-1\right)^4}. \tag{22}$$

Using the equations (14) and (15) together with the equations (21) and (22) the Jacobi identity for (K, K_{II}) yields

$$\begin{aligned} J\left(K, K_{II}\right) &= K_s\left(K_{II}\right)_\beta - K_\beta\left(K_{II}\right)_s \\ &= \frac{AS\left(-37r^2B^3 - 32rB^2 - 2B - 2r^2B^2A_s + 4rBA_s - 2AA_s\right)}{4rB^2(rB-1)^4}. \end{aligned}$$

Let us consider the equality $J\left(K, K_{II}\right) = 0$. Thereby, we have three possibilities to consider:

(i) Let $A = 0$. In this case, since $k_g = 0$ and $k_n = 0$, the second fundamental form K_{II} isn't non-degenerate.

(ii) Let $-37r^2B^3 - 32rB^2 - 2B - 2r^2B^2A_s + 4rBA_s - 2AA_s = 0$. In this case, B must vanish. Namely $k_g = 0$ and $k_n = 0$, we get $A = 0$. Consequently, this case coincident with the first case. So it isn't possible.

(iii) Let $S = 0$. In this case, we conclude that $k_g = \text{constant}$ and $k_n = \text{constant}$.

Conversely, suppose that the normal and geodesic curvatures are constants of the tube surface M with non-degenerate second fundamental form. Then it is easily seen that $J\left(K, K_{II}\right) = 0$.

Theorem 3.3 *Let M be a tube surface defined by (8) with non-degenerate second fundamental form in E^3. Then, M is (H, K_{II})-Weingarten surface if and only if $k_g = \text{constant}$ and $k_n = \text{constant}$.*

Proof: Considering the equations (16) and (17) together with the equations (21) and (22) the Jacobi identity for (H, K_{II}) becomes

$$\begin{aligned} J\left(H, K_{II}\right) &= H_s\left(K_{II}\right)_\beta - H_\beta\left(K_{II}\right)_s \\ &= \frac{A\left(37r^2B^3S + 32rB^2S - 2BS + 2r^2B^3AA_s - 4rBAA_s + 2AA_s\right)}{8rB^2(rB-1)^6}. \end{aligned}$$

Now, we will observe the equality $J(H, K_{II}) = 0$. Therefore, we investigate two cases, separately.

(i) If $A = 0$. This case can be satisfied for $k_g = 0$ and $k_n = 0$. Thus the second fundamental form can't be non-degenerate. This is a contradiction.

(ii) Let $37r^2B^3S + 32rB^2S - 2BS + 2r^2B^3AA_s - 4rBAA_s + 2AA_s = 0$. In this case, S and A_s must be zero. This case is possible, when $k_g =$ constant and $k_n =$ constant.

Conversely, suppose that the normal and geodesic curvatures are constants of the tube surface M with non-degenerate second fundamental form. Then, M is a $J(H, K_{II})$ Weingarten surface.
Hence we have the following theorems.

Theorem 3.4 *Let M be a tube surface defined by (8) in the Euclidean 3-space. M is a (H, K)-type linear Weingarten surface. Then the center curve $\gamma(s)$ is both asymptotic and geodesic on M if $a \neq -3r^2c$.*

Proof: If M is a (H, K)-type linear Weingarten surface, then the following equation can be written

$$aK + bH = c$$

where a, b, c are constants. From (12) and (13), we get

$$\left(-2a + 2rb - 2cr^2\right) k_g \cos\beta + \left(-2a + 2rb - 2cr^2\right) k_n \sin\beta - b - 2rc = 0. \quad (23)$$

Since $\cos\beta, \sin\beta$ and 1 are linear independent, all of the coefficients for equation (23) must be zero. This yields to

$$\begin{aligned} (a + 3r^2c)\, k_g &= 0 \\ (a + 3r^2c)\, k_n &= 0 \\ b + 2rc &= 0. \end{aligned}$$

Thus, for $a \neq -3r^2c$ we obtain that $k_g = 0$ and $k_n = 0$. This means that the center curve $\gamma(s)$ is both asymptotic and geodesic.

Theorem 3.5 *Let M be a tube surface defined by (8) with non-degenerate fundamental form in Euclidean 3-space. Then there is no (K, K_{II})-type linear Weingarten surface.*

Proof: A tube surface with the Darboux frame is called (K, K_{II})-type linear Weingarten surface if the mean and the second mean curvatures satisfy the following equality

$$aK + bK_{II} = c$$

where a, b, c are constants. Using equations (12) and (20), we get

$$\sum_{i=1}^{4} p_i \cos^i \beta + \sum_{j=2}^{4} q_j \sin^j \beta = 0$$

such that

$$p_1 = \left(-4ar^2 + 4br^3 - 4cr^4\right) 4k_g k_n^3 \sin^3 \beta + \left(4ar - 6br^2 + 8cr^3\right) 3k_g k_n^2 \sin^2 \beta + \left(2br - 4cr^2\right) k_g k_n \sin \beta$$

$$p_2 = \left(-4ar^2 + 4br^3 - 4cr^4\right) 6k_g^2 k_n^2 \sin^2 \beta + \left(4ar - 6br^2 + 8cr^3\right) 3k_g^2 k_n \sin \beta + \left(2rb - 4cr^2\right) k_g^2 + brk_n^2$$

$$p_3 = \left(-4ar^2 + 4br^3 - 4cr^4\right) 4k_g^3 k_n \sin \beta + \left(4ar - 6br^2 + 8cr^3\right) k_g^3$$

$$p_4 = \left(-4ar^2 + 4br^3 - 4cr^4\right) k_g^4.$$

and

$$\begin{aligned} q_2 &= \left(2br - 4cr^2\right) k_n^2 + brk_g^2 \\ q_3 &= \left(4ar - 6br^2 + 8cr^3\right) k_n^3 \\ q_4 &= \left(-4ar^2 + 4br^3 - 4cr^4\right) k_n^4. \end{aligned}$$

Since the coefficients p_i and q_i are zero for $i = 1, 2, 3, 4$ and $j = 2, 3, 4$, one have $k_n = 0$ and $k_g = 0$. This means that the tube surface M can't be non-degenerate.

This completes the proof.

Theorem 3.6 *Let M be a tube surface defined by (8) with non-degenerate fundamental form in Euclidean 3-space. Then there is no (H, K_{II})-type linear Weingarten surface.*

Proof: Let us consider that M is a (H, K_{II})-type linear Weingarten surface, then one can write

$$aH + bK_{II} = c$$

where a, b, c are constants. Substituting the equations (13) and (20) in the above equation, we have

$$\sum_{i=1}^{4} r_i \cos^i \beta + \sum_{j=2}^{4} m_j \sin^j \beta = 0$$

such that

$$r_1 = (4ar^3 + 4br^3 - 4cr^4)\, 4k_g k_n^3 \sin^3\beta + (-6ar^2 - 6br^2 + 8cr^3)\, 3k_g k_n^2 \sin^2\beta + (4ar + 2br - 4cr^2)\, k_g k_n \sin\beta$$

$$r_2 = (4ar^3 + 4br^3 - 4cr^4)\, 6k_g^2 k_n^2 \sin^2\beta + (-6ar^2 - 6br^2 + 8cr^3)\, 3k_g^2 k_n \sin\beta + (2ar + 2br - 4cr^2)\, k_g^2 + brk_n^2$$

$$r_3 = (4ar^3 + 4br^3 - 4cr^4)\, 4k_g^3 k_n \sin\beta + (-6ar^2 - 6br^2 + 8cr^3)\, k_g^3$$

$$r_4 = (4ar^3 + 4br^3 - 4cr^4)\, k_g^4.$$

and

$$m_2 = (2ar + 2br - 4cr^2)\, k_n^2 + brk_g^2$$
$$m_3 = (-6ar^2 - 6br^2 + 8cr^3)\, k_n^3$$
$$m_4 = (4ar^3 + 4br^3 - 4cr^4)\, k_n^4.$$

Since the coefficients r_i and m_j are zero for $i = 1, 2, 3, 4$ and $j = 2, 3, 4$, we obtain $k_g = 0$ and $k_n = 0$. This means that the tube surface M can't be non-degenerate.

This completes the proof.

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